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A NEW ALGEBRAIC INVARIANT FOR WEAK EQUIVALENCE OF SOFIC SUBSHIFTS

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Abstract: It is studied how taking the inverse image by a sliding block code affects the syntactic semigroup of a sofic subshift. Two independent approaches are used: ζ-semigroups as recognition structures for sofic subshifts, and relatively free profinite semigroups. A new algebraic invariant is obtained for weak equivalence of sofic subshifts, by determining which classes of sofic subshifts naturally defined by pseudovarieties of finite semigroups are closed under weak equivalence. Among such classes are the classes of almost finite type subshifts and aperiodic subshifts. The algebraic invariant is compared with other robust conjugacy invariants.

Keywords: Sofic subshift, conjugacy, weak equivalence, ζ-semigroup, pseudovariety.

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1. Introduction

Dynamical systems were first introduced in order to study systems of differential equations used to model physical phenomena. When discretizing both time and space, the physical system becomes a "symbolic" dynamical system that yields information on the real one. Symbolics dynamics is a very active area that borrows its methods from various fields such as combinatorics, algebra, automata theory, probabilities, etc, and has applications in coding theory, data storage and transmission, linear algebra...

The symbolic dynamical systems or *subshifts*, are sets of bi-infinite words, topologically closed and invariant under a *shift* operation. When trying to classify these systems, there happens to be a natural notion of equivalence between them, called *conjugacy*. Despite a rich litterature on the subject, the decidability of conjugacy remains wide open, namely for the class of finite type subshifts, the most studied subclass of sofic subshifts. To try to cope with this major difficulty, some weaker notions of equivalence of

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subshifts were introduced: see [18, 6]. The *shift equivalence* has been the most important of them; it is decidable, although the algorithm is quite intricate. In this paper, we focus on *weak equivalence*, defined by Béal and Perrin in [6], which relies on inverse images of sliding block codes (the morphisms between subshifts). We deduce an algebraic invariant for the weak equivalence of sofic subshifts. Moreover, we exhibit a pair of two sofic subshifts for which that invariant is used to easily prove that they are not weak equivalent, while various robust invariants fail to detect that they are not conjugate. The significance of this example is more appreciated once we realize that the weak equivalence relation really deserves its name, in the sense that there are very general sufficient conditions for two subshifts to be weak equivalent [6].

We briefly sketch the nature of our algebraic invariant. There is a natural bijection between subshifts and factorial prolongable languages. The sofic subshifts are precisely those whose corresponding language is rational. A well established method of classification of rational languages is by grouping them in *varieties of rational languages.* By the well known Eilenberg's Correspondence Theorem, varieties of languages are in a natural correspondence with pseudovarieties of finite semigroups. In this way a pseudovariety of finite semigroups defines naturally a class of sofic subshifts. In [12] it was determined which of these classes are closed under taking conjugate subshifts. It was also proved that such classes are closed under shift equivalence. In this paper we prove they are also closed under weak equivalence. The arguments used in [12] are based in the equational description of a pseudovariety using pseudoidentities. These arguments are somewhat heavy and it seems difficult to adapt them for the weak equivalence case, hence here we use different approaches.

The paper is organized as follows. Preliminary definitions and results are made in Section 2. Division between subshifts and weak equivalence are introduced in Section 3. After a preparatory section about transducers, we arrive to Section 5, dealing with the perspective of the first author Master's Thesis [11] of seeing finite ζ-semigroups (a generalization of ω-semigroups) as recognition structures for sofic subshifts. From a Theorem of [11] about how the operation of taking the inverse image of a subshift by a sliding block is reflected in the corresponding syntactic ζ-semigroups, we deduce a similar result concerning syntactic semigroups in the usual sense. With this result, we obtain in Section 6 our algebraic invariant, and using it we list some important classes of sofic subshifts closed under taking weak equivalent subshifts. The dynamic significance of this algebraic invariant is evaluated in Section 7. The content from Section 6 is recovered in Section 8 with results about relatively free profinite semigroups, with great proof economy. This approach complements the one using ζ-semigroups, which demanded a longer and heavier preparation, but produced more intermediate results, of a more precise nature. Another advantage of ζ-semigroups is that with a little additional effort one can use this approach to generalize results about semigroups to results about *ordered semigroups*: this is done in Section 9.

As general references for symbolic dynamics see [3, 18]. For semigroup theory, rational languages and finite automata see [21, 1, 2].

2. Preliminaries

2.1. Subshifts and codes. Let A be an alphabet. All alphabets in this paper are assumed to be finite. Let $A^{\mathbb{Z}}$ be the set of sequences of letters of A indexed by $\mathbb{Z}$. A factor of an element $(x_i)_{i\in\mathbb{Z}}$ of $A^{\mathbb{Z}}$ is a finite sequence $x_k x_{k+1} \cdots x_{k+n-1} x_{k+n}$, denoted by $x_{[k,k+n]}$, where $k \in \mathbb{Z}$ and $n \geq 0$. We endow $A^{\mathbb{Z}}$ with the product topology with respect to the discrete topology of A. Recall that the topology of $A^{\mathbb{Z}}$ is characterized by the fact that a sequence $(x^{(n)})_n$ of elements of $A^{\mathbb{Z}}$ converges to x if and only if for every positive integer k there is p_k such that $n \geq p_k$ implies $(x^{(n)})_{[-k,k]} = x_{[-k,k]}$. Note that $A^{\mathbb{Z}}$ is a compact Hausdorff space. From here on, *compact* will mean both compact and Hausdorff.

Denote by $A^{\tilde{\omega}}$ (respectively A^{ω}) the set of sequences of letters of A indexed by the set of negative integers (respectively non-negative integers). The map $\varphi : x \mapsto (\ldots x_{-3}x_{-2}x_{-1}, x_0x_1x_2\ldots)$ is an homeomorphism from $A^{\mathbb{Z}}$ to $A^{\tilde{\omega}} \times A^{\omega}$. The sequence $\varphi^{-1}(z,t)$ is usually denoted by $z.t$. Given an element u of A^+, we denote by u^{ω} the element of A^{ω} given by the right-infinite concatenation $uuu\ldots$. Dually, $u^{\tilde{\omega}} = \ldots uuuu$. Finally, u^{ζ} denotes the element $u^{\tilde{\omega}} \cdot u^{\omega}$ of $A^{\mathbb{Z}}$. We denote by $A^{<n}$ (respectively $A^{\leq n}$) the set of elements of A^* with length less (respectively less or equal) than n.

The *shift* in $A^{\mathbb{Z}}$ is the bijective function σ_A (or just σ) from $A^{\mathbb{Z}}$ to $A^{\mathbb{Z}}$ defined by $\sigma_A((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}$. A *shift dynamical system* or *subshift* of $A^{\mathbb{Z}}$ is a closed subset $\mathcal{X}$ of $A^{\mathbb{Z}}$ such that $\sigma_A(\mathcal{X}) \subseteq \mathcal{X}$ and $\sigma_A^{-1}(\mathcal{X}) \subseteq \mathcal{X}$.

If $\mathcal{X}$ is a subshift of $A^{\mathbb{Z}}$ then we denote by $L(\mathcal{X})$ the set of finite factors of elements of $\mathcal{X}$. There is a subset $\mathcal{F}$ of A^+ such that $L(\mathcal{X}) = A^+ \setminus A^*\mathcal{F}A^*$; a set $\mathcal{F}$ in such conditions is called a set of *forbidden words* for $\mathcal{X}$. A subshift is of *finite type* if it has a finite set of forbidden words. An element x of $A^{\mathbb{Z}}$

belongs to $\mathcal{X}$ if and only if every finite factor of x belongs to $L(\mathcal{X})$. The correspondence $\mathcal{X} \mapsto L(\mathcal{X})$ is a bijection between the set of subshifts of $A^{\mathbb{Z}}$ and the set of factorial prolongable languages of A^+.

A *sliding block code* (or more briefly, a *code*) F between the subshifts $\mathcal{X}$ of $A^{\mathbb{Z}}$ and $\mathcal{Y}$ of $B^{\mathbb{Z}}$ is a function $F : \mathcal{X} \to \mathcal{Y}$ for which there are integers $k, l \geq 0$ and a function $f : A^{k+l+1} \to B$ such that $F(x) = (f(x_{[i-k,i+l]}))_{i\in\mathbb{Z}}$. If we can choose f such that $k + l + 1 = n$ then we say that F has window size n. We say that f is a *block map* of F with *memory* k and *anticipation* l. The sliding block code F depends only on the restriction of f to $A^{k+l+1} \cap L(\mathcal{X})$.

It is well known [16] that a map $F : \mathcal{X} \subseteq A^{\mathbb{Z}} \to \mathcal{Y} \subseteq B^{\mathbb{Z}}$ between subshifts is a code if and only if it is a continuous function such that $F \circ \sigma_A = \sigma_B \circ F$. The identity transformation of a subshift is a code, the composition of two codes is a code and the inverse of a bijective code is a code. A bijective code is called a *conjugacy*. Two subshifts are *conjugate* if there is a conjugacy between them. A *conjugacy invariant* is a property of subshifts that is preserved for taking conjugate subshifts. See [18] for information about ordinary conjugacy invariants like the zeta function.

Two codes $\varphi_1 : \mathcal{X}_1 \to \mathcal{Y}_1$ and $\varphi_2 : \mathcal{X}_2 \to \mathcal{Y}_2$ are said to be *conjugate* if there are conjugacies $f : \mathcal{X}_1 \to \mathcal{X}_2$ and $g : \mathcal{Y}_1 \to \mathcal{Y}_2$ such that $\varphi_2 \circ f = g \circ \varphi_1$. We say that the pair (f, g) is a *conjugacy* between φ_1 and φ_2.

Given an alphabet A and $k \geq 1$, consider the alphabet A^k. To avoid ambiguities, we represent an element $w_1 \dots w_n$ of $(A^k)^+$ (with $w_i \in A^k$) by $\langle w_1, \dots, w_n\rangle$. For $k \geq 0$ let Φ_k be the function from A^+ to $(A^{k+1})^*$ defined by

$$\Phi_k(a_1 \dots a_n) = \begin{cases} 1 & \text{if } n \leq k, \\ \langle a_{[1,k+1]}, a_{[2,k+2]}, \dots a_{[n-k-1,n-1]}, a_{[n-k,n]}\rangle & \text{if } n > k, \end{cases}$$

where $a_i \in A$ and $a_{[i,j]} = a_i a_{i+1} \dots a_{j-1} a_j$. For a map $f : A^k \to B$, let $\hat{f}$ be the unique monoid homomorphism from $(A^k)^*$ to B^* extending f. Let $\bar{f}$ be $\hat{f} \circ \Phi_{k-1}$. Our interest in $\bar{f}$ relies on the fact that if $F : A^{\mathbb{Z}} \to B^{\mathbb{Z}}$ is a code with memory k and anticipation l then $F(x)_{[i,j]} = \bar{f}(x_{[i-k,i+l]})$.

By *graph* we mean an oriented graph. A *labeled graph* (G, π) is a pair such that G is a graph and π is a function mapping edges of G into letters of an alphabet A. We consider (G, π) as an automaton such that all states are initial and final, recognizing the words that are labels of paths of G through the map π. We say that a labeled graph *presents* the subshift $\mathcal{X}$

if it recognizes $L(\mathcal{X})$. A (labeled) graph is *essential* if all vertices lie in a bi-infinite path on the graph. A subshift $\mathcal{X}$ is *sofic* if $L(\mathcal{X})$ is rational. Note that finite type subshifts are sofic. One can see that $\mathcal{X}$ is sofic if and only if $L(\mathcal{X})$ is recognized by an essential finite labeled graph. For a finite graph G, let E be the set of its edges. The subset X_G of $E^{\mathbb{Z}}$ whose finite factors are paths of G is a finite type subshift of $E^{\mathbb{Z}}$. Given a subshift $\mathcal{Y}$ presented by a labeled graph (G, π), let π_* the map from X_G to $\mathcal{Y}$ that maps a sequence $(e_i)_{i\in\mathbb{Z}}$ into $(\pi(e_i))_{i\in\mathbb{Z}}$. Then π_* is an onto code with window size zero. We call π_* the *cover* associated with (G, π).

A subshift $\mathcal{X}$ of $A^{\mathbb{Z}}$ is *irreducible* if for all $u, v \in L(\mathcal{X})$ there is $w \in A^*$ such that $uwv \in L(\mathcal{X})$. A sofic subshift is irreducible if and only if it is presented by a strongly connected finite labeled graph [15]. We consider now a stronger property. A subshift $\mathcal{X}$ of $A^{\mathbb{Z}}$ is *mixing* if for all $u, v \in L(\mathcal{X})$ there is an integer N such that for all $n \geq N$ there is $w \in A^*$ with length n such that $uwv \in L(\mathcal{X})$. Being irreducible or mixing is a property preserved for taking images under codes.

