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Branched Coverings of a Quasi Locally Connected Topos

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Branched coverings of a locally connected topos

A branched covering $\psi : \mathscr{F} \longrightarrow \mathscr{E}$ of a *locally connected* space/topos $\mathscr{E}$ is interpreted in topology (Fox'57) and topos theory (Bunge-Funk'06) as the *complete spread* associated to a *locally constant covering* φ of the *unbranched part* $\mathscr{E}/S$, where $\mathscr{E}/S \rightarrowtail \mathscr{E}$ is a pure inclusion.

$$\begin{tikzcd} \mathscr{G} \arrow[r, "\rho"] \arrow[d, "\varphi"'] & \mathscr{F} \arrow[d, "\psi"] \\ \mathscr{E}/S \arrow[r, tail, "i"'] & \mathscr{E} \end{tikzcd}$$

The geometric morphism ρ is pure, and the *comprehensive factorization* $\langle \rho, \varphi \rangle$ of the composite $i \cdot \varphi : \mathscr{G} \longrightarrow \mathscr{E}$ corresponds to the *Lawvere distribution* $\mu = g_! \cdot \varphi^* \cdot i^* : \mathscr{E} \longrightarrow \mathscr{S}$. The diagram can be shown to be a pullback.

Goals

The locally connected assumption on $\mathscr{E}$ was subsequently removed from the spread completion construction in topology (Michael'63) and topos theory (Bunge-Funk '07). In the latter, it is replaced by that of a *quasi locally connected topos*. (All Grothendieck toposes are quasi locally connected.)

My goals here are as follows, for a quasi locally connected topos $\mathscr{E}$:

1. to investigate the *fundamental (pro)groupoid* of $\mathscr{E}$,
2. to investigate *branched coverings* of $\mathscr{E}$, and
3. to examine the *connection* between these two notions.

Basic definitions

Let $\mathbf{Top}_{\mathscr{S}}$ be the 2-category of toposes bounded over $\mathscr{S}$, geometric morphisms over $\mathscr{S}$, and natural isomorphisms between (inverse image parts of) geometric morphisms.

An object $\mathscr{E} \xrightarrow{e} \mathscr{S}$ of $\mathbf{Top}_{\mathscr{S}}$ (with $e^* \dashv e_*$ and e^* finite limits preserving) is said to be *locally connected* if there is an $\mathscr{S}$-indexed left adjoint

$$e_! \dashv e^* : \mathscr{S} \longrightarrow \mathscr{E}.$$

In particular, the BCC holds, in the sense that, for any $Y \xrightarrow{p} X$ in $\mathscr{S}$, the transpose (below, right) of a pullback square (below, left) is again a pullback.

$$\begin{tikzcd} B \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner"] & e^*Y \arrow[d, "e^*p"] \\ A \arrow[r, "m"] & e^*X \end{tikzcd} \qquad\qquad \begin{tikzcd} e_!B \arrow[r] \arrow[d, "e_!q"'] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d, "p"] \\ e_!A \arrow[r, "\widehat{m}"] & X \end{tikzcd}$$

The functor $E^* : \mathbf{Loc} \longrightarrow \mathscr{E}$

For any *locale* X in $\mathscr{S}$, let E^*X be defined so that, for every object A of $\mathscr{E}$, there are natural bijections

$$\frac{\dfrac{A \longrightarrow E^*X \text{ in } \mathscr{E}}{\text{geometric morphisms } \mathscr{E}/A \longrightarrow Sh(X) \text{ over } \mathscr{S}}}{\text{locale morphisms } L(A) \longrightarrow X}\ ,$$

where $L(A)$ is the locale reflection of A.

These bijections are not equivalences of categories when the 2-cell structure of **Loc** is taken into account: we say that locale morphisms $m : W \longrightarrow X$ and $l : W \longrightarrow X$ satisfy $m \leq l$ if $m^*U \leq l^*U$, for any $U \in \mathscr{O}(X)$. Then $\mathscr{E}(A, E^*X)$ is discrete in this sense, but $\mathbf{Loc}(L(A), X)$ may not be - for instance, take X to be Sierpinski space. Thus, E^* forgets 2-cells.

