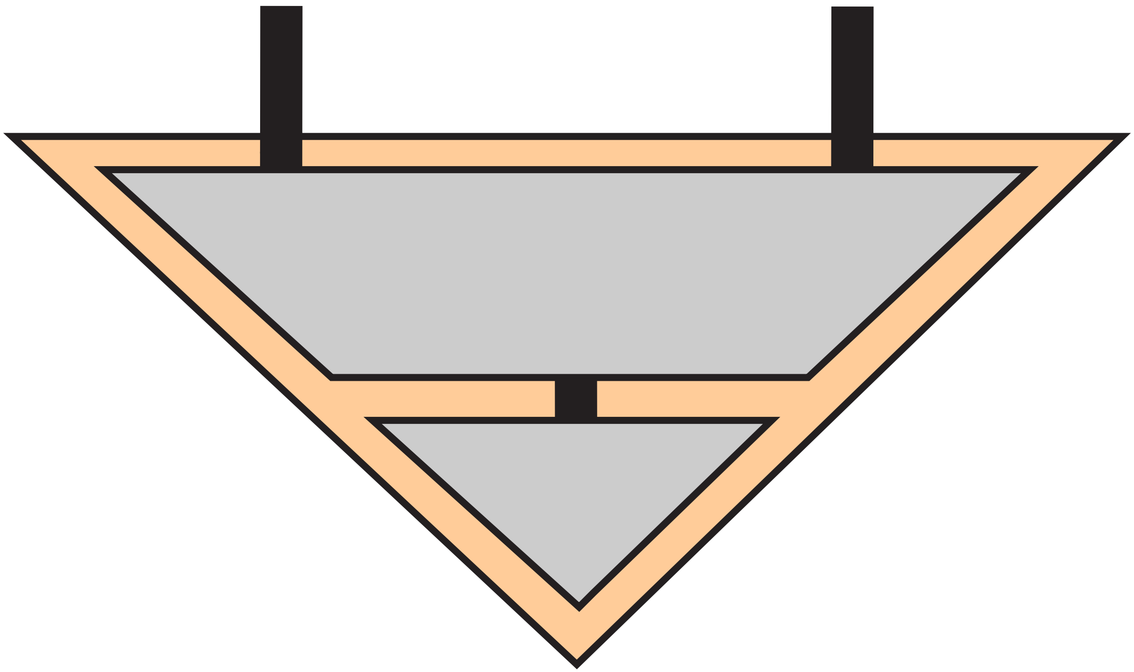
Bob Coecke
University of Oxford



Quantum physics as it is practised in the lab

WHY CATEGORIES?

Kinds/types of systems:

$$A\,,\ B\,,\ C\,,\ \ldots$$

- e.g. electron, atom, n qubits, classical data, ...

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Operations/experiments on systems:

$$A \xrightarrow{f} A\,,\quad A \xrightarrow{g} B\,,\quad B \xrightarrow{h} C\,,\ \ldots$$

- e.g. preparation, acting force field, measurement, ...

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$$A \xrightarrow{h\circ g} C \quad := \quad A \xrightarrow{\ g\ } B \xrightarrow{\ h\ } C \qquad\qquad A \xrightarrow{1_A} A$$

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Multiplicity of systems/operations:

$$A \otimes B \qquad\qquad A \otimes C \xrightarrow{f\otimes g} B \otimes D$$

Bifunctoriality ≡ independence of basic operations

$$
\begin{array}{ccc}
A \otimes C & \xrightarrow{f \otimes 1_C} & B \otimes C \\
{\scriptstyle 1_A \otimes g}\downarrow & & \downarrow{\scriptstyle 1_B \otimes g} \\
A \otimes D & \xrightarrow[f \otimes 1_D]{} & B \otimes D
\end{array}
$$

Bifunctoriality $\equiv$ independence of basic operations

$$
\begin{array}{ccc}
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{\scriptstyle 1_A\otimes g}\big\downarrow & & \big\downarrow{\scriptstyle 1_B\otimes g} \\
A \otimes D & \xrightarrow[f\otimes 1_D]{} & B \otimes D
\end{array}
$$

$\Rightarrow$ Compatibility with relativity

Symmetry $\equiv$ re-arrange systems & operations

$$\begin{array}{ccc} A \otimes C & \xrightarrow{f \otimes g} & B \otimes D \\ \downarrow{\scriptstyle \sigma_{A,C}} & & \downarrow{\scriptstyle \sigma_{B,D}} \\ C \otimes A & \xrightarrow[g \otimes f]{} & D \otimes B \end{array}$$