A state v of the minimal automaton of $L(\mathcal{X})$ is a *K-state* if there is $x \in \mathcal{X}$ such that the set of words labeling a path from the initial state to v contains infinitely many words of the form $x_{-n}x_{-(n-1)}\dots x_{-1}$, with $n \geq 1$. The *Krieger cover* of a sofic subshift $\mathcal{X}$ is the cover associated with the essential labeled graph obtained from the minimal automaton of $L(\mathcal{X})$ by deleting all the states that are not K-states [19, Section 5]. Krieger proved that two sofic subshifts are conjugate if and only if their Krieger covers are conjugate [17]. If the sofic subshift $\mathcal{X}$ is irreducible then the labeled graph representing its Krieger cover has a unique terminal strongly connected component which is an essential labeled graph presenting $\mathcal{X}$ [7]. The corresponding cover is the *Fischer cover* of $\mathcal{X}$. Two irreducible sofic subshifts are conjugate if and only if their Fischer covers are conjugate.

2.2. Semigroups. Recall that an element e of a semigroup S is *idempotent* if $e^2 = e$. If S is finite then for every $s \in S$ the set of the powers s^n (with n positive integer) has a unique idempotent of S.

A semigroup S *divides* a semigroup T if S is a homomorphic image of subsemigroup of T. We also say that S is a *divisor* of T. This situation is denoted by $S \prec T$. A *pseudovariety of semigroups* is a class of finite semigroups closed under taking divisors and finite direct products. The following classes are pseudovarieties of semigroups:

(1) the class Com of finite commutative semigroups;
(2) the class Sl of finite commutative idempotent semigroups;
(3) the class N of finite *nilpotent* semigroups, that is, finite semigroups with a zero as sole idempotent;
(4) the class Inv of finite semigroups whose idempotents commute;
(5) the class A of finite *aperiodic* semigroups, that is, finite semigroups whose subgroups are trivial;
(6) the class D_k of finite semigroups satisfying the identity $xy_1\cdots y_k = y_1\cdots y_k$;
(7) the class D of finite semigroups whose idempotents are right zeros; one has $\mathsf{D} = \bigcup_{k\geq 1}\mathsf{D}_k$;
(8) for every pseudovariety V of semigroups, the class LV of semigroups whose subsemigroups that are monoids belong to V.

Let L be a language of A^+. The *context* of a word u of A^+ relatively to L is the set $C_L(u) = \{(x,y)\in A^*\times A^* \mid xuy\in L\}$. The *syntactic semigroup* of L is the quotient of A^+ by the congruence $\equiv_L$ defined by $u\equiv_L v \Leftrightarrow C_L(u) = C_L(v)$.

A language L of A^+ is *recognized* by a semigroup homomorphism $\varphi : A^+\to S$ if there exists a subset I of S such that $L = \varphi^{-1}(I)$. We say that L is recognized by S if there is a semigroup homomorphism $\varphi : A^+ \to S$ recognizing L. The syntactic semigroup of L recognizes L and divides all semigroups recognizing L. A *recognizable* or *rational* language is a language recognized by a finite semigroup. It is well known that a language L is rational if and only if L is recognized by a finite automaton, if and only if its syntactic semigroup is finite. Consider a pseudovariety of semigroups V. A V-*recognizable* language of A^+ is a language recognized by a semigroup from V. A language is V-recognizable if and only if its syntactic semigroup belongs to V.

Let S and T be semigroups. The set S^T of maps from T to S, viewed as a direct product of copies of S, is a semigroup; the product fg between two elements f and g of S^T is defined by the rule $fg(t) = f(t)g(t)$.

For a semigroup T, denote by T^1 the monoid that equals T if T is a monoid, and if not then $T^1 = T\cup\{1\}$ for some extra element 1, with the semigroup operation of T^1 extending that of T and 1 being the neutral element of T^1.

For this paragraph, see [1, Chapter 10] or [14]. Given semigroups S and T, let $t_0\in T^1$ and $f\in S^{T^1}$. Denote by ${}^{t_0}f$ the element of S^{T^1} given by the correspondence $t\mapsto f(tt_0)$. The *wreath product* of S and T, denoted by $S\circ T$, is the semigroup with underlying set $S^{T^1} \times T$ and the following operation:

$$(f_1, t_1) \cdot (f_2, t_2) = (f_1 \cdot {}^{t_1}f_2, t_1 \cdot t_2).$$

The *semidirect product* of two pseudovarieties V and W, denoted by $\mathsf{V} * \mathsf{W}$, is the class of divisors of semigroups of the form $S \circ T$, with $S \in \mathsf{V}$ and $T \in \mathsf{W}$. The class $\mathsf{V} * \mathsf{W}$ is also a pseudovariety. The semidirect product of pseudovarieties is an associative operation. One has $\mathsf{D} * \mathsf{D} \subseteq \mathsf{D}$, $\mathsf{V} * \mathsf{D} \subseteq \mathsf{LV}$ and $\mathsf{LV} = \mathsf{LV} * \mathsf{D}$. Also, $\mathsf{LSl} = \mathsf{Sl} * \mathsf{D}$.

3. Weak equivalence

Confronted with the difficulty of deciding conjugacy, some other equivalence relations between subshifts were introduced such as the *weak equivalence* defined by M.-P. Béal and D. Perrin in [6]. Let A, B be two alphabets, let \$ be a symbol that does not belong to B and let $B_\$ = B \cup \{\$\}$. We say that a subshift $\mathcal{X}$ of $A^{\mathbb{Z}}$ *divides* a subshift $\mathcal{Y}$ of $B^{\mathbb{Z}}$ if there exists a sliding block code $F : A^{\mathbb{Z}} \to {B_\$}^{\mathbb{Z}}$ such that $\mathcal{X} = F^{-1}(\mathcal{Y})$; we also say $\mathcal{X}$ is a *divisor* of $\mathcal{Y}$, and use the notation $\mathcal{X} \prec \mathcal{Y}$. The division of subshifts is reflexive and transitive. Two subshifts $\mathcal{X}$ and $\mathcal{Y}$ are *weak equivalent* if $\mathcal{X} \prec \mathcal{Y}$ and $\mathcal{Y} \prec \mathcal{X}$.

The reason why an extra letter like \$ is needed in the definition of division is that otherwise this notion produces a dependency on the involved alphabets that prevents conjugate subshifts of being weak equivalent. Let us see an example that illustrates what we are saying. Let A be the three-letter alphabet $A = \{a, b, c\}$ and let $B = \{a, b\}$. Consider an arbitrary subshift $\mathcal{X}$ of $B^{\mathbb{Z}}$ containing a^ζ and b^ζ. Let F be any sliding block code from $A^{\mathbb{Z}}$ to $B^{\mathbb{Z}}$. Then $F(c^\zeta) \in \{a^\zeta, b^\zeta\}$, thus $\mathcal{X} \neq F^{-1}(\mathcal{X})$. Hence, the definition of division without the extra symbol \$ implies that $\mathcal{X}$ as a subshift of $A^{\mathbb{Z}}$ does not divide $\mathcal{X}$ as a subshift of $B^{\mathbb{Z}}$; therefore, such alternative definition is inadequate. On the other hand, the definition of division we adopted is adequate for studying subshifts up to conjugacy, as it is stated in the following theorem. Its proof, although unpublished, was put forward to us by M.-P. Bal.

Theorem 3.1. *Two conjugate subshifts are weak equivalent.*

Proof: Before proceeding, first recall that the topology of $A^{\mathbb{Z}}$ is generated by the following metric:

$$d(x, y) = 2^{-r(x,y)} \text{ where:}$$

$$\text{if } x \neq y \text{ then } r(x, y) = \min\{n \geq 0 \mid x_{[-n,n]} \neq y_{[-n,n]}\}, \text{ and } r(x, x) = 0.$$

Let $\mathcal{X} \subseteq A^{\mathbb{Z}}$ and $\mathcal{Y} \subseteq B^{\mathbb{Z}}$ be two conjugate subshifts and let $G : \mathcal{X} \to \mathcal{Y}$ be a conjugacy. There exists an integer k and a block map $g : A^k \to B$ corresponding to G, with memory r and anticipation s. Clearly, we can extend G into a sliding block code $\hat{G} : A^{\mathbb{Z}} \to B^{\mathbb{Z}}$ built on the block map g. Denote by $\mathcal{F}(\mathcal{X})$ and $\mathcal{F}(\mathcal{Y})$ the sets of forbidden words that respectively define $\mathcal{X}$ and $\mathcal{Y}$. For each integer n, let $\mathcal{F}_n(\mathcal{X})$ be the set $\mathcal{F}(\mathcal{X}) \cap A^{\leq n}$, and denote by $\mathcal{X}_n$ the subshift of finite type defined by the set of forbidden words $\mathcal{F}_n(\mathcal{X})$. Note that $\mathcal{X} \subseteq \mathcal{X}_n$.

We claim that there is n such that the restriction of $\hat{G}$ to $\mathcal{X}_n$ is injective. Indeed, suppose that for each integer n, there exists a pair $(x^{(n)}, y^{(n)})$ of distinct elements of $\mathcal{X}_n$ such that $\hat{G}(x^{(n)}) = \hat{G}(y^{(n)})$. Since the sets $\mathcal{X}_n$ are closed under the shift operation, one can choose $x^{(n)}$ and $y^{(n)}$ such that they do not have the same letter of index 0. Then $d(x^{(n)}, y^{(n)}) = 1$. Since $A^{\mathbb{Z}}$ is compact, the sequences $(x^{(n)})_n$ and $(y^{(n)})_n$ admit sub-sequences converging to elements x and y of $A^{\mathbb{Z}}$, respectively. Even if it means considering sub-sequences, we may assume that $(x^{(n)})_n$ converges to x and $y^{(n)}$ to y. Let e_n be the greatest even number less than n. The central factor $(x^{(n)})_{[-e_n/2, e_n/2]}$ of length $e_n{+}1$ belongs to $L(\mathcal{X})$ since $x^{(n)}$ is in $\mathcal{X}_n$. Therefore, there exists $\tilde{x}^{(n)}$ in $\mathcal{X}$ such that $d(\tilde{x}^{(n)}, x^{(n)}) < 2^{-e_n/2}$. Since $\mathcal{X}$ is compact, taking subsequences if necessary, one may suppose that $\tilde{x}^{(n)}$ converges to an element $\tilde{x}$ of $\mathcal{X}$. Since the metric d is continuous, we have

$$d(\tilde{x}, x) = \lim_{n \to +\infty} d(\tilde{x}^{(n)}, x^{(n)}) \leq \lim_{n \to +\infty} 2^{-e_n/2} = 0,$$

thus $\tilde{x} = x$. Hence $x \in \mathcal{X}$, and similarly $y \in \mathcal{X}$. Since $\hat{G}$ is continuous, we have $G(x) = G(y)$. On the other hand, since for every n we have $d(x^{(n)}, y^{(n)}) = 1$, we also have $d(x, y) = 1$, thus $x \neq y$. This is a contradiction with the hypothesis that G is a conjugacy, which proves the claim.

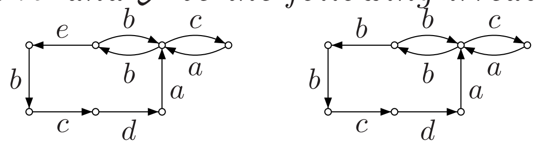
Now let n_0 be such that the restriction of $\hat{G}$ to $\mathcal{X}_{n_0}$ is injective. We define a block map $h : A^k \to B_\$$ by $\begin{cases} h(u) = g(u), \text{ if } u \notin \mathcal{F}_{n_0}(\mathcal{X}), \\ h(u) = \$, \text{ if } u \in \mathcal{F}_{n_0}(\mathcal{X}). \end{cases}$

Then, define the corresponding sliding block code $H : A^{\mathbb{Z}} \to {B_\$}^{\mathbb{Z}}$ with memory r and anticipation s. We claim that $\mathcal{X} = H^{-1}(\mathcal{Y})$. Clearly, $H(\mathcal{X}) = \hat{G}(\mathcal{X}) = \mathcal{Y}$, thus $\mathcal{X} \subseteq H^{-1}(\mathcal{Y})$. Conversely, let $x \in H^{-1}(\mathcal{Y})$. Since $\$ \notin L(\mathcal{Y})$, all factors of x with length n_0 do not belong to $\mathcal{F}_{n_0}(\mathcal{X})$, thus $x \in \mathcal{X}_{n_0}$. Hence $H(x) = \hat{G}(x)$. On the other hand, since $\mathcal{Y} = \hat{G}(\mathcal{X})$, there is $x' \in \mathcal{X}$ such that $\hat{G}(x) = \hat{G}(x')$. Since the restriction of $\hat{G}$ to $\mathcal{X}_{n_0}$ is injective, we have $x = x'$. Therefore $\mathcal{X} = H^{-1}(\mathcal{Y})$, thus $\mathcal{X} \prec \mathcal{Y}$. By symmetry, we get that $\mathcal{X}$ and $\mathcal{Y}$ are weak equivalent. ■

The properties of beeing mixing or irreducible are not weak equivalence invariants [6]. On the other hand, all finite type subshifts with a constant sequence are weak equivalent [6, Proposition 4].