Locally discrete locales

These remarks justify the restriction of our discusssion to categories of locally discrete locales.

A locale Z is *locally discrete* if for every locale X the partial ordering in $\mathbf{Loc}(X,Z)$ is discrete. Likewise, a map $Z \xrightarrow{p} B$ is *locally discrete* if for every $X \xrightarrow{q} B$, $\mathbf{Loc}/B(q,p)$ is discrete.

Let **LD** denote the category of locally discrete locales. It is easy to verify that **LD** may be regarded as an $\mathscr{S}$-indexed category. As such, **LD** has Σ satisfying the BCC, and small hom-objects.

Zero-dimensional locales

A morphism $m : B \longrightarrow A$ in a topos $f : \mathscr{F} \longrightarrow \mathscr{S}$ is $\mathscr{S}$*-definable* (Barr-Paré '80) if it can be put in a pullback square as follows.

$$\begin{tikzcd} B \arrow[r, "m"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d] \\ f^*J \arrow[r, "f^*n"'] & f^*I \end{tikzcd}$$

We say that a geometric morphism $\mathscr{F} \xrightarrow{\psi} \mathscr{E}$ over $\mathscr{S}$ is *a spread* if it has an $\mathscr{S}$-definable family that generates $\mathscr{F}$ relative to $\mathscr{E}$.

A locale X in $\mathscr{S}$ is said to be *zero-dimensional* if its topos of sheaves *Sh*$(X) \longrightarrow \mathscr{S}$ is a spread.

Denote by **Z** the full subcategory of **Loc** whose objects are the zero-dimensional locales.

A counterexample

If $Y \longrightarrow X$ is an etale map of locales and X is discrete, then Y is discrete. It may seem intuitively true that a similar condition holds for zero-dimensional locales, but the following example (courtesy of Peter Johnstone) shatters this naïve belief.

Let $X = \{0\} \cup \{\frac{1}{n} \mid n \geq 1\}$ with the topology induced by its inclusion in $\mathbb{R}$. Let $Y = X + X/\sim$, obtained by identifying the two $\frac{1}{n}$'s, for every n. The map $Y \longrightarrow X$ identifying the two 0's is etale, X is an object of **Z**, but Y is not. The topology on Y is T_1, but not Hausdorff.

A suitable category **V** *of locales*

V shall denote an $\mathscr{S}$-indexed category of locales in $\mathscr{S}$ with Σ with the BCC, which in addition satisfies the following conditions:

(1) $\mathscr{S} \hookrightarrow \mathbf{V} \hookrightarrow \mathbf{LD}$,

(2) If $Y \longrightarrow Z$ is a **V**-map, and Z is in **V**, then so is Y.

(3) If Y is in **V**, then any locale morphism $Y \longrightarrow Z$ is a **V**-map.

(4) **V** is closed under *open sublocales*, and

(5) **V** is closed under pullbacks

$$\begin{tikzcd} W \arrow[r, "q"] \arrow[d, "n"'] \arrow[dr, phantom, "\lrcorner"] & Z \arrow[d, "m"] \\ Y \arrow[r, "p"] & X \end{tikzcd}$$

where p is *etale*. That is, if m is a **V**-map, then so is n.

Examples

The categories $\mathscr{S}$, **Z**, **Loc**$_{prodis}$, and **LD** are all instances of such a **V**, whereas **Loc** is not.

Simplification. For the purposes of this lecture, we may think of **V** as either $\mathscr{S}$ or **Z**. The $\mathscr{S}$-maps are usually called etale maps, the **Z**-maps are called spreads.