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$$\begin{array}{ccc} A \otimes C & \xrightarrow{f \otimes g} & B \otimes D \\ \downarrow{\scriptstyle \sigma_{A,C}} & & \downarrow{\scriptstyle \sigma_{B,D}} \\ C \otimes A & \xrightarrow[g \otimes f]{} & D \otimes B \end{array}$$

... re-associate, introduce/discard systems & operations

PRACTICING PHYSICS
Physical System
Physical Operation

PROGRAMMING
Data Types
Programs

LOGIC & PROOF THEORY
Propositions
Proofs

Practising physics in the lab = operationalism

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NOT categorifying the mathematical models of QM

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NOT speculating about a grand unificational theory

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Model interaction of the scientist with his subject

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The particular capabilities of doing so $\equiv$ structure

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NOT categorifying the mathematical models of QM

NOT speculating about a grand unificational theory

Model interaction of the scientist with his subject

The particular capabilities of doing so $\equiv$ structure

- ***Quantum structure*: non-local correlations**
- ***Classical structure*: ability to clone/delete**

The immediate pay-off

Distinct types of systems

The immediate pay-off

Distinct types of systems

Two-dimensional compositionality

The immediate pay-off

Distinct types of systems

Two-dimensional compositionality

Full comprehension w.r.t. classical data flow

The immediate pay-off

Distinct types of systems

Two-dimensional compositionality

Full comprehension w.r.t. classical data flow

Radical increase of degrees of axiomatic freedom

[von Neumann 1932] Formalized quantum mechanics in "Mathematische Grundlagen der Quantenmechanik"

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[Birkhoff & von Neumann 1936] "The LOGIC of Quantum Mechanics", *Annals of Mathematics*.

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Several quantum logic programmes emerged, ...

Birkhoff-von Neumann paradigm:

$$\frac{\textbf{Quantum logic}}{\textbf{Classical logic}} \simeq \frac{\textbf{NO distributivity}}{\textbf{distributivity}}$$

Birkhoff-von Neumann paradigm:

$$\frac{\textbf{Quantum logic}}{\textbf{Classical logic}} \simeq \frac{\textbf{NO deduction}}{\textbf{deduction}}$$

Birkhoff-von Neumann paradigm:

$$\frac{\textbf{Quantum logic}}{\textbf{Classical logic}} \simeq \frac{\textbf{NO deduction}}{\textbf{deduction}}$$

We are solving:

$$\frac{???}{\textbf{quantum theory}} \simeq \frac{\textbf{natural deduction}}{\textbf{truth tables}}$$

Physicist use Dirac notation, not Hilbert space axioms.

$\vert\psi\rangle$	$\langle\phi\vert$	$\langle\phi\vert\psi\rangle$	$\vert\psi\rangle\langle\psi\vert$
ket	*bra*	*bra-ket*	*projector*

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$|\psi\rangle$	$\langle\phi|$	$\langle\phi|\psi\rangle$	$|\psi\rangle\langle\psi|$
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Physicist's desire for pictures: Feynman, Penrose, ...

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graphical language for $\otimes$-categories:

$\otimes \sim$ *horizontal* $\quad \circ \sim$ *vertical*

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---

provable from categorical axioms

$\Longleftrightarrow$

derivable in graphical language

Physicist use Dirac notation, not Hilbert space axioms.

$\lvert\psi\rangle$	$\langle\phi\rvert$	$\langle\phi\vert\psi\rangle$	$\lvert\psi\rangle\langle\psi\rvert$
ket	*bra*	*bra-ket*	*projector*

Physicist's desire for pictures: Feynman, Penrose, ...

graphical language for $\otimes$-categories:

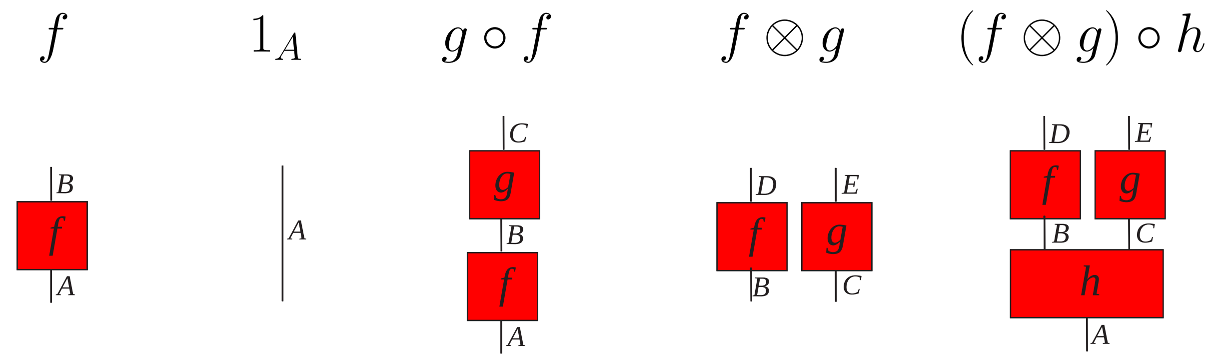
$\otimes \sim$ *horizontal* $\quad \circ \sim$ *vertical*