It is important to notice that the relation of division between subshifts cannot be reduced to a similar relation between the corresponding languages of finite factors. Let us be more precise. Let $\mathcal{X}$ and $\mathcal{Y}$ be subshifts of $A^{\mathbb{Z}}$ and $B^{\mathbb{Z}}$, respectively. Write $\mathcal{X} \lhd \mathcal{Y}$ if there is an integer n and a map $f : A^n \to B_\$$ such that $L(\mathcal{X}) \setminus A^{<n} = \bar{f}^{-1}(L(\mathcal{Y}))$. Then we have the following result:

Proposition 3.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be the following irreducible sofic subshifts:*



Then $\mathcal{X}$ and $\mathcal{Y}$ are conjugate but $\mathcal{X} \not\lhd \mathcal{Y}$.

Proof: Let $A = \{a, b, c, d, e\}$ and $B = A \setminus \{e\}$. Consider the block map $h : A \to B$ that maps e to b and leaves the remaining letters unchanged. Let H be the code between $\mathcal{X}$ and $\mathcal{Y}$ having h as block map with memory and anticipation zero. We prove that H is a conjugacy. It is clearly an onto map. Suppose it is not one-to-one. Then there are $z, t \in \mathcal{X}$ such that $z_0 \neq t_0$ and $H(z) = H(t)$. Then $h(z_0) = h(t_0)$, thus $\{z_0, t_0\} = \{b, e\}$. Suppose $z_0 = b$ and $t_0 = e$. The only word of $L(\mathcal{X})$ with length four having e as the first letter is $ebcd$, thus $t_{[0,3]} = ebcd$. Then,

$$H(z) = H(t) \Rightarrow \bar{h}(z_{[0,3]}) = \bar{h}(t_{[0,3]}) \Rightarrow \bar{h}(z_{[0,3]}) = bbcd \Rightarrow z_{[0,3]} \in \{bbcd, becd\}.$$

But $\{bbcd, becd\} \cap L(\mathcal{X}) = \emptyset$.

Suppose there is $n \geq 1$ and $f : A^n \to B_\$$ such that $L(\mathcal{X}) \setminus A^{<n} = \bar{f}^{-1}(L(\mathcal{Y}))$. Consider the letters $\alpha = \bar{f}(ab^{n-1})$, $\gamma = \bar{f}(b^{n-1}c)$ and $\beta = \bar{f}(b^n)$. Then $\bar{f}(b^{2n}) = \beta^{n+1}$. Since $b^{2n} \in L(\mathcal{X})$, we have $\beta^{n+1} \in L(\mathcal{Y})$. This implies $\beta = b$.

Let $N \geq n$. Then $\alpha b = \bar{f}(ab^n)$, $b\gamma = \bar{f}(b^n c)$ and $\bar{f}(ab^N c) = \alpha b^{N-(n-1)}\gamma$. Since $ab^n, b^n c \in L(\mathcal{X})$, we have $\alpha b, b\gamma \in L(\mathcal{Y})$, which implies $\alpha \in \{a, b\}$ and $\gamma \in \{b, c\}$. Then $\alpha b^i \gamma \in L(\mathcal{Y})$ for every $i \geq 2$, and so $ab^N c \in \bar{f}^{-1}(L(\mathcal{Y}))$ for every $N \geq n$. Since $\bar{f}^{-1}(L(\mathcal{Y})) \subseteq L(\mathcal{X})$, we reach the absurd conclusion that there is an odd integer N such that $ab^N c \in L(\mathcal{X})$. ■

4. The transducer of a block map

Consider an alphabet A and a non-negative integer k. Given $u \in A^+$ define $\mathrm{t}_k(u)$ as follows: if the length of u is less than k then $\mathrm{t}_k(u) = u$, otherwise $\mathrm{t}_k(u)$ is the unique word v of length k such that $u = wv$ for some $w \in A^*$. The *De Bruijn automaton* $T_k(A)$ is the complete deterministic automaton over A whose states are the words of $A^{\leq k}$ and whose action $\delta : A^{\leq k} \times A^+ \to A^{\leq k}$ is given by $\delta(w, u) = \mathrm{t}_k(wu)$. We shall use the more familiar notation $w \cdot u$ for $\delta(w, u)$, but note that in general $w \cdot u$ is not the same as the concatenation wu. We will consider also the sub-automaton $\tilde{T}_k(A)$ built from $T_k(A)$ by deleting states corresponding to words of $A^{<k}$. The restriction of δ to $A^{\leq k} \times A^{\leq k}$ gives to $A^{\leq k}$ a semigroup structure. Denote this semigroup by $\mathcal{D}_k$.

Lemma 4.1. *The transition semigroups of* $T_k(A)$ *and* $\tilde{T}_k(A)$ *are isomorphic to* $\mathcal{D}_k$.

Proof: We only prove the lemma for $\tilde{T}_k(A)$, the other case being even more easy. Let μ be the transition map of $\tilde{T}_k(A)$. Clearly if $|u| \geq k$ then $\mu(u) = \mu(\mathrm{t}_k(u))$. If $u \in \mathcal{D}_k$, then the image of $\mu(u)$ is the set $A^{k-|u|}u$, and if $v \in A^*$ and $l \geq 0$ are such that $A^l v = A^{k-|u|}u$ then $l = k - |u|$ and $v = u$. Therefore the restriction of μ to $\mathcal{D}_k$ is an isomorphism between $\mathcal{D}_k$ and $\tilde{T}_k(A)$. ■

Note that $\mathcal{D}_k \in \mathsf{D}_k$ and that the idempotents of $\mathcal{D}_k$ are the words of length k.

By a *transducer* with input alphabet A and output alphabet B we mean an automaton $\mathfrak{A}$ over the alphabet $A \times B$. Usually in this context an element (u, v) of $A^* \times B^*$ is represented by u/v. If in the transition edges of $\mathfrak{A}$ we replace the letter a/b by a (resp. b) the resulting automaton is called the *input automaton* of $\mathfrak{A}$ (resp. *output automaton*). Consider a map $f : A^k \to B$. In the De Bruijn automaton $T_{k-1}(A)$ replace an edge from u to v labeled a by the pair $(a, \bar{f}(ua))$. Then the resulting automaton $\mathcal{T}(f)$ over the alphabet $A \times B$ is a transducer having $T_{k-1}(A)$ as input automaton. With the sub-automaton $\tilde{T}_{k-1}(A)$ define in a similar way the transducer $\tilde{\mathcal{T}}(f)$. See Figure 1.

Consider a transducer $\mathfrak{A}$ whose input automaton over the alphabet A has a complete and deterministic action $\cdot$ of A over its states (for example, the transducer $\mathcal{T}(f)$). Then, if $u \in A^+$ and q is a state of $\mathfrak{A}$, we denote by $q * u$ the label in the output automaton of the unique path p in $\mathfrak{A}$ from q to $q \cdot u$; we say that u and $q * u$ are the *input* and *output label* of p, respectively. For example, given a map $f : A^k \to B$, on the transducer $\mathcal{T}(f)$ we have

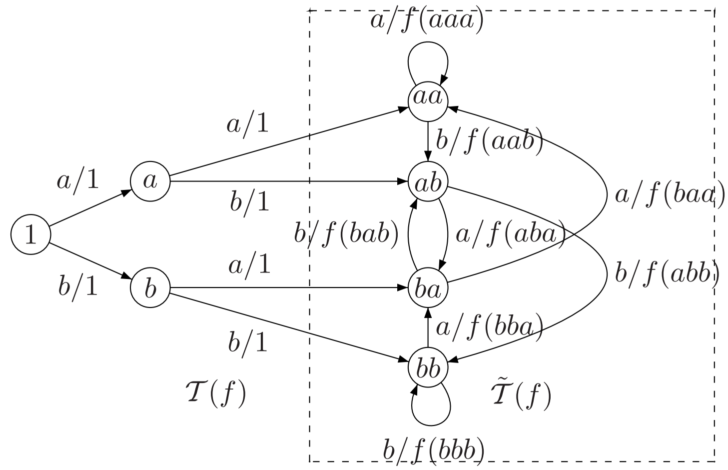


FIGURE 1. Transducer of a block map $f : A^3 \to B$.

$\bar{f}(u) = 1 * u$. A function $\varphi : A^+ \to B^*$ for which there is some transducer and some state q such that $\varphi(u) = q * u$ for all $u \in A^+$ is a called a *sequential function*. The following theorem is a particular instance of a general result about sequential functions [14, Chapter IX, Proposition 1.1].

Theorem 4.2. *For alphabets A and B and a positive integer k, consider a map $f : A^k \to B$. Let Y be a rational language of B^+ with syntactic semigroup S. Then the syntactic semigroup of $\bar{f}^{-1}(Y)$ divides $S \circ \mathcal{D}_{k-1}$.*

5. ζ-semigroups

5.1. Motivation and definitions. Let $\mathcal{X}$ be a subshift of $A^{\mathbb{Z}}$. For the sake of conciseness, the syntactic semigroup of $L(\mathcal{X})$ will be called the *syntactic semigroup of* $\mathcal{X}$. When we consider the inverse image by a sliding block code of a sofic subshift do we have a result similar to Theorem 4.2? In this section we prove the answer is yes. As a consequence of Proposition 3.2 we know it is not possible to do an immediate reduction to Theorem 4.2. The passage from finite sequences to bi-infinite sequences suggests trying a similar passage at the syntactic semigroup level. This motivates the introduction of ζ-semigroups, a generalization of ω-semigroups.

We quickly review here basic definitions about ω-semigroups. For an exhaustive overview, see [20]. An *ω-semigroup* is a two-component algebra $S = (S_+, S_\omega)$ equipped with a binary product on S_+, a map $S_+ \times S_\omega \to S_\omega$ called the *mixt product*, and a map $\pi : S_+^\omega \to S_\omega$ called the *infinite product*, and such that the following conditions are satisfied:

(1) the set S_+ equipped with its product is a semigroup,

(2) for each s, t in S_+ and u in S_ω, $s(tu) = (st)u$,
(3) for each non-decreasing sequence $(i_n)_{n>0}$ of $\mathbb{N}$ and each sequence $(s_n)_{n\in\mathbb{N}}$ of S_+^ω, $\pi(s_0 s_1 \cdots s_{(i_1-1)},\, s_{i_1} \cdots s_{(i_2-1)}, \cdots) = \pi(s_0,\, s_1,\, s_2, \cdots)$.
(4) for all s in S_+ and for each sequence $(s_n)_{n\in\mathbb{N}}$ of S_+^ω, $s\,\pi(s_0,\, s_1,\, s_2, \cdots) = \pi(s,\, s_0,\, s_1,\, s_2, \cdots)$.

An ω*-semigroup morphism* from $S = (S_+, S_\omega)$ into $T = (T_+, T_\omega)$ is a pair $\varphi = (\varphi_+, \varphi_\omega)$ such that $\varphi_+ : S_+ \to T_+$ is a semigroup morphism and $\varphi_\omega : S_\omega \to T_\omega$ preserves both infinite product and mixt product. The $\tilde{\omega}$*-semigroups* and $\tilde{\omega}$*-semigroup morphisms* are similarly defined, all products operating on the left.

