On the structure of covers of sofic shifts

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November 26 2010

Overview

Presentations of sofic shifts

Irreducible sofic shifts

Generalizing the Fischer cover

Sofic shifts

Sofic shifts and presentations

$E = (E^0, E^1, r, s)$ is a finite directed graph.
$\mathfrak{a}$ is a finite set (alphabet).
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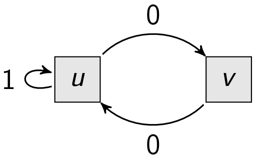
$$\mathsf{X}_{(E,\mathcal{L})} = \left\{ (\mathcal{L}(x_i))_i \in \mathfrak{a}^{\mathbb{Z}} \mid x_i \in E^1, r(x_i) = s(x_{i+1}) \right\}.$$

Example: The even shift



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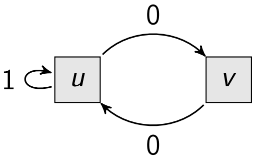

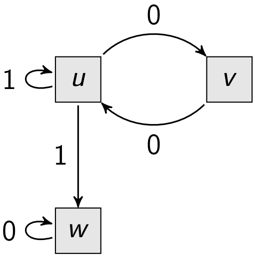

A nice presentation: The Krieger cover

X sofic shift.

$X^+ = \{x_0x_1x_2\ldots \mid x \in \mathsf{X}\}$ (right rays)
$X^- = \{\ldots x_{-3}x_{-2}x_{-1} \mid x \in \mathsf{X}\}$ (left rays)

For $x^+ \in \mathsf{X}^+$, define the **predecessor set** of x^+ to be $P_\infty(x^+) = \{y^- \in \mathsf{X}^- \mid y^-x^+ \in \mathsf{X}\}$.

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The left Krieger cover of X is a labelled graph $(E_K, \mathcal{L}_K)$

Vertices: $E_K^0 = \{P_\infty(x^+) \mid x^+ \in \mathsf{X}^+\}$,

Edges: Draw an edge labelled $a \in \mathfrak{a}$ from $P \in E_K^0$ to $P' \in E_K^0$ if and only if there exists $x^+ \in \mathsf{X}^+$ such that $P = P_\infty(ax^+)$ and $P' = P_\infty(x^+)$.

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Past set cover: Use predecessor sets of words (finite factors) instead of predecessor sets of rays.

Example: Krieger cover of the even shift

$$P_\infty(0^{2n}1x^+) = \{y^-10^{2k} \in \mathsf{X}^- \mid k \in \mathbb{N}_0\} \cup \{0^\infty\}$$

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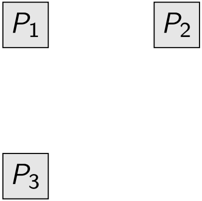

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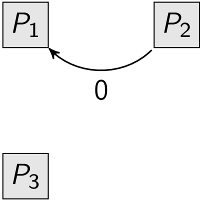

Edge:
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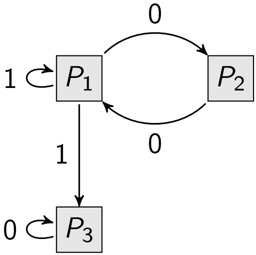

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Irred. sofic shifts and the Fischer cover

For now:
Assume that X is **irreducible**, i.e. there exists an irreducible (transitive, strongly connected) presentation of X.

A presentation $(E, \mathcal{L})$ of X is **left-resolving** if no vertex in E^0 receives two edges with the same label.

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Theorem (Fischer)

There is a unique minimal left-resolving presentation $(E_K, \mathcal{L}_K)$ *of* X *when* X *is irreducible.*

This presentation is the **left Fischer cover** of X.

Layers in the Krieger cover

For $v \in E_F^0$ define $P_\infty(v)$ to be the set of left rays which have a presentation terminating at v.

For $x^+ \in \mathsf{X}^+$ define $S(x^+)$ to be the set of vertices in E_F^0 that are sources of presentations of x^+.

Note: $P_\infty(x^+) = \cup_{v \in S(x^+)} P_\infty(v)$

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A vertex $P_\infty(x^+) \in E_K^0$ is in the nth **layer** of the left Krieger cover if n is the smallest number such that there exist $v_1, \ldots, v_n \in E_F^0$ with $P_\infty(x^+) = P_\infty(v_1) \cup \cdots \cup P_\infty(v_n)$.

x^+ is said to be **1/n-synchronizing**.

