Cover relations on categories

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We define a *precover relation* on a category $\mathbb{C}$ to be any binary relation $\sqsubset$ on the class of morphisms of $\mathbb{C}$ such that if $f \sqsubset g$ then f and g have the same codomain. If $\mathbb{C}$ has finite limits, then a precover relation on $\mathbb{C}$ is in some sense the weakest structure on $\mathbb{C}$ which allows to interpret in $\mathbb{C}$ first order sentences of the form $\forall_{x_1,\dots,x_n}(\phi \Rightarrow \psi)$, where ϕ and ψ are formulas made up of atomic ones using the connective $\wedge$ and the quantifier $\exists$. One of the uses of categorical interpretations of such first order sentences is that it allows to reformulate purely categorically certain term conditions of Universal Algebra. For instance, the condition that defines a Mal'tsev category can be obtained in this way from the term condition that defines a Mal'tsev variety.

A *cover relation* is a precover relation having the following two properties: (i) if $f \sqsubset g$ and f, g are composable with h, then $hf \sqsubset hg$, (ii) if $f \sqsubset g$ and e is composable with f then $fe \sqsubset g$. We show that by imposing natural axioms on cover relations we obtain, on the one hand, a special class of factorization systems, and, on the other hand, a special class of monoidal structures.