Dirac notation in two-dimensions

Categorical Quantum Axiomatics

BACKGROUND LANGUAGE

Penrose, Freyd-Yetter, Joyal-Street, Turaev, ...


f
1_A
$g \circ f$
$f \otimes g$
$(f \otimes g) \circ h$
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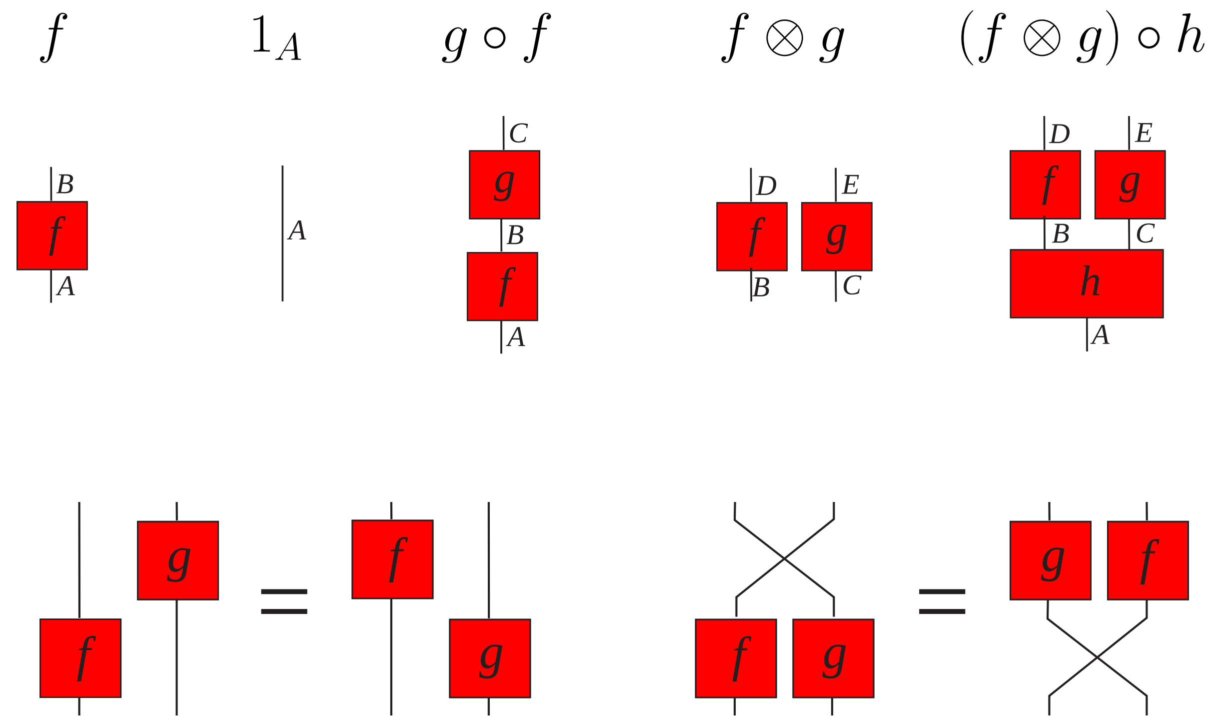

f
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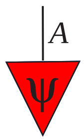
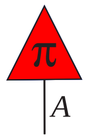
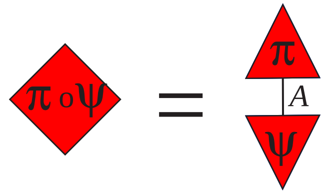
$\psi : \mathrm{I} \longrightarrow A$ $\qquad$ $\pi : A \longrightarrow \mathrm{I}$ $\qquad$ $\pi \circ \psi : \mathrm{I} \longrightarrow \mathrm{I}$







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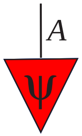

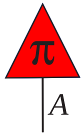

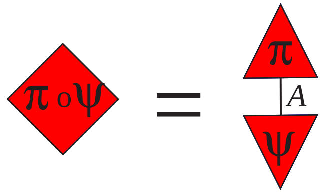

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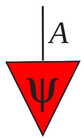