Theorem 5.1 ([20, Chapter II, Theorem 5.1] and [10, Lemma 4]). *Let S_+ be a finite semigroup and S_ω a finite set. Suppose there are maps $S_+S_\omega \to S_\omega$ and $S_+ \to S_\omega$, denoted respectively $(s,t) \mapsto st$ and $s \mapsto s^\omega$, satisfying the following conditions:*

$$s(tu) = (st)u \text{ for every } s, t \in S_+ \text{ and every } u \in S_\omega, \tag{5.1}$$

$$(s^n)^\omega = s^\omega \text{ for every } s \in S_+ \text{ and every } n > 0, \tag{5.2}$$

$$s(ts)^\omega = (st)^\omega \text{ for every } s, t \in S_+ \text{ such that } st \text{ and } ts \text{ are idempotents.} \tag{5.3}$$

Then the pair $S = (S_+, S_\omega)$ can be equipped, in a unique way, with a structure of ω-semigroup such that for every $s \in S_+$ the product $sss\cdots$ is equal to s^ω.

A ζ*-semigroup* is a four-component algebra $S = (S_+, S_\omega, S_{\tilde{\omega}}, S_\zeta)$ such that (S_+, S_ω) is an ω-semigroup, $(S_+, S_{\tilde{\omega}})$ is an $\tilde{\omega}$-semigroup, and with a mapping $\rho : S_{\tilde{\omega}} \times S_\omega \to S_\zeta$ such that if $s \in S_{\tilde{\omega}}$, $t \in S_+$, and $u \in S_\omega$ then $\rho(s,\, tu) = \rho(st,\, u)$. A ζ-semigroup is *finite* if all its four components are finite.

Example 5.2. *Denote by A^ζ the quotient of $A^\mathbb{Z}$ under the equivalence relation:*

$$u \sim_\sigma v \Leftrightarrow \exists n \in \mathbb{Z} \mid u = \sigma^n(v).$$

The algebra $A^\infty = (A^+, A^\omega, A^{\tilde{\omega}}, A^\zeta)$ equipped with the usual concatenation is then a ζ-semigroup, called the free ζ-semigroup *on A.*

Let S and T be ζ-semigroups. A ζ*-semigroup morphism* from S into T is a quadruplet $\varphi = (\varphi_+, \varphi_\omega, \varphi_{\tilde{\omega}}, \varphi_\zeta)$ such that $(\varphi_+, \varphi_\omega)$ (resp. $(\varphi_+, \varphi_{\tilde{\omega}})$) is an ω-semigroup morphism (resp. $\tilde{\omega}$-semigroup morphism), and φ_ζ is a map from S_ζ into T_ζ such that for every s in $S_{\tilde{\omega}}$ and t in S_ω one has $\varphi_\zeta(st) = \varphi_{\tilde{\omega}}(s)\varphi_\omega(t)$. Note that φ is entirely determined by φ_+.

A subset P of A^ζ is *recognized* by a ζ-semigroup homomorphism $\varphi : A^\zeta \to S$ if there is a subset I of S_ζ such that $P = \varphi_\zeta^{-1}(I)$. We say that P is recognized by a ζ-semigroup S if there is a ζ-semigroup homomorphism $\varphi : A^\zeta \to S$ recognizing P.

5.2. The syntactic ζ-semigroup. Let $u \in A^+$. In absence of confusion the $\sim_\sigma$-class of u^ζ is also denote by u^ζ. Consider a subset P of A^ζ. The *syntactic congruence on* P is the 4-tuple of equivalence relations $(\sim_+, \sim_\omega, \sim_{\tilde\omega}, \sim_\zeta)$ defined by

$$(1)\ \forall s,t \in A^+, s \sim_+ t \Longleftrightarrow \begin{cases} \forall x \in A^{\tilde\omega},\ \forall y \in A^\omega, & xsy \in P \ \Leftrightarrow\ xty \in P \\ \forall x \in A^{\tilde\omega},\ \forall y \in A^+, & x(sy)^\omega \in P \ \Leftrightarrow\ x(ty)^\omega \in P \\ \forall x \in A^+,\ \forall y \in A^\omega, & (xs)^{\tilde\omega} y \in P \ \Leftrightarrow\ (xt)^{\tilde\omega} y \in P \\ \forall x \in A^+, & (xs)^\zeta \in P \Leftrightarrow (xt)^\zeta \in P \end{cases}$$

$$(2)\ \forall s,t \in A^\omega,\ \ s \sim_\omega t \iff \big[\, \forall x \in A^{\tilde\omega},\ \ xs \in P \ \Leftrightarrow\ xt \in P \,\big]$$

$$(3)\ \forall s,t \in A^{\tilde\omega},\ \ s \sim_{\tilde\omega} t \iff \big[\, \forall x \in A^\omega,\ \ xs \in P \ \Leftrightarrow\ xt \in P \,\big]$$

$$(4)\ \forall s,t \in A^\zeta,\ \ s \sim_\zeta t \iff \big[\, s \in P \ \Leftrightarrow t \in P \,\big].$$

The proof of the following lemma consists on mere routines.

Lemma 5.3. *For any subset P of A^∞ we have the following:*

(1) $\sim_+$ is a semigroup congruence;
(2) if $u \sim_+ v$ then $u^\omega \sim_\omega v^\omega$ and $u^{\tilde\omega} \sim_{\tilde\omega} v^{\tilde\omega}$, for all $u, v \in A^+$;
(3) if $u \sim_+ v$ and $s \sim_\omega t$ then $us \sim_\omega vt$, for all $s,t \in A^\omega$, $u, v \in A^+$;
(4) if $u \sim_+ v$ and $s \sim_{\tilde\omega} t$ then $su \sim_{\tilde\omega} tv$, for all $s,t \in A^{\tilde\omega}$, $u, v \in A^+$;
(5) if $s \sim_{\tilde\omega} t$ and $s' \sim_\omega t'$ then $ss' \sim_\zeta tt'$, for all $s,t \in A^{\tilde\omega}$, $s', t' \in A^\omega$.

Denote by $\mathcal{S}(P)$ the 4-tuple $(A^+/{\sim_+}, A^\omega/{\sim_\omega}, A^{\tilde\omega}/{\sim_{\tilde\omega}}, A^\zeta/{\sim_\zeta})$ of quotient sets. Denote by π_P the quotient map from A^∞ to $\mathcal{S}(P)$, defined as 4-tuple of quotient maps in the obvious way.

Proposition 5.4. *If $\mathcal{S}(P)$ is finite then, in a unique way, π_P defines in $\mathcal{S}(P)$ a structure of ζ-semigroup for which π_P is a homomorphism of ζ-semigroups. Moreover, π_P recognizes P.*

Proof: We want to apply Theorem 5.1. By Lemma 5.3, $(A^+/{\sim_+})$ is a semigroup, the map $(A^+/{\sim_+}) \times (A^\omega/{\sim_\omega}) \to (A^\omega/{\sim_\omega})$ given by $\pi_P(u) \cdot \pi_P(s) = \pi_P(us)$ (where $u \in A^+$ and $s \in A^\omega$) is well defined and satisfies condition (5.1) in Theorem 5.1, and we can define $\pi_P(u)^\omega$ as being $\pi_P(u^\omega)$ (where $u \in A^+$). Then clearly $(\pi_P(u)^n)^\omega = \pi_P(u)^\omega$ and $\pi_P(u)(\pi_P(v)\pi_P(u))^\omega = \pi_P(uv)^\omega$ for all

$u,v \in A^+$. Hence $(A^+/\sim_+, A^\omega/\sim_\omega)$ is an ω-semigroup. Dually, $(A^+/\sim_+, A^{\tilde\omega}/\sim_{\tilde\omega})$ is an $\tilde\omega$-semigroup.

Let $\rho_P : (A^\omega/\sim_\omega)\times(A^{\tilde\omega}/\sim_{\tilde\omega}) \to A^\zeta/\sim_\zeta$ be the map defined by $\rho_P(\pi_P(u),\pi_P(v)) = \pi_P(uv)$, where $u \in A^{\tilde\omega}, v \in A^\omega$. This map is well defined, by Lemma 5.3 (5). Moreover, for all $s \in A^{\tilde\omega}$, $t \in A^+$ and $u \in A^\omega$, we have

$$\rho_P(\pi_P(s)\,\pi_P(t),\pi_P(u)) = \pi_P(stu) = \rho_P(\pi_P(s),\pi_P(t)\,\pi_P(u)).$$

Hence $\mathcal{S}(P)$ has a structure of ζ-semigroup for which π_P is a homomorphism of ζ-semigroups. Since π_P is onto, such structure is unique. Finally, it is obvious that $\pi_P^{-1}(\pi_P(P)) = P$. ■

We call $\mathcal{S}(P)$ the *syntactic ζ-semigroup of P*, if $\mathcal{S}(P)$ is finite.

Let $\mathcal{X}$ be a subshift of $A^\mathbb{Z}$. Since $\mathcal{X}$ is saturated by the relation $\sim_\sigma$, we do not lose information if we identify $\mathcal{X}$ with $\mathcal{X}/\sim_\sigma$. For this reason and for the sake of conciseness, we indistinctly consider $\mathcal{X}$ as a subset of both $A^\mathbb{Z}$ and A^ζ.

Next we proceed to establish a relationship between the syntactic ζ-semigroup of a sofic subshift and its classical syntactic semigroup. For a subshift $\mathcal{X}$ of $A^\mathbb{Z}$ and a word u of A^+, the set $\{(x,y) \in A^{\tilde\omega} \times A^\omega \mid x.uy \in \mathcal{X}\}$ is denoted by $C_\mathcal{X}(u)$.

Lemma 5.5. *Consider a subshift $\mathcal{X}$ of $A^\mathbb{Z}$. For all $u,v \in A^+$, we have $C_{L(\mathcal{X})}(u) \subseteq C_{L(\mathcal{X})}(v)$ if and only if $C_\mathcal{X}(u) \subseteq C_\mathcal{X}(v)$.*

Proof: Consider the set $\mathcal{X}_u$ of the elements of $A_\$^{\tilde\omega} \times A_\$^\omega$ of the form $(\$^{\tilde\omega}r, s\$^\omega)$, with $(r,s) \in C_{L(\mathcal{X})}(u)$. We are going to prove the following equality:

$$\overline{\mathcal{X}_u} \cap A^{\tilde\omega} \times A^\omega = C_\mathcal{X}(u). \tag{5.4}$$

Let $(x,y) \in \overline{\mathcal{X}_u}\cap(A^{\tilde\omega}\times A^\omega)$, and consider a sequence $(\$^{\tilde\omega}x_n, y_n\$^\omega)_n$ of elements of $\mathcal{X}_u$ converging to (x,y). Let $z_n = \$^{\tilde\omega}x_n.uy_n\$^\omega$. Then $(z_n)_n$ converges to $x.uy$. Consider a positive integer k. Since \$ does not occur in $x.uy$, for sufficiently large n the lenghts of x_n and uy_n are greater than k. Then, for sufficently large n, the word $z_{[-k,k]}$ is a factor of x_nuy_n, thus $z_{[-k,k]} \in L(\mathcal{X})$. Since k is arbitrary, we conclude that $z \in \mathcal{X}$, thus $(x,y) \in C_{L(\mathcal{X})}(u)$.

Conversely, if $(x,y) \in C_\mathcal{X}(u)$ then $(x_{[-n,-1]}, x_{[0,n]}) \in C_{L(\mathcal{X})}(u)$ for all n. Then $(\$^{\tilde\omega}x_{[-n,-1]}, x_{[0,n]}\$^\omega)_n$ is a sequence of elements of $\mathcal{X}_u$ converging to (x,y), thus $(x,y) \in \overline{\mathcal{X}_u} \cap (A^{\tilde\omega} \times A^\omega)$. This finishes the proof of (5.4).

If $C_{L(\mathcal{X})}(u) \subseteq C_{L(\mathcal{X})}(v)$ then $\mathcal{X}_u \subseteq \mathcal{X}_v$, thus $C_\mathcal{X}(u) \subseteq C_\mathcal{X}(v)$ by (5.4). Conversely, suppose that $C_{L(\mathcal{X})}(u) \nsubseteq C_{L(\mathcal{X})}(v)$. Let $(x,y) \in C_{L(\mathcal{X})}(u) \setminus C_{L(\mathcal{X})}(v)$.

Then, since $x.uy \in L(\mathcal{X})$, there are $p \in A^{\tilde{\omega}}$ and $q \in A^{\omega}$ such that $px.uyq \in \mathcal{X}$. Moreover, $px.vyq \notin \mathcal{X}$ because $x.vy \notin L(\mathcal{X})$. Therefore $(px, yq) \in C_{\mathcal{X}}(u) \setminus C_{\mathcal{X}}(v)$. ∎

Proposition 5.6. *Let $\mathcal{X}$ be a subshift of $A^{\mathbb{Z}}$. Then the syntactic semigroup of $\mathcal{X}$ is $\mathcal{S}(\mathcal{X})_+$. Moreover, $\mathcal{X}$ is sofic if and only if $\mathcal{S}(\mathcal{X})$ is finite.*

Proof: Let $\equiv$ be the syntactic congruence of $L(\mathcal{X})$. Let $u, v \in A^+$. Suppose $u \sim_+ v$. By the first condition defining $\sim_+$, one has $C_{\mathcal{X}}(u) = C_{\mathcal{X}}(v)$. Then $u \equiv v$ by Lemma 5.5.