V-*determined toposes*

A topos $\mathscr{E} \xrightarrow{e} \mathscr{S}$ is said to be **V**-*determined* if there is an $\mathscr{S}$-indexed left adjoint

$$E_! \dashv E^* : \mathbf{V} \longrightarrow \mathscr{E},$$

such that the BCC for etale maps in **V** *holds* , in the sense that, for any $Y \xrightarrow{p} X$ etale map in **V**, the transpose (below, right) of a pullback square (below, left) is again a pullback.

$$\begin{tikzcd} B \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner"] & E^*Y \arrow[d, "E^*p"] \\ A \arrow[r, "m"] & E^*X \end{tikzcd} \qquad \begin{tikzcd} E_!B \arrow[r] \arrow[d, "E_!q"'] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d, "p"] \\ E_!A \arrow[r, "\widehat{m}"] & X \end{tikzcd}$$

Definition. We call *quasi locally connected* a **Z**-determined topos. (An $\mathscr{S}$-determined topos is precisely a locally connected one.)

V-initial geometric morphisms

Consider the transpose $\widehat{\alpha} : F^* \Longrightarrow \rho_* G^*$ of $\alpha : \rho^* F^* \Longrightarrow G^*$ under $\rho^* \dashv \rho_*$.

$$
\begin{tikzcd}
F^* \arrow[d, Rightarrow, "\eta F^*"'] \arrow[dr, Rightarrow, "\widehat{\alpha}"] & \\
\rho_* \rho^* F^* \arrow[r, Rightarrow, "\rho_* \alpha"'] & \rho_* G^*
\end{tikzcd}
$$

Definition. $\mathscr{G} \xrightarrow{\rho} \mathscr{F}$ is **V**-*initial* if $\widehat{\alpha}$ is an isomorphism. ρ_* *preserves* **V**-*coproducts* if ηF^* is an isomorphism.

Proposition. An inclusion *local homeomorphism* $\mathscr{E}/A \xrightarrow{\varphi_A} \mathscr{E}$ over $\mathscr{S}$ is **Z**-initial iff $(\varphi_A)_*$ preserves **Z**-coproducts. An arbitrary geometric morphism $\mathscr{F} \xrightarrow{\rho} \mathscr{E}$ over $\mathscr{S}$ is $\mathscr{S}$-initial iff ρ_* preserves $\mathscr{S}$-coproducts.

The comprehensive **V***-factorization*

We recall the following crucial result.

Theorem. (Bunge-Funk '07) For any geometric morphism $\mathscr{F} \xrightarrow{\varphi} \mathscr{E}$, with $\mathscr{F}$ a **V**-determined topos, there is a 'unique' factorization relative to the **V**-distribution $\mu = F_! \cdot \varphi^*$

$$\begin{tikzcd}
\mathscr{F} \arrow[rr, "\rho"] \arrow[dr, "\varphi"'] & & \mathscr{X} \arrow[dl, "\psi"] \\
& \mathscr{E} &
\end{tikzcd}$$

into a **V**-initial first factor $\mathscr{F} \xrightarrow{\rho} \mathscr{G}$, followed by a **V**-fibration $\psi : \mathscr{X} \longrightarrow \mathscr{E}$ (where $\psi = \{\mu\}$ is the support[*] of μ). The middle topos $\mathscr{X}$ is **V**-determined,

[*]A construction of the support of a **V**-distribution can be given relative to a site presentation of $\mathscr{E}$, but it does not depend on the chose site.

Toposes 'over **V***'*

Note that E^*X (defined earlier for a locale X) is the *interior* of the topos pullback below, left. It exists for any localic geometric morphism.

$$
\begin{tikzcd}
\mathscr{E} \times \mathit{Sh}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathit{Sh}(X) \arrow[d] \\
\mathscr{E} \arrow[r, "e"] & \mathscr{S}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{E}/E^*X \arrow[r] \arrow[d] & \mathit{Sh}(X) \arrow[d] \\
\mathscr{E} \arrow[r, "e"] & \mathscr{S}
\end{tikzcd}
$$

We have a commutative square of toposes above, right.