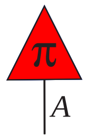

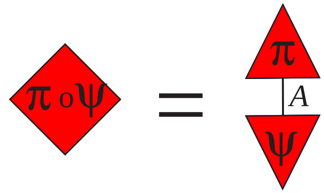

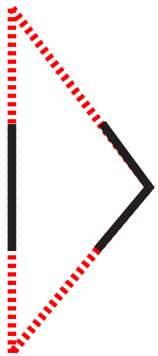

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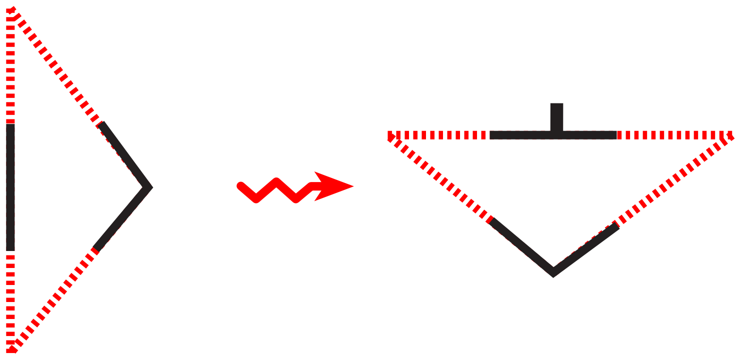

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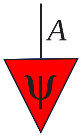

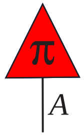

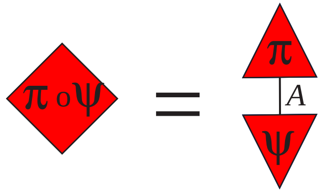

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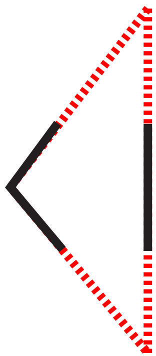

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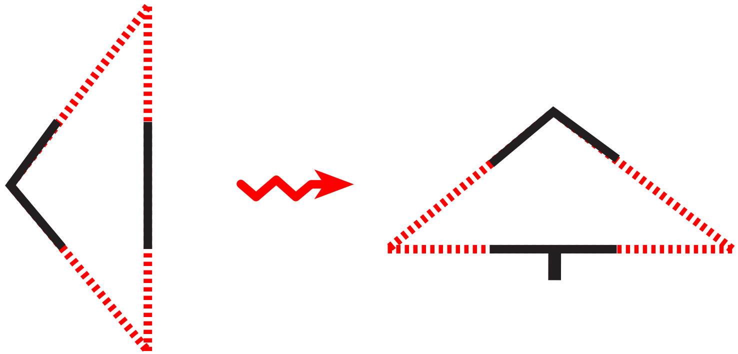

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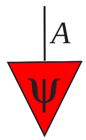

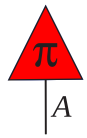

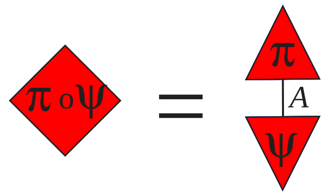

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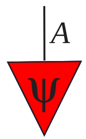

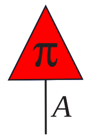

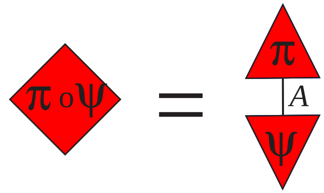

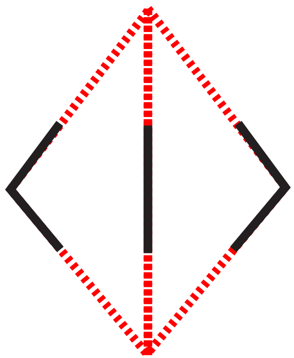

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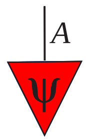
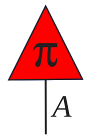
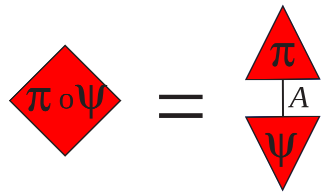
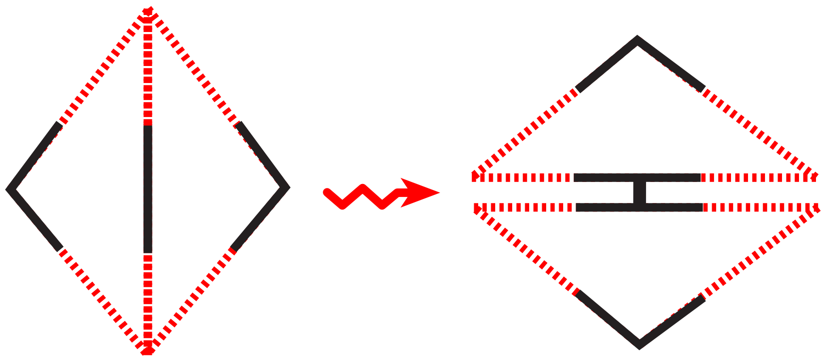