Same definition can be used for the **past set cover**.

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$$P_1 = \{y^- 10^{2k} \in \mathsf{X}^- \mid k \in \mathbb{N}_0\} \cup \{0^\infty\} = P_\infty(u)$$
$$P_2 = \{y^- 10^{2k+1} \in \mathsf{X}^- \mid k \in \mathbb{N}_0\} \cup \{0^\infty\} = P_\infty(v)$$
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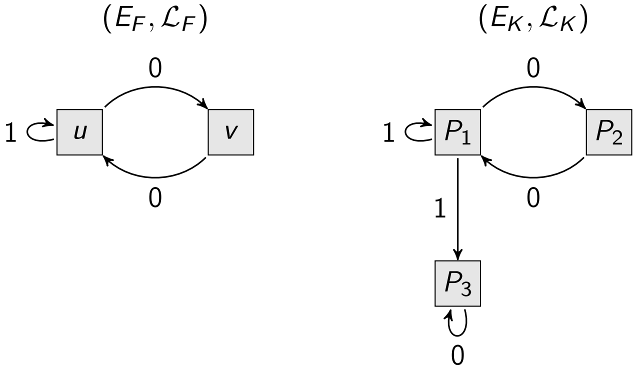
Left Fischer cover and left Krieger cover:



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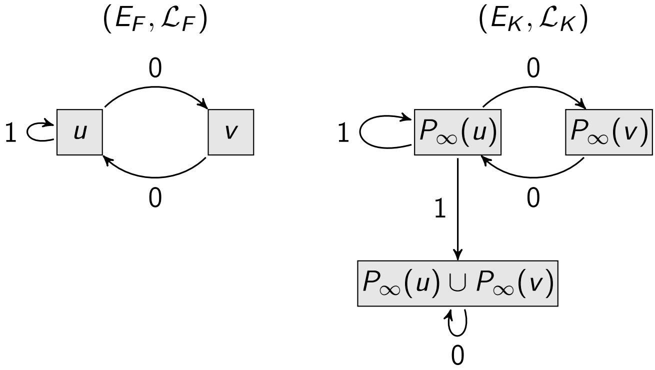

Theorem (Krieger)

The left Fischer cover is (isomorphic as a labelled graph to) the first layer of the left Krieger cover.

Proof.
Identify $u \in E_F^0$ with $P_\infty(u) \in E_K^0$. □

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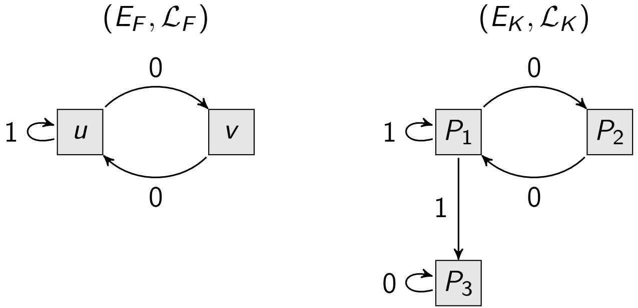
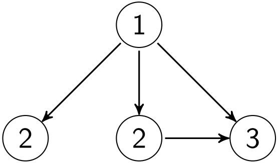


Structure of the layers

Proposition

If there is an edge in E_K which starts at a vertex in the mth layer and ends at a vertex in the nth layer then $m \leq n$.



Proof.
Blackboard. □

Flow equivalence

Conjugacy: Shift commuting homeomorphism $\Phi \colon X_1 \to X_2$.

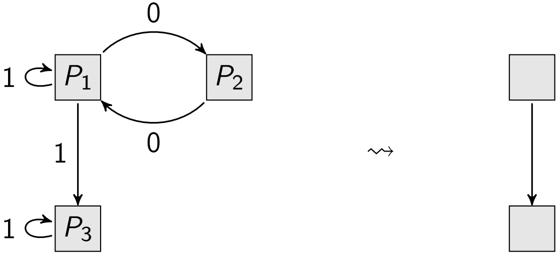
Symbol expansion: Given $a \in \mathfrak{a}$ and $\diamond \notin \mathfrak{a}$ replace every occurrence of a by $a\diamond$ in each $x \in X$.