For every object D of $\mathscr{E}$, there is a canonical geometric morphism

$$\rho_D : \mathscr{E}/D \longrightarrow \mathscr{E}/E^*E_!D \longrightarrow \mathit{Sh}(E_!D).$$

The zero-dimensional locale $E_!D$ is the locale whose points (if any) are the *quasicomponents* of D. A topos $\mathscr{E}$ is quasi locally connected iff ρ_D is a pure surjection for every D.

Laxness

1. If $\mathscr{G} \xrightarrow{\rho} \mathscr{F}$ is a geometric morphism over $\mathscr{S}$, then there is a natural transformation

$$\alpha : \rho^* F^* \Longrightarrow G^*.$$

2. The 2-cell α is an *iso* if either $\mathbf{V} = \mathscr{S}$, or else $\mathbf{V} = \mathbf{Z}$ and ρ is a *local homeomorphism*.

The fundamental pushout topos of a quasi locally connected topos

For an epimorphism $U \twoheadrightarrow 1$ in $\mathscr{E}$, denote by $\mathscr{G}_U(\mathscr{E})$ the topos defined by the following pushout *

$$
\begin{array}{ccc}
\mathscr{E}/U & \xrightarrow{\varphi_U} & \mathscr{E} \\
{\scriptstyle \rho_U}\downarrow & & \downarrow{\scriptstyle \sigma_U} \\
\mathit{Sh}(E_! U) & \xrightarrow[p_U]{} & \mathscr{G}_U(\mathscr{E})\,,
\end{array}
$$

where $\mathscr{E}/U \xrightarrow{\varphi_U} \mathscr{E}$ is the canonical local homeomorphism, and where ρ_U is the *surjective* pure factor described above.

*The analogue for a locally connected topos $\mathscr{E}$ was introduced and exploited in Bunge '91 (see also Bunge '04).

Locally constant coverings over **Z**

Definition. An object A of $\mathscr{E}$ is said to be *locally constant over* **Z**, and the corresponding geometric morphism $\mathscr{E}/A \longrightarrow \mathscr{E}$ a *locally constant covering of* $\mathscr{E}$ *over* **Z**, provided there is an object $U \twoheadrightarrow 1$ of $\mathscr{E}$, an etale map $\alpha : Y \longrightarrow X$ in **Loc** with X zero-dimensional, a morphism $\eta : U \longrightarrow E^*X$ in $\mathscr{E}$, and a morphism $\zeta : A \times U \longrightarrow E^*Y$ in $\mathscr{E}$, for which the square

$$\begin{tikzcd} A \times U \arrow[r, "\zeta"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & E^*Y \arrow[d, "E^*\alpha"] \\ U \arrow[r, "\eta_U"'] & E^*X \end{tikzcd}$$

is a pullback. We say that A is *split by* U.

Denote by $\mathscr{C}_U(\mathscr{E})$ the obvious *category of U-split locally constant coverings of $\mathscr{E}$ over* **Z**.

The fundamental groupoid topos

Theorem. Let $\mathscr{E}$ be a *Grothendieck topos*. Then:
(1) There is an equivalence $\mathscr{G}_U(\mathscr{E}) \cong \mathscr{C}_U(\mathscr{E})$ which commutes with $\sigma^* : \mathscr{G}_U(\mathscr{E}) \longrightarrow \mathscr{E}$ and the forgetful $\mathscr{C}_U(\mathscr{E}) \longrightarrow \mathscr{E}$, both of which are surjections but not necessarily connected.

(2) The localic point p_U : *Sh*$(E_! U) \longrightarrow \mathscr{G}_U(\mathscr{E})$ is a spread and is of effective descent. The fundamental pushout topos $\mathscr{G}_U(\mathscr{E})$ is the classifying topos of an etale complete zero-dimensional groupoid ${\pi_1}^U(\mathscr{E})$.