$$f : A \to B \qquad \longleftrightarrow \qquad f^\dagger : B \to A$$

Example model

Hilbert spaces

Linear maps

Composition of linear maps

Tensor product of Hilbert spaces and linear maps

Adjoint of linear maps

Example model

Hilbert spaces

Linear maps

Composition of linear maps

Tensor product of Hilbert spaces and linear maps

Adjoint of linear maps

Expressiveness

unitary, isometry, positivity, self-adjoint, projector

QUANTUM STRUCTURE

Abramsky-Coecke (2004) IEEE-LiCS

Kelly-Laplaza (1980) ***Coherence for compact closed categories*****.**

Selinger (2007) †-***Compact categories and CPMs*****.**

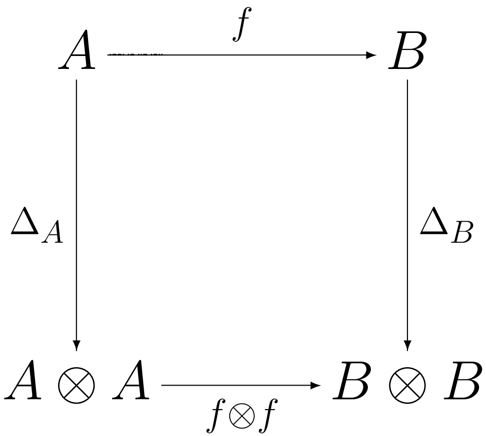
Natural diagonal ?

$$\{\Delta_A : A \longrightarrow A \otimes A\}_A$$

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ {\scriptstyle \Delta_A}\downarrow & & \downarrow{\scriptstyle \Delta_B} \\ A \otimes A & \xrightarrow[f \otimes f]{} & B \otimes B \end{array}$$

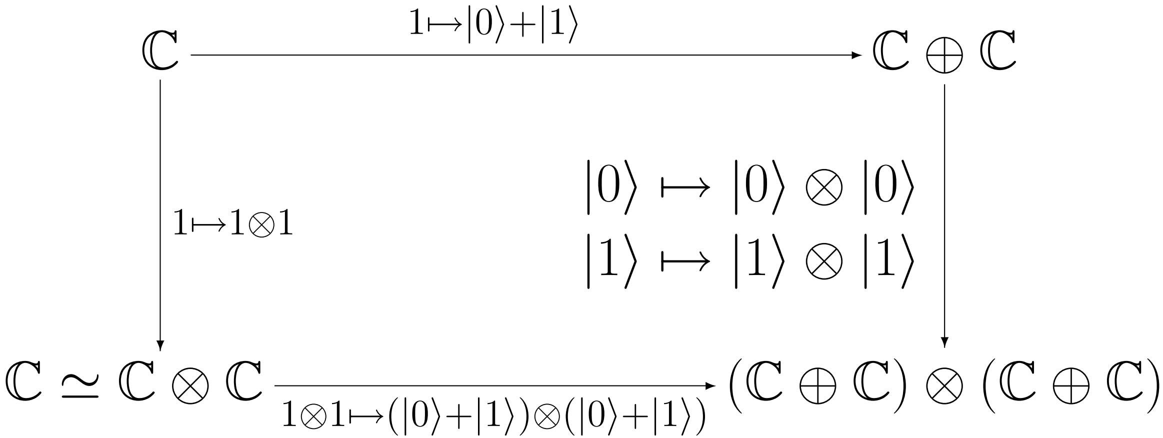
Cloning ?

$$\{\Delta_A : A \longrightarrow A \otimes A\}_A$$



No-cloning of quantum states

$$\{\Delta_{\mathcal{H}} : |\, i\, \rangle \mapsto |\, i\, \rangle \otimes |\, i\, \rangle\}_{\mathcal{H}}$$