Flow equivalence: Equivalence relation generated by

- Conjugacy
- Symbol expansion
- Symbol contraction

A flow invariant

Proper communicating graph:



Theorem (Bates, Eilers, Pask)

The proper communicating graph of the left Krieger cover of a sofic shift is a flow invariant.

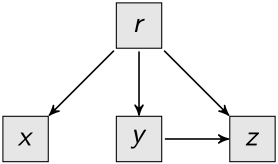
Range

Proposition

A directed graph E is the proper communicating graph of the left Krieger cover of an irreducible sofic shift if and only if it is finite, contains no closed walk, and has maximal vertex.

"$\Rightarrow$": Clear.
"$\Leftarrow$": By construction.

E



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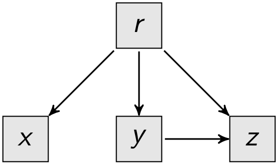
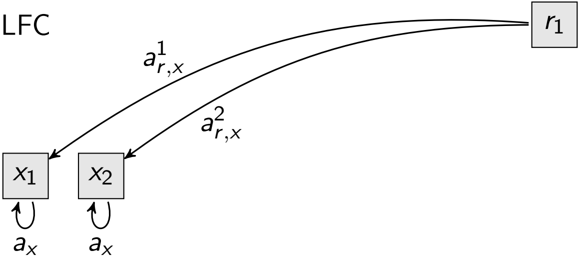
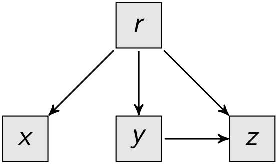


LFC



E



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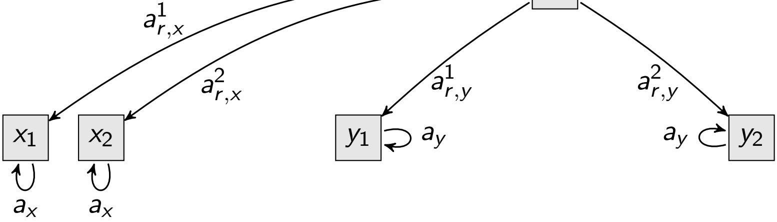

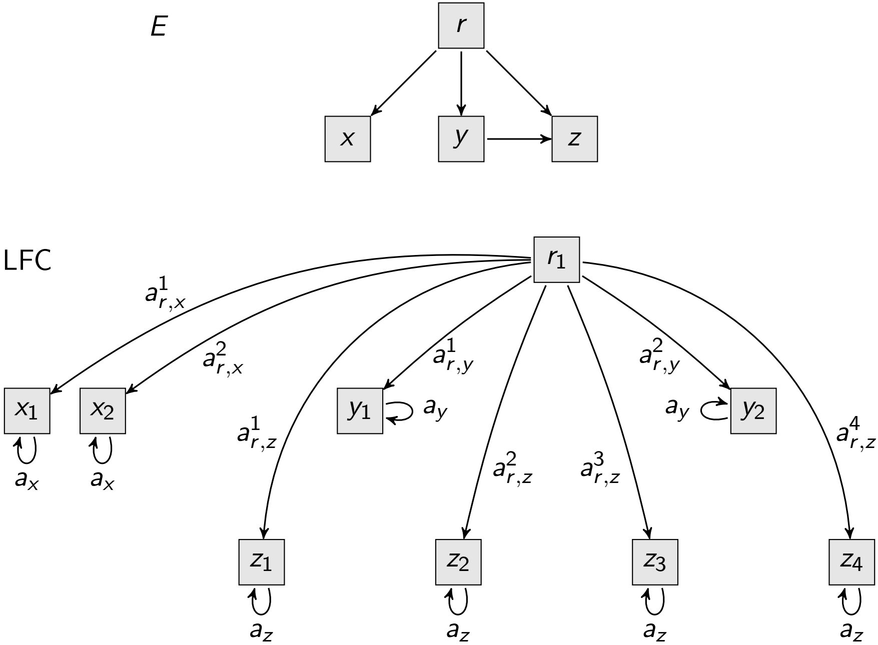
E
r
x
y
z
LFC
r_1
$a^1_{r,x}$
$a^2_{r,x}$
$a^1_{r,y}$
$a^2_{r,y}$
$a^1_{r,z}$
$a^2_{r,z}$
$a^3_{r,z}$
$a^4_{r,z}$
x_1
x_2
y_1
y_2
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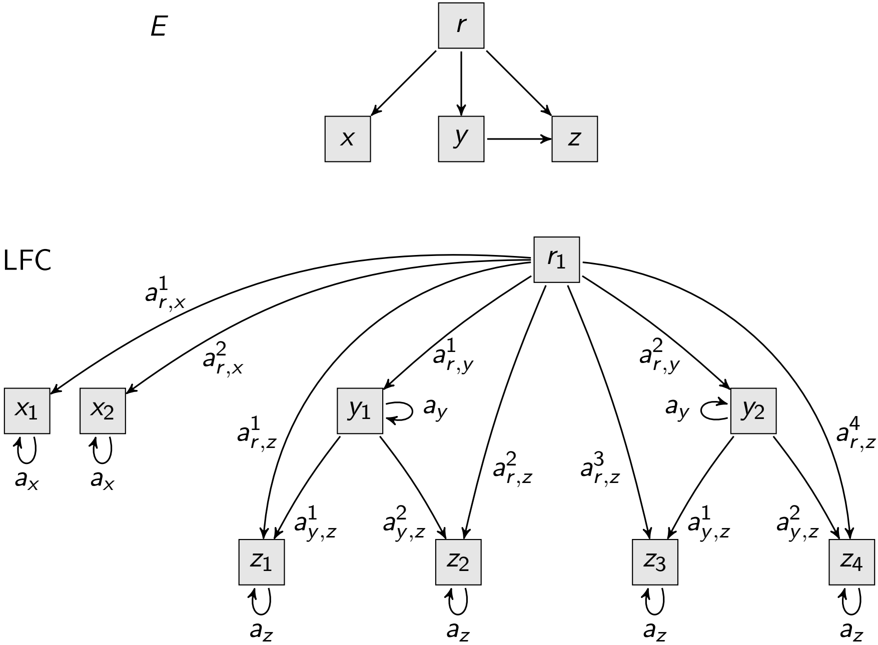