Conversely, suppose $u \equiv v$. Then $C_{\mathcal{X}}(u) = C_{\mathcal{X}}(v)$, by Lemma 5.5. Let $x \in A^{\tilde{\omega}}$ and $y \in A^+$. Then $(uy)^n \equiv (vy)^n$ for all positive integer n. Hence,

$$\begin{aligned} x(uy)^{\omega} \in \mathcal{X} &\Leftrightarrow [\forall n > 0,\, x_{[-n,-1]}(uy)^n \in L(\mathcal{X})] \\ &\Leftrightarrow [\forall n > 0,\, x_{[-n,-1]}(vy)^n \in L(\mathcal{X})] \\ &\Leftrightarrow x(vy)^{\omega} \in \mathcal{X}. \end{aligned}$$

Analogously, for every $x \in A^+$ and $y \in A^{\omega}$ we have

$$(xu)^{\tilde{\omega}} y \in \mathcal{X} \Leftrightarrow (xv)^{\tilde{\omega}} y \in \mathcal{X}.$$

And since for every $x \in A^+$ and $y \in A^+$ we have $(xu)^n \equiv (xv)^n$ and

$$(xu)^{\zeta} \in \mathcal{X} \Leftrightarrow [\forall n > 0,\, (xu)^n \in L(\mathcal{X})],$$

we also conclude that $(xu)^{\zeta} \in \mathcal{X} \Leftrightarrow (xv)^{\zeta} \in \mathcal{X}$. Therefore $u \sim_+ v$. This concludes the proof that the syntactic semigroup of $\mathcal{X}$ is $\mathcal{S}(\mathcal{X})_+$.

Independently of $\mathcal{X}$ being sofic, it is always true that $\mathcal{S}(\mathcal{X})_{\zeta}$ has at most two elements. The subshift $\mathcal{X}$ is sofic if and only if its syntactic semigroup (which is $\mathcal{S}(\mathcal{X})_+$) is finite. In particular, if $\mathcal{S}(\mathcal{X})$ is finite then $\mathcal{X}$ is sofic. Moreover, it is also known ([17]; see also [18, Exercise 3.2.8]) that the number of $\sim_{\omega}$-classes is finite if and only if $\mathcal{X}$ is sofic (that number is the number of states of the graph defining the left Krieger cover [17]); the same result holds for the $\sim_{\tilde{\omega}}$-classes. ∎

5.3. Wreath product. The set of idempotents of a semigroup T is denoted by $E(T)$. Note that if $T \in \mathsf{D}$ then $E(T)$ is a subsemigroup of T.

Definition 5.7. *Let S be a finite ζ-semigroup, and T a semigroup from D. Denote by $S \circ T$ the 4-tuple $\big(S_+^{E(T)} \times T, S_{\omega}^{E(T)}, S_{\tilde{\omega}} \times E(T), S_{\zeta}\big)$ endowed with the following structure:*

(1) $S_+^{E(T)} \times T$ *is the semigroup defined by* $(f_1, t_1) \cdot (f_2, t_2) = (f, t_1 t_2)$ *with* $f(e) = f_1(e) f_2(e t_1)$*;*
(2) *for all* $(f, t) \in S_+^{E(T)} \times T$ *and for all* $g \in S_\omega^{E(T)}$ *we have*
  (a) $(f, t) \cdot g = h$*, with* $h(e) = f(e) g(et)$*,*
  (b) $(f, t)^\omega = h$*, with* $h(e) = f'(e)\big(f'(t')\big)^\omega$*, where* (f', t') *is the idempotent power of* (f, t)*;*
(3) *for all* $(s, e) \in S_{\tilde\omega} \times E(T)$ *and for all* $(f, t) \in S_+^{E(T)} \times T$
  (a) $(s, e) \cdot (f, t) = (sf(e), et)$*,*
  (b) $(f, t)^{\tilde\omega} = \big(f'(t')^{\tilde\omega}, t'\big)$*, where* (f', t') *is the idempotent power of* (f, t)*;*
(4) *for all* $(s, e) \in S_{\tilde\omega} \times E(T)$ *and for all* $g \in S_\omega^{E(T)}$ *we have* $(s, e) \cdot g = sg(e)$*.*

Proposition 5.8. *If* S *is a finite* ζ*-semigroup and* $T \in \mathsf{D}$ *then* $S \circ T$ *is a* ζ*-semigroup.*

We call $S \circ T$ the *wreath product* of S and T. This construction is inspired by a similar one by O. Carton on ω-semigroups [10]. Note that the semigroup $(S \circ T)_+$ is the homomorphic image of $S_+ \circ T$ by the homomorphism $(f, t) \mapsto (f_{|E(T)}, t)$.

of Proposition 5.8: Conditions 1 and 2 of Definition 5.7 endow the pair $\big(S_+^{E(T)} \times T, S_\omega^{E(T)}\big)$ with the structure of ω-semigroup: the proof of this fact is entirely analogous to the proof in [10] of the consistency of the definition of wreath product of a finite ω-semigroup and a finite semigroup *.

We claim that Conditions 1 and 3 of Definition 5.7 endow $\big(S_+^{E(T)} \times T, S_{\tilde\omega} \times E(T)\big)$ with the structure of $\tilde\omega$-semigroup. To prove the claim, we use the dual of Theorem 5.1. From Conditions 1 and 3 one almost immediately deduce identities (5.1) and (5.2). Let (f_1, t_1) and (f_2, t_2) be elements of $S_+^{E(T)} \times T$ such that $(f_1, t_1)(f_2, t_2)$ and $(f_2, t_2)(f_1, t_1)$ are idempotents. Let $(i, j) \in \{(1, 2), (2, 1)\}$. Then $t_j t_i \in E(T)$. Hence $t_i t_j t_i = t_j t_i$, because $T \in \mathsf{D}$. Moreover, by Condition 3b we have:

$$\big((f_i, t_i) \cdot (f_j, t_j)\big)^{\tilde\omega} = \big((f_i(t_i t_j) f_j(t_i t_j t_i))^{\tilde\omega}, t_i t_j\big) = \big((f_i(t_i t_j) f_j(t_j t_i))^{\tilde\omega}, t_i t_j\big).$$

*In fact, according to that definition, $\big(S_+^{E(T)} \times T, S_\omega^{E(T)}\big)$ is a homomorphic image of the wreath product of (S_+, S_ω) and T.

Then, by the late equality and Condition 3a, we have

$$\begin{aligned}\big((f_1,t_1)\cdot(f_2,t_2)\big)^{\tilde\omega}\cdot(f_1,t_1) &= \big((f_1(t_1t_2)f_2(t_2t_1))^{\tilde\omega},t_1t_2\big)\cdot(f_1,t_1)\\ &= \big((f_1(t_1t_2)f_2(t_2t_1))^{\tilde\omega}f_1(t_1t_2),t_1t_2t_1\big)\\ &= \big((f_2(t_2t_1)f_1(t_1t_2))^{\tilde\omega},t_2t_1\big)\\ &= \big((f_2,t_2)\cdot(f_1,t_1)\big)^{\tilde\omega}\end{aligned}$$

Hence the identity (5.3) is proved, and the claim holds.

Finally, let $(s,e)\in S_{\tilde\omega}\times E(T)$, $(f,t)\in S^{E(T)}\times T$ and $g\in S_\omega^{E(T)}$. Then

$$\big((s,e)\cdot(f,t)\big)\cdot g=(sf(e),et)\cdot g=sf(e)g(et).$$

On the other hand, let h be the map $(f,t)\cdot g$. Then

$$(s,e)\cdot\big((f,t)\cdot g\big)=(s,e)\cdot h=sh(e)=sf(e)g(et).$$

Hence $\big((s,e)\cdot(f,t)\big)\cdot g=(s,e)\cdot\big((f,t)\cdot g\big)$. ∎

Lemma 5.9. *Let P be a subset of B^ζ recognized by a ζ-semigroup homomorphism $\varphi: B^\infty\to Z$, where Z is a finite ζ-semigroup. Consider the transducer $\tilde{\mathcal{T}}(f)$, where f is a map from A^k to B. Let ψ be the unique ζ-semigroup homomorphism from A^∞ to $Z\circ\mathcal{D}_{k-1}$ such that*

$$\psi_+(a)=(g_a,a),\qquad \text{where } g_a: e\mapsto\varphi_+(e*a).$$

Then ψ has the following properties:

(1) if $u\in A^+$ then $\psi_+(u)=(g_u,\mathrm{t}_{k-1}(u))$, where $g_u: e\mapsto\varphi_+(e*u)$;
(2) if $u\in A^\omega$ then $\psi_\omega(u)=(e\mapsto\varphi_\omega(e*u))$;
(3) if $u\in A^{\tilde\omega}$ then $\psi_{\tilde\omega}(u)=(\varphi_{\tilde\omega}(u*q(u)),q(u))$, where $q(u)$ is the final state of the unique left-infinite path in $\tilde{\mathcal{T}}(f)$ with input label u, and $u*q(u)$ is the corresponding output label.

Proof: The proof of the first two properties is entirely similar to the proofs of Lemmas 7 and 8 in [10]. We prove the third property. By [20, Chapter II, Theorem 2.2], there is a factorization $u=\cdots u_2u_1u_0$ such that $\psi_+(u_i)=\psi_+(u_1)=\psi_+(u_1)^2$ for all $i>0$ and $\psi_+(u_1)\psi_+(u_0)=\psi_+(u_0)$. Hence $\psi_{\tilde\omega}(u)=\psi_+(u_1)^{\tilde\omega}\,\psi_+(u_0)$. By the first part of the theorem we have $\psi_+(u_i)=(g_{u_i},\mathrm{t}_{k-1}(u_i))$

for all $i \geq 0$. Then, since $\psi(u_1)$ is idempotent, we use Condition 3 of Definition 5.7 deducing the following:

$$\begin{aligned}
\psi_{\tilde{\omega}}(u) &= \big(g_{u_1}(\mathrm{t}_{k-1}(u_1))^{\tilde{\omega}}, \mathrm{t}_{k-1}(u_1)\big) \cdot (g_{u_0}, \mathrm{t}_{k-1}(u_0)) \\
&= \big(g_{u_1}(\mathrm{t}_{k-1}(u_1))^{\tilde{\omega}} \cdot g_{u_0}(\mathrm{t}_{k-1}(u_1)),\ \mathrm{t}_{k-1}(u_1u_0)\big) \\
&= \big((\varphi_{+}(\mathrm{t}_{k-1}(u_1) * u_1))^{\tilde{\omega}} \cdot \varphi_{+}(\mathrm{t}_{k-1}(u_1) * u_0)\big),\ \mathrm{t}_{k-1}(u_1u_0)\big) \\
&= \big(\varphi_{\tilde{\omega}}\big[(\mathrm{t}_{k-1}(u_1) * u_1)^{\tilde{\omega}} \cdot (\mathrm{t}_{k-1}(u_1) * u_0)\big],\ \mathrm{t}_{k-1}(u_1u_0)\big). \qquad (5.5)
\end{aligned}$$

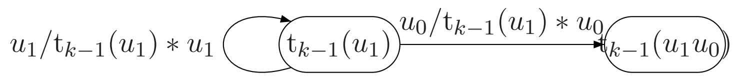


FIGURE 2. A path in the transducer $\tilde{\mathcal{T}}(f)$.

For all $i > 0$, since $\mathrm{t}_{k-1}(u_1) = \mathrm{t}_{k-1}(u_i)$, we have $\mathrm{t}_{k-1}(u_1) \cdot u_1 = \mathrm{t}_{k-1}(u_{i+1}) \cdot u_i$ and $\mathrm{t}_{k-1}(u_1) * u_1 = \mathrm{t}_{k-1}(u_{i+1}) * u_i$. Hence (see Figure 2) we have $q(u) = \mathrm{t}_{k-1}(u_1u_0)$ and $(\mathrm{t}_{k-1}(u_1) * u_1)^{\tilde{\omega}} \cdot (\mathrm{t}_{k-1}(u_1) * u_0) = u * q(u)$. The result now follows from (5.5). ■

Let $F : A^{\mathbb{Z}} \to B^{\mathbb{Z}}$ be a sliding block code. Since F commutes with the shift operation, one can define the function from A^{ζ} to B^{ζ} mapping $x/{\sim_\sigma}$ into $F(x)/{\sim_\sigma}$. We also denote such map by F, and call it (sliding block) code. The following result appears in the first author Master's thesis [11, Theorem 2.7].