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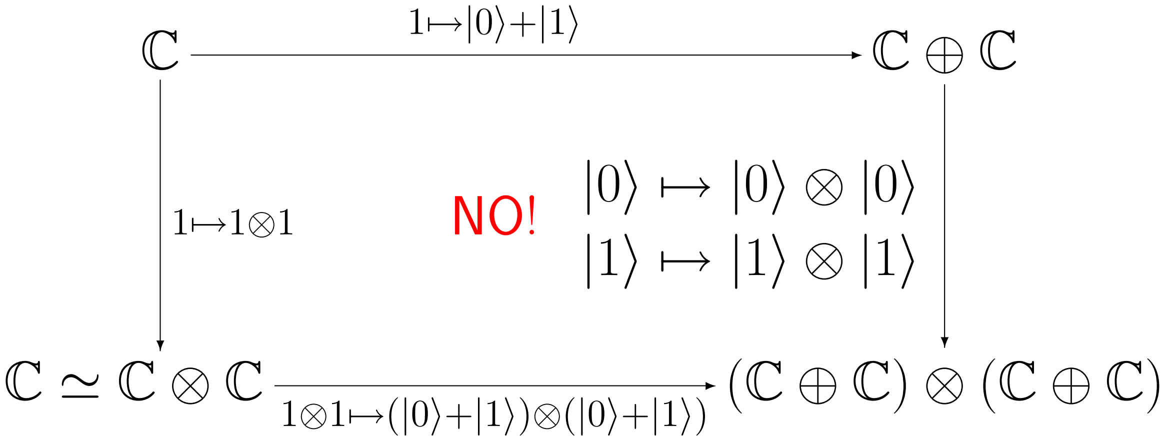


$$|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle \neq (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)$$

Bell-states cause trouble!

No-cloning in $(\mathbf{Rel}, \times)$

$$\{\Delta_X : x \mapsto (x, x)\}_X$$

$$\begin{array}{ccc}
\{*\} & \xrightarrow{\{(*,0),(*,1)\}} & \{0,1\} \\
\Big\downarrow {\scriptstyle \{(*,(*,*))\}} & \textbf{NO!} & \Big\downarrow {\scriptstyle \{(0,(0,0)),(1,(1,1))\}} \\
\{*\} \times \{*\} & \xrightarrow[\{(*,0),(*,1)\}\times\{(*,0),(*,1)\}]{} & \{0,1\} \times \{0,1\}
\end{array}$$

$$\{(0,0),(1,1)\} \neq \{0,1\} \times \{0,1\}$$

Object with quantum structure

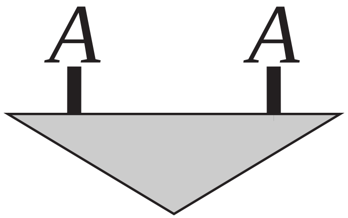
A pair

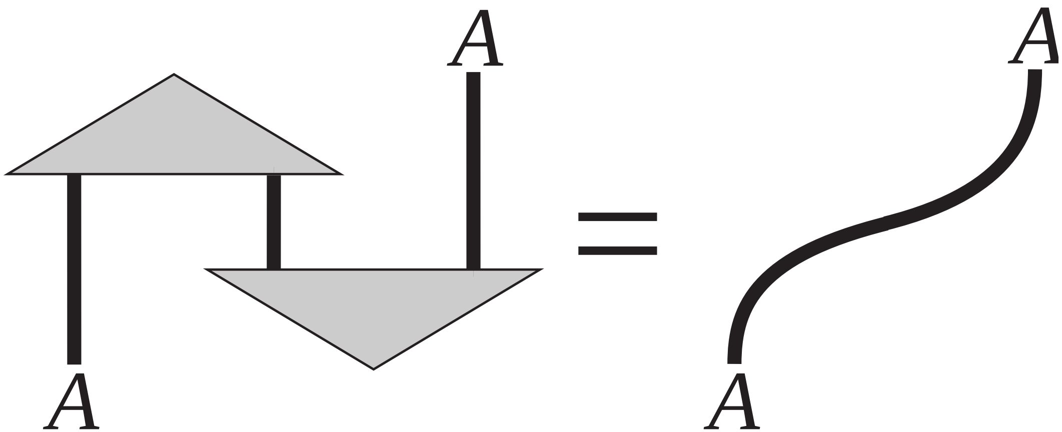
$$(A\,,\eta : \mathrm{I} \to A \otimes A)$$

such that:

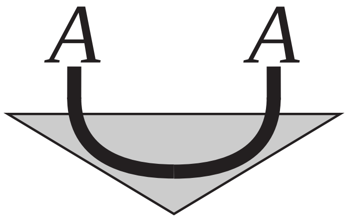
$$\begin{array}{ccccc}
A & \xleftarrow{\;\simeq\;} & \mathrm{I} \otimes A & \xleftarrow{\;\eta^{\dagger} \otimes 1_A\;} & (A \otimes A) \otimes A \\
{\scriptstyle 1_A}\uparrow & & & & \uparrow{\scriptstyle \simeq} \\
A & \xrightarrow[\;\simeq\;]{} & A \otimes \mathrm{I} & \xrightarrow[\;1_A \otimes \eta\;]{} & A \otimes (A \otimes A)
\end{array}$$