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r
x
y
z
LFC
r_1
$a^1_{r,x}$
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x_1
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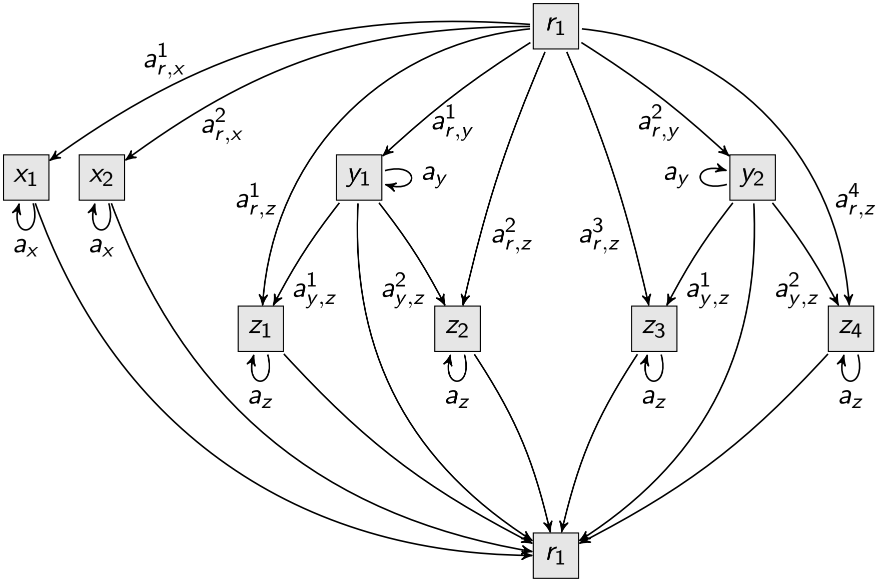

r_1
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$a^2_{r,x}$
$a^1_{r,y}$
$a^2_{r,y}$
x_1
x_2
y_1
a_y
a_y
y_2
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a_x
a_x
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$a^1_{y,z}$
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z_1
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Goal:

- Find generalization of the left Fischer cover
- Use generalized left Fischer cover to define layers

Generalizing the left Fischer cover

Idea: Find a suitable subgraph of the left Krieger cover.