(3) The limit topos $\mathscr{G}(\mathscr{E})$ for a generating (non-strict) filtering 2-system of covers $U \twoheadrightarrow 1$ is the classifying topos of a zero-dimensional progroupoid $\pi_1(\mathscr{E})$.

(4) Unlike the locally connected case, there no implicit Galois theory in the general case.

Locally **Z***-trivial coverings*

Defintion. An object A of a topos $\mathscr{E}$ over **V** is said to be a *locally* **Z***-trivial object*, and the corresponding geometric morphism $\mathscr{E}/A \longrightarrow \mathscr{E}$ a *locally* **Z***-trivial covering* provided there is an object $U \twoheadrightarrow 1$ of $\mathscr{E}$, an etale map $\alpha : Y \longrightarrow X$ in **Z**, a morphism $\eta : U \longrightarrow E^*X$ in $\mathscr{E}$, and a morphism $A \times U \xrightarrow{\zeta} E^*Y$ in $\mathscr{E}$, for which the square

$$\begin{tikzcd} A \times U \arrow[r, "\zeta"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner"] & E^*Y \arrow[d, "E^*\alpha"] \\ U \arrow[r, "\eta_U"'] & E^*X \end{tikzcd}$$

is a pullback.

Theorem. Any locally **Z**-trivial covering of $\varphi : \mathscr{G} \longrightarrow \mathscr{E}$ is a **Z**-fibration (and a local homeomorphism). We say that such a geometric morphism is **Z**-*unramified* (or **Z**-*unbranched*).

Branched coverings of a quasi locally connected topos

Definition. Let $\mathscr{E}$ be quasi locally connected over $\mathscr{S}$. A commutative diagram

$$\begin{tikzcd} \mathscr{G} \arrow[r, "\rho"] \arrow[d, "\varphi"'] & \mathscr{F} \arrow[d, "\psi"] \\ \mathscr{E}/S \arrow[r, rightarrowtail, "i"'] & \mathscr{E} \end{tikzcd}$$

is said to be a *branched covering* of $\mathscr{E}$ if (1) ψ is a **Z**-fibration, (2) $i : \mathscr{E}/S \rightarrowtail \mathscr{E}$ a **Z**-initial inclusion, (3) $\varphi : \mathscr{G} \longrightarrow \mathscr{E}/S$ a locally **Z**-trivial covering, and (4) ρ is **Z**-initial.

The diagram can be shown to be a *pullback*. This is crucial for any notion of a branched covering: the restriction to the non-singular part is an *unbranched* **Z***-covering*, that is, both a local homeomorphism and a **Z**-fibration.

Topological invariants

For any quasi locally connected topos $\mathscr{E}$, there is an obvious category $\mathscr{Z}_U(\mathscr{E})$ of locally **Z**-trivial coverings of $\mathscr{E}$ split by a given $U \twoheadrightarrow 1$ in $\mathscr{E}$, as well as an also obvious category ${\mathscr{B}_U}^S(\mathscr{E})$ of branched coverings of $\mathscr{E}$ with non-singular part $\mathscr{E}/S \rightarrowtail \mathscr{E}$.

Theorem. Let $\mathscr{E}$ be a quasi locally connected topos over $\mathscr{S}$. Then, for each $U \twoheadrightarrow 1$ in $\mathscr{E}$, there is an equivalence ${\mathscr{B}_U}^S(\mathscr{E}) \cong \mathscr{Z}_U(\mathscr{E}/S)$ given, in one direction, by pulling back along $\mathscr{E}/S \rightarrowtail \mathscr{E}$ and, in the other direction, by taking the comprehensive **Z**-factorization of $\mathscr{G} \longrightarrow \mathscr{E}/S \rightarrowtail \mathscr{E}$.