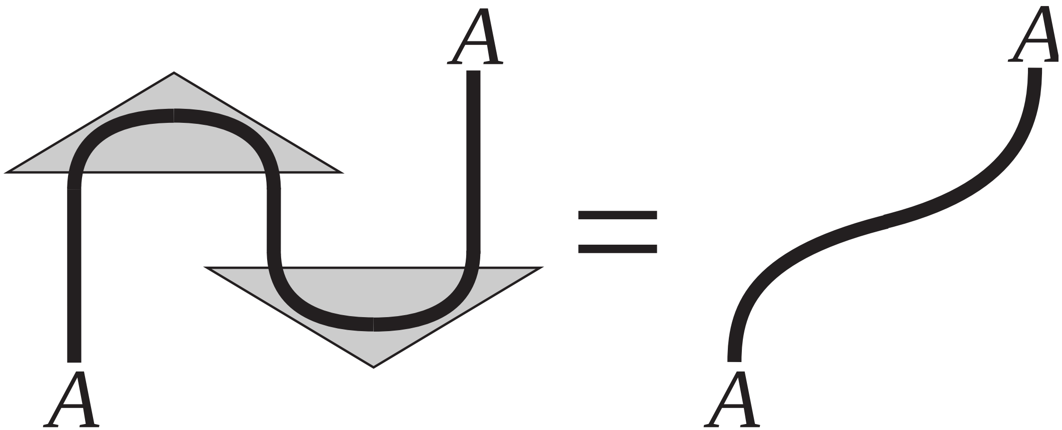
Object with quantum structure



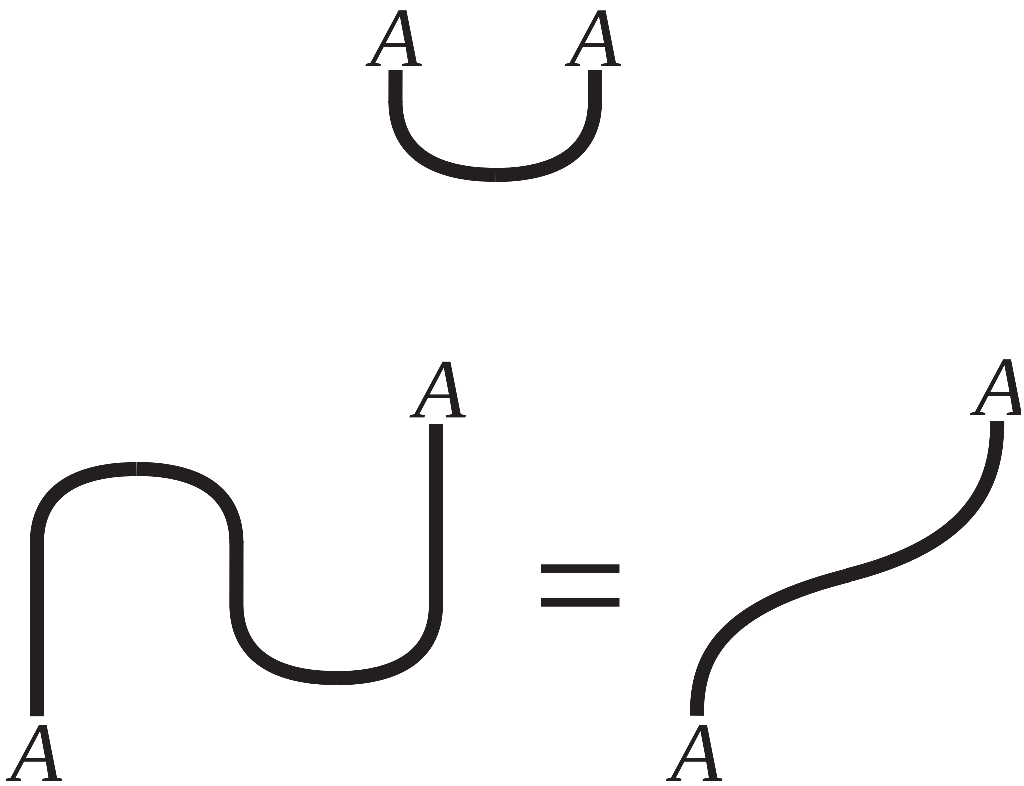


Object with quantum structure

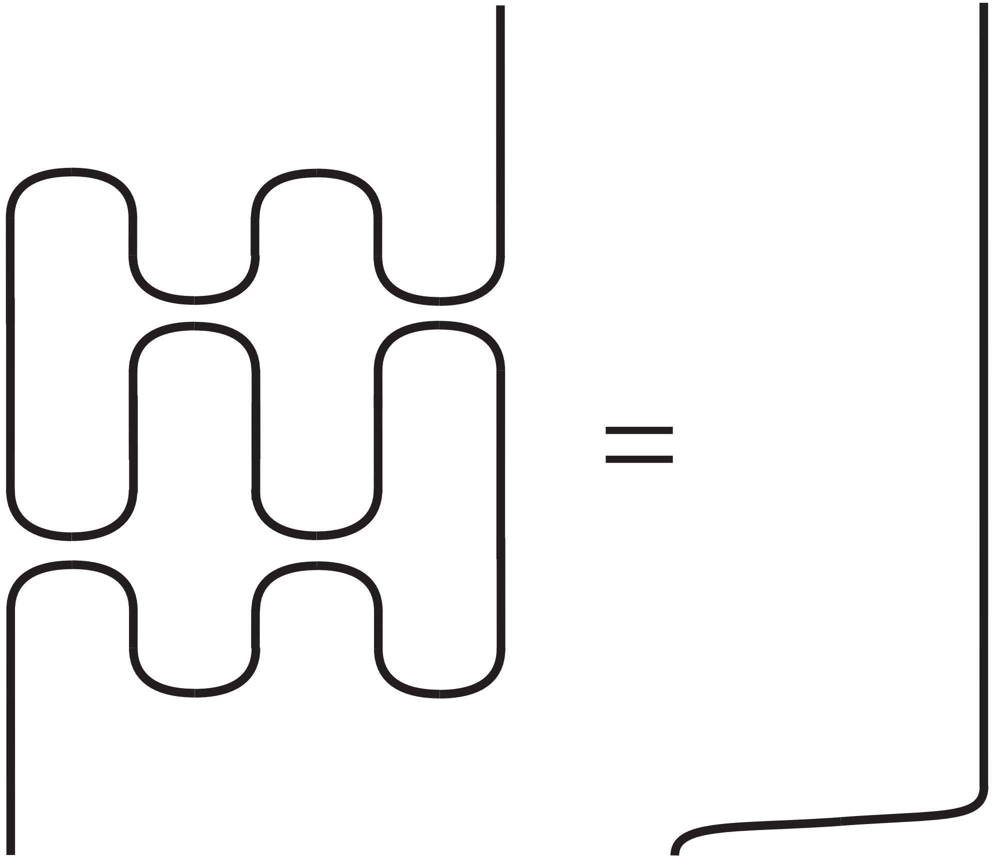