$P \in E_K^0$ is said to be **decomposable** if there exist $n > 1$ and $P_1, \ldots, P_n \in E_K^0 \setminus \{P\}$ such that $P_1 \cup \cdots \cup P_n = P$.

Lemma

If P is non-decomposable then the subgraph of $(E_K, \mathcal{L}_K)$ induced by $E_K^0 \setminus \{P\}$ is not a presentation of X.

Generalized left Fischer cover

$E_G^0 = \{P \in E_K^0 \mid \text{Path in } E_K \text{ from } P \text{ to non-decomp. } P'\}$.
$(E_G, \mathcal{L}_G)$ the labelled subgraph of $(E_K, \mathcal{L}_K)$ induced by E_G^0.

Properties of the generalized left Fischer cover

1. $(E_G, \mathcal{L}_G)$ is an essential, left-resolving, and predecessor-separated presentation of X.
2. If X is irreducible then $(E_G, \mathcal{L}_G) = (E_F, \mathcal{L}_F)$.
3. When X_1, X_2 have disjoint alphabets then the generalized left Fischer cover of $X_1 \cup X_2$ is obtained as the disjoint union of the generalized left Ficher covers of X_1 and X_2.

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Proof.

1. Given $y^- \in X^-$ choose $x^+ \in X^+$ such that $y^- \in P_\infty(x^+)$. Choose non-decomp. $P_1, \ldots, P_n \in E_K^0$ such that $P_\infty(x^+) = \cup_{i=1}^n P_i$, and i such that $y^- \in P_i$. Now there is a path in $(E_K, \mathcal{L}_K)$ labelled y^- terminating at P_i. This is also a path in $(E_G, \mathcal{L}_G)$.
   Inherited: Left-resolving and predecessor-separated.
2. $P \in E_K^0$ non-decomposable $\Leftrightarrow P \in E_F^0$.
3. Inherited from the left Krieger cover.

Canonical

Theorem

The generalized left Fischer cover is canonical, i.e. if $\Phi \colon \mathsf{X}_1 \to \mathsf{X}_2$ *is a conjugacy and* $\pi_i \colon \mathsf{X}_{E_{G_i}} \to \mathsf{X}_{(E_{G_i}, \mathcal{L}_{G_i})} = \mathsf{X}_i$ *is the covering map of the generalized left Fischer cover of* X_i *then there is a conjugacy* $\phi \colon \mathsf{X}_{E_{G_1}} \to \mathsf{X}_{E_{G_2}}$ *such that* $\Phi \circ \pi_1 = \pi_2 \circ \phi$.

Proof uses strategy and techniques used by Nasu to prove an analogous result for the Krieger cover.

Bipartite codes

$\Phi\colon \mathsf{X}_1 \to \mathsf{X}_2$ is a **bipartite code** if there exist injective maps $f_1\colon \mathfrak{a}_1 \to \mathfrak{c}\mathfrak{d}$ and $f_2\colon \mathfrak{a}_2 \to \mathfrak{d}\mathfrak{c}$ such that

$$x \in \mathsf{X}_1, y = \Phi(x), f_1(x_i) = c_i d_i \quad \Rightarrow \quad f_2(y_i) = d_i c_{i+1} \text{ for all } i$$

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Recoding:
Replace X_1 by $\hat{\mathsf{X}}_1 = \{((f_1(x_i))_i \mid x \in \mathsf{X}_1\} \subseteq (\mathfrak{c}\mathfrak{d})^{\mathbb{Z}}$.
Replace X_2 by $\hat{\mathsf{X}}_2 = \{((f_2(y_i))_i \mid y \in \mathsf{X}_2\} \subseteq (\mathfrak{d}\mathfrak{c})^{\mathbb{Z}}$.
Replace Φ by $\hat{\Phi}\colon \hat{\mathsf{X}}_1 \to \hat{\mathsf{X}}_2$, $\hat{\Phi}((c_i d_i)_i) = (d_i c_{i+1})_i$.

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Theorem (Nasu)

Any conjugacy is a product of bipartite codes.