Theorem 5.10. *Let $F : A^{\zeta} \to B^{\zeta}$ be a sliding block code with window size k and let P be a subset of B^{ζ} recognized by a finite ζ-semigroup Z. Then $F^{-1}(P)$ is recognized by $Z \circ \mathcal{D}_{k-1}$.*

Proof: Consider a block map $f : A^k \to B$ for F. Let φ be a ζ-semigroup homomorphism from B^{∞} to Z recognizing P, and let ψ be the ζ-semigroup homomorphism from A^{∞} to $Z \circ \mathcal{D}_{k-1}$ as defined in Lemma 5.9. We are going to prove that for all u in A^{ζ} we have $\psi_{\zeta}(u) = \varphi_{\zeta}(F(u))$. In fact, if $u = st$ with $s \in A^{\tilde{\omega}}$ and $t \in A^{\omega}$, then

$$\begin{aligned}
\psi_{\zeta}(u) &= \psi_{\tilde{\omega}}(s) \cdot \psi_{\omega}(t) \\
&= \big(\varphi_{\tilde{\omega}}(s * q(s)), q(s)\big) \cdot (e \mapsto \varphi_{\omega}(e * t)) \\
&= \varphi_{\tilde{\omega}}(s * q(s)) \cdot \varphi_{\omega}(q(s) * t) \\
&= \varphi_{\zeta}\big((s * q(s)) \cdot (q(s) * t)\big).
\end{aligned}$$

Hence $\psi_\zeta(u)$ is the image by φ_ζ of the output label of the unique bi-infinite path with input label u. Since $\tilde{\mathcal{T}}(f)$ realizes the map F, this output label is precisely $F(u)$ and $\psi_\zeta(u) = \varphi_\zeta(F(u))$. Let I be a subset of Z_ζ such that $P = \varphi_\zeta^{-1}(I)$. Then $F^{-1}(P) = \psi_\zeta^{-1}(I)$, thus $F^{-1}(P)$ is recognized by $Z \circ \mathcal{D}_{k-1}$. ■

Lemma 5.11. *Consider a subset P of A^ζ and a ζ-semigroup homomorphism $\psi : A^\infty \to T$. Suppose I is a subset of T_ζ such that $P = \psi_\zeta^{-1}(I)$. Consider the sets*

$$L(P) = \{u \in A^+ \mid \exists x \in A^{\tilde{\omega}}\ \exists y \in A^\omega : xuy \in P\},$$
$$I_\psi = \{t \in T_+ \mid \exists x \in A^{\tilde{\omega}}\ \exists y \in A^\omega : \psi_{\tilde{\omega}}(x)\, t\, \psi_\omega(y) \in I\}.$$

Then $L(P) = \psi_+^{-1}(I_\psi)$.

Proof: $u \in \psi_+^{-1}(I_\psi) \Leftrightarrow \big[\exists x \in A^{\tilde{\omega}}\ \exists y \in A^\omega : \psi_\zeta(xuy) \in I\big] \Leftrightarrow u \in L(P)$. ■

Theorem 5.12. *Let $F : A^\mathbb{Z} \to B^\mathbb{Z}$ be a code with window size k and let $\mathcal{Y}$ be a sofic subshift of $B^\mathbb{Z}$ with syntactic semigroup S. Then the syntactic semigroup of the subshift $F^{-1}(\mathcal{Y})$ divides $S \circ \mathcal{D}_{k-1}$.*

Proof: Let Z be the syntactic ζ-semigroup of $\mathcal{Y}$. By Theorem 5.10 the subshift $F^{-1}(\mathcal{Y})$ is recognized by $Z \circ \mathcal{D}_{k-1}$. Then by Lemma 5.11 the language $L(F^{-1}(\mathcal{Y}))$ is recognized by $(Z \circ \mathcal{D}_{k-1})_+$. Hence, if R is the syntactic semigroup of $F^{-1}(\mathcal{Y})$ then $R \prec (Z \circ \mathcal{D}_{k-1})_+$. Since $(Z \circ \mathcal{D}_{k-1})_+ \prec Z_+ \circ \mathcal{D}_{k-1}$, and the division between semigroups is transitive, we deduce $R \prec Z_+ \circ \mathcal{D}_{k-1}$. By Proposition 5.6 we have $S = Z_+$. ■

6. Classes of sofic subshifts closed under taking divisors

A *full shift* is a subshift of the form $A^\mathbb{Z}$, for some alphabet A. The syntactic semigroup of a language L of A^+ may depend on the alphabet A. For example, the syntactic semigroup of A^+ as a language of A^+ is the trivial semigroup, while if $A \subsetneq B$ then the syntactic semigroup of A^+ as language of B^+ is the unique monoid $\{0, 1\}$ with the usual multiplication. The pseudovariety Sl is the least pseudovariety containing this monoid. To avoid ambiguities, we consider the syntactic semigroup of a full shift to be the trivial semigroup. On the other hand, if $\mathcal{X}$ is a subshift of $A^\mathbb{Z}$ different from a full shift then the syntactic semigroup of $L(\mathcal{X})$ is independent of A, basically because all elements of the non-empty set $A^+ \setminus L(\mathcal{X})$ are in the same class of the syntactic congruence, which is a zero of the syntactic semigroup [8]. For a pseudovariety V, consider the class $\mathscr{S}(\mathsf{V})$ of subshifts $\mathcal{X}$ whose syntactic semigroup belongs to V.

Theorem 6.1. *Let* V *be pseudovariety of semigroups containing* SI. *Then the class* $\mathscr{S}(\mathsf{V} * \mathsf{D})$ *is closed under taking divisors.*

Proof: Suppose $\mathcal{Y}$ is a subshift of $A^{\mathbb{Z}}$ belonging to $\mathscr{S}(\mathsf{V} * \mathsf{D})$. Let $\mathcal{X}$ be a subshift of $B^{\mathbb{Z}}$ dividing $\mathcal{Y}$. Then there is an integer k and a code $F : A^{\mathbb{Z}} \to {B_\$}^{\mathbb{Z}}$ with window size k such that $\mathcal{X} = F^{-1}(\mathcal{Y})$. Since $\mathsf{SI} \subseteq \mathsf{V}$ the syntactic semigroup of $L(\mathcal{Y})$ as a language of $(B_\$)^+$ also belongs to $\mathsf{V} * \mathsf{D}$. By Theorem 5.12, we have $\mathcal{X} \in \mathscr{S}((\mathsf{V} * \mathsf{D}) * \mathsf{D}_{k-1})$. But $\mathsf{V} * \mathsf{D} * \mathsf{D}_{k-1} = \mathsf{V} * \mathsf{D}$, because $\mathsf{D} * \mathsf{D} \subseteq \mathsf{D}$. ∎

Therefore, when V is a pseudovariety containing SI, the class $\mathscr{S}(\mathsf{V} * \mathsf{D})$ defines an algebraic invariant for weak equivalence. It is proved in [12] that this class defines a shift equivalence invariant. Let $\mathscr{S}_I(\mathsf{V})$ be the class of irreducible subshifts in $\mathscr{S}(\mathsf{V})$. For every pseudovariety V of semigroups we have $\mathsf{LV} = \mathsf{LV} * \mathsf{D}$, thus if $\mathsf{SI} \subseteq \mathsf{V}$ then $\mathscr{S}_I(\mathsf{LV})$ is closed under taking weak equivalent irreducible subshifts. There are infinitely many such classes [12]. Theorem 6.1 has the following converse:

Theorem 6.2 ([12])**.** *Let* V *be pseudovariety of semigroups. Let* $\mathscr{O}$ *be any of the operators* $\mathscr{S}$ *or* $\mathscr{S}_I$. *If* $\mathscr{O}(\mathsf{V})$ *is closed under taking conjugate subshifts then* $\mathsf{LSI} \subseteq \mathsf{V}$ *and* $\mathscr{O}(\mathsf{V}) = \mathscr{O}(\mathsf{V} * \mathsf{D})$.

Theorem 6.1 can be used as a method of proving that a certain class of subshifts is closed under division and therefore under weak equivalence. For example, the class of sofic subshifts is the class $\mathscr{S}(\mathsf{S})$, where S is the pseudovariety of all finite semigroups. Hence, an immediate corollary of Theorem 6.1 is that the class of sofic subshifts is closed under divisions. The class of finite type subshifts is also closed under division, but it is not of the form $\mathscr{S}(\mathsf{V})$; on the other hand, the class of *irreducible* finite type subshifts is equal to $\mathscr{S}_I(\mathsf{LCom})$ [12].

Two elements x and y of $A^{\mathbb{Z}}$ are *right-asymptotic* if there is an integer n such that $x_{[n,+\infty[} = y_{[n,+\infty[}$. A code $\varphi : \mathcal{X} \to \mathcal{Y}$ between two subshifts is *left-closing* if distinct right-asymptotic elements of $\mathcal{X}$ have distinct images by φ. For an algorithm to decide whether the cover associated to a labeled graph is left-closing or not see [3]. Clearly one can consider the dual definition of *right-closing* code. A code is *bi-closing* if it is simultaneously right-closing and left-closing. An *almost finite type subshift* is the image of an irreducible finite type subshift by a bi-closing code. Almost finite type subshifts form a class of irreducible sofic subshifts strictly containing the irreducible finite type subshifts. An irreducible sofic subshift is of almost finite type if and only if its right Fischer cover is left-closing [3, Proposition 2.16]. It is known that this class is closed under conjugation [3, Proposition 4.1]. Independently from this result, in [4] it was proved that almost finite type subshifts belong to $\mathscr{S}_I(\mathsf{LInv})$, and the second author proved in [12] that in fact all elements of $\mathscr{S}_I(\mathsf{LInv})$ are almost finite type subshifts. Therefore, since $\mathsf{SI} \subseteq \mathsf{Inv}$, from Theorem 6.1 we deduce the following sharper result:

Theorem 6.3. *The class of almost finite type subshifts is closed under taking irreducible divisors.*

The class of *aperiodic subshifts* is a class of almost finite type subshifts that deserves some attention [3]. It is proved in [5] that this class is equal to $\mathscr{S}_I(\mathsf{A})$. Since $\mathsf{SI} \subseteq \mathsf{A} = \mathsf{LA}$, Theorem 6.1 also has the following corollary:

Theorem 6.4. *The class of aperiodic subshifts is closed under taking irreducible divisors.*

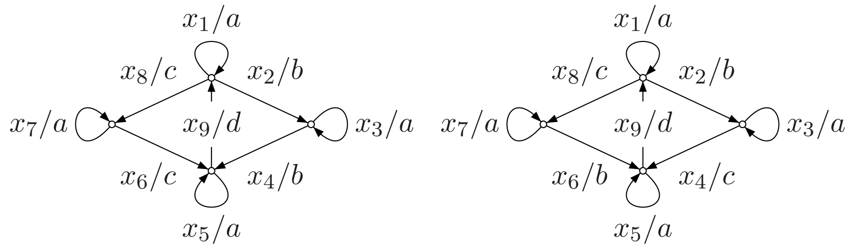
7. Comparison with other invariants

For a code $\varphi : \mathcal{X} \to \mathcal{Y}$, let $\mathcal{M}(\varphi)$ be the set $\{x \in \mathcal{X} : |\varphi^{-1}\varphi(x)| > 1\}$. Clearly, $\sigma(\mathcal{M}(\varphi)) \subseteq \mathcal{M}(\varphi)$ and $\sigma^{-1}(\mathcal{M}(\varphi)) \subseteq \mathcal{M}(\varphi)$. Hence $\overline{\mathcal{M}(\varphi)}$ is a subshift, called *multiplicity subshift of* φ. In general $\mathcal{M}(\varphi)$ is not closed. On the other hand, if $\varphi : \mathcal{X} \to \mathcal{Y}$ is a bi-closing code then $\mathcal{M}(\varphi)$ is closed [9]. The multiplicity subshift of a sofic subshift is effectively computable: see the Appendix, page 27. Note that if (f, g) is a conjugacy between φ and ψ, then $(f_{|\overline{\mathcal{M}(\varphi)}}, g_{|\varphi(\overline{\mathcal{M}(\varphi)})})$ is a conjugacy between $\varphi : \overline{\mathcal{M}(\varphi)} \to \varphi\big(\overline{\mathcal{M}(\varphi)}\big)$ and $\psi : \overline{\mathcal{M}(\psi)} \to \psi\big(\overline{\mathcal{M}(\psi)}\big)$.