The inclusion ${\mathscr{B}_U}^S(\mathscr{E}) \hookrightarrow \mathscr{C}_U(\mathscr{E}/S)$ is (in general) a *proper inclusion*. In the case of *discrete* locales the inclusion is an equivalence.

An inconvenient truth in the quasi locally connected case

Let $\mathscr{E}$ be a quasi locally connected. Let

$$j : \mathscr{E}_{\mathbf{Z}} \hookrightarrow \mathscr{E}$$

be the subtopos of sheaves for the largest topology for which all objects E^*X for $X \in \mathbf{Z}$ are sheaves (use methods of Paré '80). Clearly, j_* preserves $\mathbf{Z}$-coproducts and is the smallest such.

Since for each local homeomorpism $i : \mathscr{E}/S \hookrightarrow \mathscr{E}$ is $\mathbf{Z}$-initial iff i_* preserves $\mathbf{Z}$-coproducts, $\mathscr{E}_{\mathbf{Z}} \hookrightarrow \mathscr{E}$ factors through every $\mathbf{Z}$-initial local homeomorphism $i : \mathscr{E}/S \hookrightarrow \mathscr{E}$. However, $\mathscr{E}_{\mathbf{Z}} \hookrightarrow \mathscr{E}$ itself can 'never' be a local homeomorphism.

It follows that *the smallest* $\mathbf{Z}$*-initial subtopos* of $\mathscr{E}$ does not exist in general and that therefore the intrinsic characterization of branched coverings (Bunge-Funk'06) does not hold in general either.

Concluding remarks

1. In the locally connected case, to regard branched coverings as complete spreads is adequate both in topology and topos theory since their study *reduces to the fundamental groupoid of the non-singular parts*, hence topological invariants are readily available. In the quasi locally connected case, there is a *divergence* between the two, hence the matter of topological invariants is less clear (non existent?).

2. Nevertheless, a *van Kampen theorem* ('pushout to pushout') of the sort considered by Fox to be of relevance to knot theory, still holds in the general setting (basically by the generakl results of Bunge-Lack '03).

3. In the locally connected case there is a *'universal knot group'* where the various knot groups can be effectively compared. This fails in the quasi locally connected case since the smallest $\mathbf{Z}$-initial subtopos of a quasi locally connected topos does not exist in general. In particular, the *intrinsic characterization* of branched coverings (Bunge-Funk, Street Fest contribution) is no longer valid in this generality.

4. In the locally connected case, the *fundamental pushout* gives the entire theory of the fundamental (pro)groupoid at each stage, including the *Galois theory*. This, alas, is no longer the case in the quasi locally connected case. In particular, the fundamental groupoid topos is not a Galois topos.

5. Note, however, that the fundamental pushout, which (in my view) is the key to the fundamental groupoid, is defined in 'the same'

way for *any* **V**, using the comprehensive **V**-factorization. Whereas, in the locally connected case, $\pi_0(\mathscr{E}/U)$ is discrete (locale of *connected components* of U), in the quasi locally connected case, $\pi_0(\mathscr{E}/U)$ is zero-dimensional (a locale whose opens are the *clopen subsets* of U, and whose points are the quasicomponents).

6. From the point of view of (Lawvere) distributions, what changes is the target, which is the category **V** of locales in each case. Among other things, we lose the classifying topos of distributions on $\mathscr{E}$ and the theory of the symmetric Kock-Zöberlein topos doctrine.

7. In essence, in order to study branched coverings in the general (quasi locally connected) case, one has to get *out of topos theory*.

This study (should it be worthwhile) would involve, instead of *toposes* of $\mathscr{S}$-valued sheaves (on zero-dimensional locales), *categories* of $\mathbf{Z}$-valued sheaves (also on zero-dimensional locales).

8. We conclude that the locally connected case is truly advantageous over the quasi locally connected (or general) case from many viewpoints. This, which was the motivation for the present investigation with regard to branched coverings and knot invariants, explains perhaps why the locally connected (and often also the connected) assumption is so ubiquitous in topology.

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