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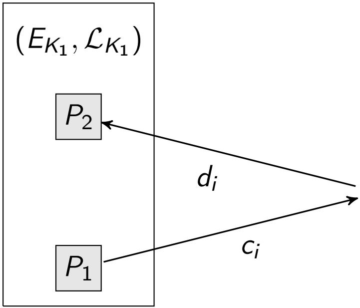

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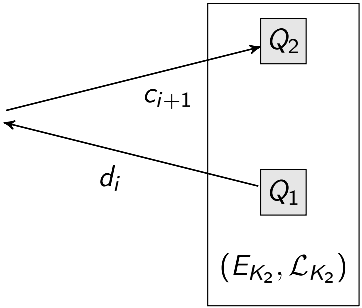

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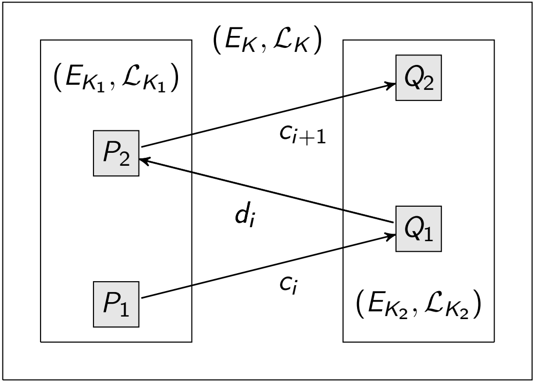

Nasu: $(E_K, \mathcal{L}_K)$ left Krieger cover of a sofic shift X.
E_K bipartite graph with induced graphs E_{K_1}, E_{K_2}.

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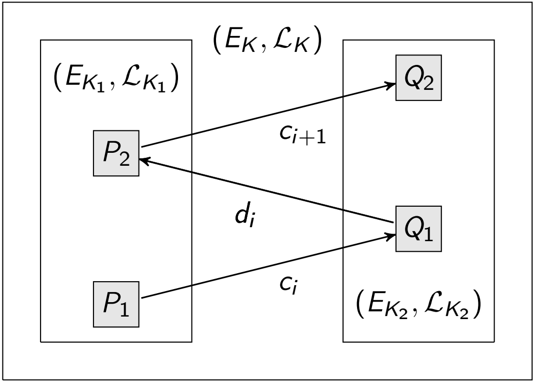

$$\Phi\colon X_1 \to X_2$$
$$\Phi((c_i d_i)_i) = (d_i c_{i+1})_i$$

$$\phi\colon X_{E_{K_1}} \to X_{E_{K_2}}$$
$$\phi((e_i f_i)_i) = (f_i e_{i+1})_i$$

$$\pi_i\colon X_{E_{K_i}} \to X_i$$
$$\Phi \circ \pi_1 = \pi_2 \circ \phi$$

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$(E_{G_i}, \mathcal{L}_{E_{G_i}})$ generalized left Fischer cover of X_i, $(E_G, \mathcal{L}_{E_G})$ generalized left Fischer cover of X.

Lemma: E_G bipartite, induced subgraphs E_{G_1} and E_{G_2}.

Corollary: Path in $E_{G_1} \leftrightarrow$ path in E_{G_2}, so GLFC is canonical.

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Proof of lemma.
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Given $P \in E^0_{G_i}$ there is a path in E_{K_i} from P to a non-decomposable $P' \in E^0_{K_i}$. This is also a path in E_K, so $P \in E^0_G$.

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Given $P_1 \in E^0_G$ there is a path in E_K from P_1 to a non-decomposable $P_2 \in E^0_K$. We are done if P_1, P_2 are in the same $E^0_{K_i}$, so assume $P_1 \in E^0_{K_1}$, $P_2 \in E^0_{K_2}$. E_K is essential, so there must be an edge from P_2 to a vertex $P' \in E^0_{K_1}$. If P' is decomposable in E_K then there must be an edge with the same label from P_2 to a non-decomposable $P'' \in E^0_K$. This gives a path from P_1 to P'' in E_{K_1}, so $P_1 \in E^0_{G_1}$. □

Layers

A vertex $P \in E_K^0$ is in the nth **layer** of the left Krieger cover if n is the smallest number such that there exist $v_1, \dots, v_n \in E_G^0$ with $P = P_\infty(v_1) \cup \cdots \cup P_\infty(v_n)$.

The first layer is the generalized left Fischer cover.

Proposition

If there is an edge in E_K which starts at a vertex in the mth layer and ends at a vertex in the nth layer then $m \leq n$.