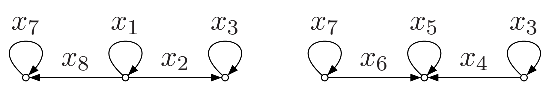
Theorem 7.1 ([9, Theorem 2.8])**.** *Suppose that $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are mixed subshifts of almost finite type. For each $i \in \{1, 2\}$, let $\pi_i : \mathcal{X}_i \to \mathcal{Y}_i$ be the right Fischer cover of $\mathcal{Y}_i$. Suppose that $(\pi_1)_{|\mathcal{M}(\pi_1)}$ and $(\pi_2)_{|\mathcal{M}(\pi_2)}$ are conjugate, and that $\mathcal{X}_1$ and $\mathcal{X}_2$ are shift equivalent. Then $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are shift equivalent.*

Since shift equivalence is a very strong conjugacy invariant for sofic subshifts, if $\pi : \mathcal{X} \to \mathcal{Y}$ is the right Fischer cover of a sofic subshift then the conjugacy class of $\pi_{|\mathcal{M}(\pi)}$ together with the shift equivalence class of $\mathcal{X}$ form a conjugacy invariant that is particularly strong when $\mathcal{Y}$ is mixed and of almost finite type.

Let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be the sofic subshifts whose right Fischer covers π_1 and π_2 are respectively realized by the following labeled graphs (where x/α means that the edge x is labeled α):



Subshifts $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are mixing almost finite type subshifts with the same zeta function. The domains of the right and left Krieger/Fischer covers are respectively equal. The next invariant to be tested is the multiplicity subshift. The multiplicity subshifts of $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are equal to the following subshift $\mathcal{X}$:



To prove that $\pi_{1|\mathcal{X}}$ is not conjugate with $\pi_{2|\mathcal{X}}$ we shall use the following lemma:

Lemma 7.2 ([9, Lema 2.3])**.** *Let $\varphi : \mathcal{X} \to \mathcal{Y}$ and $\psi : \mathcal{X} \to \mathcal{Z}$ be codes with equal domain. Then φ and ψ are conjugate if and only if there is for $\mathcal{X}$ an automorphism*[†] *F such that $\psi \circ F$ and φ have the same kernel*[‡]*.*

Let F be an automorphism of $\mathcal{X}$, with block map f with window size n. Since F permutes constant sequences, there is $i \in \{1, 3, 5, 7\}$ such that $F(x_i^\zeta) = x_1^\zeta$. Suppose $i \neq 1$. Then there are k, j such that $k \neq i$ and $x_k^{\tilde{\omega}}.x_j x_i^\omega \in \mathcal{X}$. Since $f(x_i^n) = x_1$, we have $F(x_k^{\tilde{\omega}}.x_j x_i^\omega) \sim_\sigma y.x_1^\omega$ for some $y \in A^{\tilde{\omega}}$. Since $x_l x_1 \in L(\mathcal{X})$ implies $l = 1$, we have $F(x_k^{\tilde{\omega}}.x_j x_i^\omega) = x_1^\zeta = F(x_i^\zeta)$, contradicting F being one-to-one. Hence $F(x_1^\zeta) = x_1^\zeta$. Analogously, $F(x_5^\zeta) = x_5^\zeta$, thus $\{F(x_3^\zeta), F(x_7^\zeta)\} = \{x_3^\zeta, x_7^\zeta\}$. Let $z = x_1^{\tilde{\omega}}.x_2 x_3^\omega$ and $t = x_3^{\tilde{\omega}}.x_4 x_5^\omega$. Then $z, t \in \mathcal{X}$ and $\pi_1(z) = \pi_1(t) = a^{\tilde{\omega}}.ba^\omega$. Suppose $F(x_3^\zeta) = x_7^\zeta$ and $F(x_7^\zeta) = x_3^\zeta$. Then $F(z) \sim_\sigma x_1^{\tilde{\omega}}.x_8 x_7^\omega$ and $F(t) \sim_\sigma x_7^{\tilde{\omega}}.x_6 x_5^\omega$, thus $\pi_2 F(z) \sim_\sigma a^{\tilde{\omega}}.ca^\omega$ and $\pi_2 F(t) \sim_\sigma a^{\tilde{\omega}}.ba^\omega$. In particular, $\pi_2 F(z) \neq \pi_2 F(t)$, and the same conclusion holds if $F(x_3^\zeta) = x_3^\zeta$ and $F(x_7^\zeta) = x_7^\zeta$. Therefore $\pi_{1|\mathcal{X}}$ and $\pi_{2|\mathcal{X}}$ are not conjugate, by Lema 7.2. Hence $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are not conjugate.

[†]An *automorphism* for $\mathcal{X}$ is a conjugacy from $\mathcal{X}$ to $\mathcal{X}$.

[‡]Recall that the kernel of a map $h : P \to Q$ is the set $\{(x, y) \in P \times P \mid h(x) = h(y)\}$.

The preceding arguments are somewhat ad-hoc, and depend on knowing the group of automorphisms of a subshift, a very difficult problem in general [18, Chapter 13]. On the other hand, as observed in [12], for the pseudovariety V of finite semigroups satisfying the identity $x^3 = x^2$, one has $\mathcal{Y}_1 \notin \mathscr{S}(\mathsf{LV})$ and $\mathcal{Y}_2 \in \mathscr{S}(\mathsf{LV})$. Since $\mathsf{Sl} \subseteq \mathsf{V}$, by Theorem 6.1 we conclude that $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are not weak equivalent. Hence Theorem 6.1 provides an expedite form of proving not only that $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are not conjugate, but also that they are far from being conjugate, in the sense that weak equivalence is considered a very weak conjugacy invariant.

8. A topological proof

In this section we use a different method for proving Theorem 6.1, based on profinite semigroup theory. As an introductory reference for this theory see [2].

A semigroup endowed with a compact topology for which the multiplication is continuous is called a *compact semigroup*. We consider finite semigroups as compact semigroups, endowing them with the discrete topology. A compact semigroup S is said to be *generated* by a map $\iota : A \to S$ if the subsemigroup of S generated by $\iota(A)$ is dense in S. Let V be a pseudovariety of semigroups. A *pro*-V semigroup is a projective limit of semigroups from V. A *pro*-V semigroup is therefore a compact semigroup. For the pseudovariety S of all finite semigroups, the term *profinite* is usually used instead of pro-S. For every set A there is a pro-V semigroup such that $\overline{\Omega}_A\mathsf{V}$ is generated by a map ι with domain A with the property that for every map φ from A into a semigroup S from V there is a unique continuous homomorphism $\hat{\varphi} : \overline{\Omega}_A\mathsf{V} \to S$ such that $\hat{\varphi} \circ \iota = \varphi$. The semigroup $\overline{\Omega}_A\mathsf{V}$ is the unique pro-V semigroup with these properties, up to isomorphism of compact semigroups. For this reason it is called the *free profinite semigroup relatively to* V (or *free pro-*V *semigroup*) generated by A. Assuming A is finite (as we do from here on), the topology of $\overline{\Omega}_A\mathsf{V}$ is generated by a metric. If V contains some non-trivial semigroup, then ι is injective, thus A is considered as a subset of $\overline{\Omega}_A\mathsf{V}$. And if V contains N then A^+ embeds in $\overline{\Omega}_A\mathsf{V}$ as a dense subsemigroup whose elements are isolated points.

The following proposition makes the connection between the combinatorial properties of V-recognizable languages and the topology of $\overline{\Omega}_A\mathsf{V}$, when $\mathsf{N} \subseteq \mathsf{V}$. For a more general result see [1, Theorem 3.6.1], or [2, Section 3].

Proposition 8.1. *Let* V *be a pseudovariety containing* N*. If* L *is a language of* A^+ *then* L *is* V*-recognizable if and only if the closure of* L *in* $\overline{\Omega}_A\mathsf{V}$ *open. The topology of* $\overline{\Omega}_A\mathsf{V}$ *is generated by the closures of the* V*-recognizable languages of* A^+*.*

Let V be a pseudovariety with non-trivial semigroups. Then, as we have said before, the alphabet A is a subset of $\overline{\Omega}_A\mathsf{V}$. Since $\overline{\Omega}_A\mathsf{V}$ is a profinite semigroup, there is a unique continuous homomorphism p_V from $\overline{\Omega}_A\mathsf{S}$ to $\overline{\Omega}_A\mathsf{V}$ such that $p_\mathsf{V}(a) = a$. The map p_V is called the *canonical projection* from $\overline{\Omega}_A\mathsf{S}$ to $\overline{\Omega}_A\mathsf{V}$. This map is closely related with the equational theory of pseudoidentities, since the equality $p_\mathsf{V}(u) = p_\mathsf{V}(v)$ means that the pseudoidentity $u = v$ is satisfied by V. We do not need to enter in this theory of pseudoidentities.

Theorem 8.2. *Let* V *be a pseudovariety containing* SI *and* N*. For every alphabet* A *and non-negative integer* k*, the map* $\Phi_k : A^+ \to (A^{k+1})^*$ *has a unique continuous extension from* $\overline{\Omega}_A(\mathsf{V} * \mathsf{D}_k)$ *to* $(\overline{\Omega}_{A^{k+1}}\mathsf{V})^1$*, which we denote by* Φ_k^V*.*

Proof: The theorem was already proved by Almeida for the pseudovariety S of all finite semigroups [1, Lemma 10.6.11]. The following map

$$\begin{array}{rrcl} \Phi_k^\mathsf{V}: & \overline{\Omega}_A(\mathsf{V} * \mathsf{D}_k) & \to & (\overline{\Omega}_{A^{k+1}}\mathsf{V})^1 \\ & p_{\mathsf{V}*\mathsf{D}_k}(u) & \mapsto & p_\mathsf{V}(\Phi_k^\mathsf{S}(u)), \quad u \in \overline{\Omega}_A\mathsf{S}, \end{array}$$

is well defined by Theorem 10.6.12 from [1]. Since $p_{\mathsf{V}*\mathsf{D}_k}$ and $p_\mathsf{V} \circ \Phi_k^\mathsf{S}$ are continuous and $\Phi_k^\mathsf{V} \circ p_{\mathsf{V}*\mathsf{D}_k} = p_\mathsf{V} \circ \Phi_k^\mathsf{S}$, by a well-known topological result the map Φ_k^V is also continuous. ∎

Theorem 8.3. *Let* V *be a pseudovariety containing* SI *and* N*. Consider a code* $F : A^\mathbb{Z} \to B^\mathbb{Z}$ *with window size* k*. Let* $\mathcal{Y}$ *be a subshift of* $B^\mathbb{Z}$*. If* $\mathcal{Y} \in \mathscr{S}(\mathsf{V})$ *then* $F^{-1}(\mathcal{Y}) \in \mathscr{S}(\mathsf{V} * \mathsf{D}_{k-1})$*.*

Proof: Let $f : A^k \to B$ be a block map for F. Then there is a unique continuous homomorphism $\hat{f} : \overline{\Omega}_{A^k}\mathsf{V} \to \overline{\Omega}_B\mathsf{V}$ extending f. Denote by W the pseudovariety $\mathsf{V} * \mathsf{D}_{k-1}$. Let $\tilde{f} : \overline{\Omega}_A\mathsf{W} \to (\overline{\Omega}_B\mathsf{V})^1$ be the map $\hat{f} \circ \Phi_{k-1}^\mathsf{V}$. The map $\tilde{f}$ is a (unique) continuous extension of f.

Let $\mathcal{X} = F^{-1}(\mathcal{Y})$. Since $\mathcal{Y} \in \mathscr{S}(\mathsf{V})$, the set $\overline{L(\mathcal{Y})}$ is open in $\overline{\Omega}_B\mathsf{V}$, by Proposition 8.1. By the same proposition, what we want to prove is that $\overline{L(\mathcal{X})}$ is open in $\overline{\Omega}_A\mathsf{W}$. Let $u \in \overline{L(\mathcal{X})}$. Since $L(\mathcal{X})$ is a prolongable language, with a simple compactness argument [13] one proves that for every integer l there are finite words r_l and s_l with length greater that l such that $r_l u s_l \in \overline{L(\mathcal{X})}$. Since $\overline{\Omega}_A \mathsf{W}$ is compact, taking subsequences if necessary, we can assume that $(r_l)_l$ and $(s_l)_l$ converge to some elements r and s of $\overline{\Omega}_A \mathsf{W}$, respectively. Then $rus \in \overline{L(\mathcal{X})}$. Since $\tilde{f}(L(\mathcal{X})) \subseteq L(\mathcal{Y}) \cup \{1\}$ and $\tilde{f}$ is continuous, one has $\tilde{f}(rus) \in \overline{L(\mathcal{Y})}$. Let $(u_n)_n$ be an arbitrary sequence of elements of A^+ converging to u. Then $\lim_{n\to+\infty,l\to+\infty} \tilde{f}(r_l u_n s_l) = \tilde{f}(rus)$. Since $\overline{L(\mathcal{Y})}$ is an open neighborhood of $\tilde{f}(rus)$, there is an integer N such that if $n, l > N$ then $\tilde{f}(r_l u_n s_l) \in \overline{L(\mathcal{Y})}$. Let $n > N$. Since the elements of B^+ are isolated in $\overline{\Omega}_B \mathsf{V}$, we have $\tilde{f}(r_l u_n s_l) \in L(\mathcal{Y})$ for all $l > N$. Consider arbitrary elements $p_l \in A^{\tilde{\omega}}$, $q_l \in A^{\omega}$ and let $x_{n,l} = p_l r_l . u_n s_l q_l \in A^{\mathbb{Z}}$. Let x_n be an adherent point of $(x_{n,l})_l$ in $A^{\mathbb{Z}}$. Then, given a positive integer k, for sufficiently large l the word $F(x_n)_{[-k,k]}$ is a factor of $\tilde{f}(r_l u_n s_l)$, hence it belongs to $L(\mathcal{Y})$. Since k is arbitrary, we have $F(x_n) \in \mathcal{Y}$, thus $x_n \in \mathcal{X}$. Therefore $u_n \in L(\mathcal{X})$, because u_n is a factor of x_n. Since $(u_n)_n$ is an arbitrary sequence of elements of A^+ converging to u and A^+ is dense in $\overline{\Omega}_A \mathsf{W}$, we conclude that $\overline{L(\mathcal{X})}$ is open. ∎

Since $\mathsf{V} * \mathsf{D} = (\mathsf{V} * \mathsf{D}) * \mathsf{D}_k$, and $\mathsf{SI}, \mathsf{N} \subseteq \mathsf{LSI} = \mathsf{SI} * \mathsf{D} \subseteq \mathsf{V} * \mathsf{D}$, Theorem 6.1 is a corollary of Theorem 8.3.

9. Ordered semigroups

In a partial ordered set $(X, \leq)$, an *order ideal* is a subset I of X such that if $t \leq s$ and $s \in I$ then $t \in S$. The order ideal generated by a set X is the set $\downarrow X = \{t \in S \mid \exists s \in X : t \leq s\}$.

An *ordered semigroup* is a semigroup S endowed with a partial order $\leq$ such that if $s \leq t$ then $xsy \leq xty$, for all $x, y \in S^1$. For an introduction to ordered semigroups see [22]. Usual semigroups are considered as ordered semigroups for the equality order. The morphisms between ordered semigroups are the order preserving homomorphisms of semigroups. Divisors and direct products have the obvious definitions, and there is also a theory of pseudovarieties of ordered semigroups.

The *syntactic ordered semigroup* of a language L of A^+ is the syntactic semigroup of L endowed with the partial order $\leq$ such that $u \leq v$ if and only if $C_L(v) \subseteq C_L(u)$. A language L of A^+ is recognized by a homomorphism φ from A^+ into an ordered semigroup S if there is an order ideal I in S such that $L = \varphi^{-1}(I)$. We say that L is recognized by the ordered semigroup S if there are such homomorphism φ and ideal I. If $\equiv$ is the syntactic congruence of L, then the language L is recognized by the homomorphism $\varphi : u \mapsto u/\equiv$ into its syntactic ordered semigroup.

The natural partial order for the wreath product of two ordered semigroups is defined as follows: given $(f_1, t_1), (f_2, t_2) \in S \circ T$ we have

$$(f_1, t_1) \leq (f_2, t_2) \Leftrightarrow \begin{cases} f_1(t) \leq f_2(t), \forall t \in T^1, \\ t_1 \leq t_2. \end{cases}$$

Theorem 9.1. *Let $F : A^{\mathbb{Z}} \to B^{\mathbb{Z}}$ be a code with window size k and let $\mathcal{Y}$ be a sofic subshift of $B^{\mathbb{Z}}$ with syntactic ordered semigroup S. Then the syntactic ordered semigroup of $F^{-1}(\mathcal{Y})$ divides the ordered wreath product $S \circ \mathcal{D}_{k-1}$.*

Proof: Consider the syntactic ζ-semigroup Z of $\mathcal{Y}$. Let φ be the canonical ζ-semigroup homomorphism from B^∞ to Z, and ψ be the ζ-semigroup homomorphism from A^∞ to $Z \circ \mathcal{D}_{k-1}$ as defined in Lemma 5.9. Denote by $\mathcal{X}$ the subshift $F^{-1}(\mathcal{Y})$.

Let I be a subset of Z_ζ such that $\mathcal{Y} = \varphi_\zeta^{-1}(I)$. By the proof of Theorem 5.10 we know that $\mathcal{X} = \psi_\zeta^{-1}(I)$. Consider the following set:

$$I_\psi = \{t \in (Z \circ \mathcal{D}_{k-1})_+ \mid \exists x \in A^{\tilde{\omega}}, y \in A^\omega : \psi_{\tilde{\omega}}(x)\, t\, \psi_\omega(y) \in I\}.$$

By Lemma 5.11 we have $L(\mathcal{X}) = \psi_+^{-1}(I_\psi)$. Let $u \in \psi_+^{-1}[\downarrow \psi_+(L(\mathcal{X}))]$. There is $v \in L(\mathcal{X})$ such that $\psi_+(u) \leq \psi_+(v)$. That is, $(g_u, \mathrm{t}_{k-1}(u)) \leq (g_v, \mathrm{t}_{k-1}(v))$. Equivalently,

$$\varphi_+(e * u) \leq \varphi_+(e * v), \forall e \in E(\mathcal{D}_{k-1}), \tag{9.1}$$

$$\mathrm{t}_{k-1}(u) = \mathrm{t}_{k-1}(v). \tag{9.2}$$

Since $v \in \psi_+^{-1}(I_\psi)$, there are $x \in A^{\tilde{\omega}}$, $y \in A^\omega$ such that $\psi_\zeta(xvy) \in I$. By the proof of Theorem 5.10, we have

$$\psi_\zeta(xvy) = \psi_{\tilde{\omega}}(x)\psi_\omega(vy) = \varphi_{\tilde{\omega}}(x * q(x)) \cdot \varphi_\omega(q(x) * vy) \in I. \tag{9.3}$$

By Lemma 5.9, we have

$$\begin{aligned}\varphi_\omega(q(x) * vy) &= \psi_\omega(vy)[q(x)]\\ &= \psi_+(v)\psi_\omega(y)[q(x)]\\ &= (g_v, \mathrm{t}_{k-1}(v))\psi_\omega(y)[q(x)]\\ &= g_v[q(x)]\cdot \psi_\omega(y)[q(x)\cdot \mathrm{t}_{k-1}(v)]\\ &= \varphi_+[q(x)*v]\cdot \varphi_\omega[(q(x)\cdot \mathrm{t}_{k-1}(v))*y]\\ &= \varphi_+[q(x)*v]\cdot \varphi_\omega[(q(x)\cdot \mathrm{t}_{k-1}(u))*y] \quad \text{(by (9.2))}.\end{aligned}$$

Therefore, by (9.3), we have

$$[x*q(x)]\cdot[q(x)*v]\cdot[(q(x)\cdot \mathrm{t}_{k-1}(u))*y] \in \varphi_\zeta^{-1}(I) = \mathcal{Y}. \tag{9.4}$$

By Lemma 5.5 we have $\varphi_+(w_1) \leq \varphi_+(w_2) \Leftrightarrow C_{\mathcal{Y}}(w_2) \subseteq C_{\mathcal{Y}}(w_1)$. Therefore, from (9.1) and (9.4) we deduce that

$$[x*q(x)]\cdot[q(x)*u]\cdot[(q(x)\cdot \mathrm{t}_{k-1}(u))*y] \in \varphi_\zeta^{-1}(I) = \mathcal{Y}.$$

Going backwards in the arguments, we conclude that

$$\psi_\zeta(xuy) = \varphi_\zeta([x*q(x)]\cdot[q(x)*u]\cdot[(q(x)\cdot \mathrm{t}_{k-1}(u))*y]) \in I,$$

thus $u \in \psi_+^{-1}(I_\psi) = L(\mathcal{X})$. This proves that $\psi_+^{-1}[\downarrow \psi_+(L(\mathcal{X}))] \subseteq L(\mathcal{X})$. The inclusion $L(\mathcal{X}) \subseteq \psi_+^{-1}[\downarrow \psi_+(L(\mathcal{X}))]$ is trivial. We conclude that $L(\mathcal{X})$ is recognized by $(Z \circ \mathcal{D}_{k-1})_+$. For the remaining part of the proof the arguments are the same as those used in the proof of Theorem 5.12. ■

Using Theorem 9.1, it is now easy to prove that the results about pseudovarieties of semigroups from Section 6 generalize to pseudovarieties of ordered semigroups, in a similar way to the corresponding results from [12]. The instruments for the topological approach from Section 8 seem to be insufficient for achieving the ordered case.

Appendix: the computation of the multiplicity subshift

A labeled graph is *faithfully labeled* if distinct co-terminal edges have distinct labels. Next we describe an algorithm to compute the multiplicity subshift of the cover associated to a faithfully labeled graph. This algorithm has similarities with the algorithm appearing in [3] for deciding if the cover is left-closing or not. In a graph G, we say that an edge e from r to s is a *descendant* (respectively, an *ascendant*) of a vertice p if there is in G a path from p to r (respectively, from s to p). Supposing that (G,π) is a faithfully labeled graph, denote by (p,a,q) the unique edge from p to q labeled a, if such edge exists. The *π-square* of G is the graph G^π whose vertices are the pairs of vertices of G, and where the set of edges between two vertices (p,r) and (q,s) is the set of triples $((p,r),a,(q,s))$ such that (p,a,q) and (r,a,s) are edges from G. A *diagonal* vertice of G^π is a vertice of the form (p,p). The projections defined by the rules $((p,r),a,(q,s)) \mapsto (p,a,q)$ and $((p,r),a,(q,s)) \mapsto (r,a,s)$ will be denoted by λ and ρ, respectively.

Proposition 9.2. *Consider the faithfully labeled graph (G,π). Let W be the set of the elements of X_{G^π} corresponding to bi-infinite paths over G^π passing at a non-diagonal vertice. Then $\mathcal{M}(\pi_*) = \lambda_*(W)$. The language $L(\mathcal{M}(\pi_*))$ is recognized by the labeled graph obtained from the essential part of (G^π,λ) by removing the edges which are neither ascendants or descendants of non-diagonal vertices.*

Proof: Suppose that $c = (p_i, a_i, p_{i+1})_{i\in\mathbb{Z}}$ is an element of $\mathcal{M}(\pi_*)$. Then there is in X_G an element of the form $(q_i, a_i, q_{i+1})_{i\in\mathbb{Z}}$ distinct from c. Let $\hat{c} = ((p_i,q_i),a_i,(p_{i+1},q_{i+1}))_{i\in\mathbb{Z}}$. Then $\hat{c} \in W$ and $\lambda_*(\hat{c}) = c$, thus $\mathcal{M}(\pi_*) \subseteq \lambda_*(W)$. Conversely, for every $c \in X_{G^\pi}$ we have $\pi_*(\lambda_*(c)) = \pi_*(\rho_*(c))$, and $c \in W$ if and only if $\lambda_*(c) \neq \rho_*(c)$, thus $\lambda_*(W) \subseteq \mathcal{M}(\pi_*)$.

In an essential graph, an edge is ascendant or descendant of a given vertice if and only if it belongs to a bi-infinite path passing at such vertice. Therefore, by the first part of the proof, in the essential part of (G^π,λ) the labels of paths whose edges are ascendants or descendants of non-diagonal vertices are precisely the elements of $L(\mathcal{M}(\pi_*))$. ■

Since $L(\mathcal{M}(\pi_*)) = L(\overline{\mathcal{M}(\pi_*)})$, Proposition 9.2 allows us to compute a presentation of the subshift $\overline{\mathcal{M}(\pi_*)}$.

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