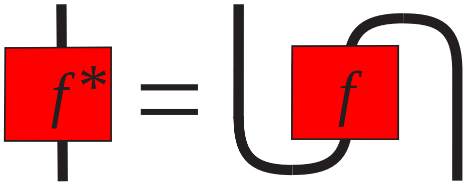
Object with quantum structure



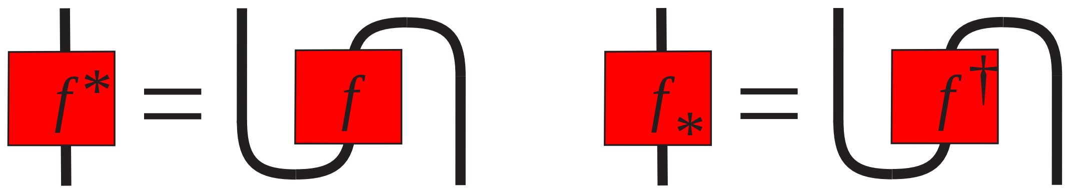
“Clean” normalization



Another contravariant involution

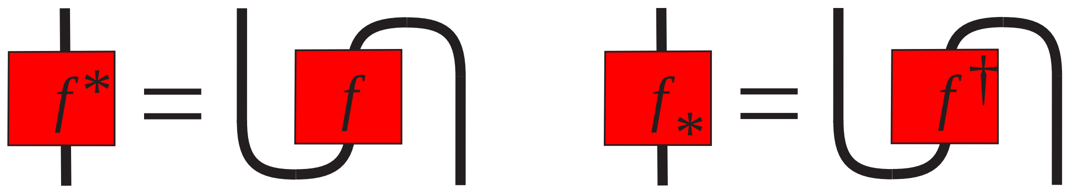


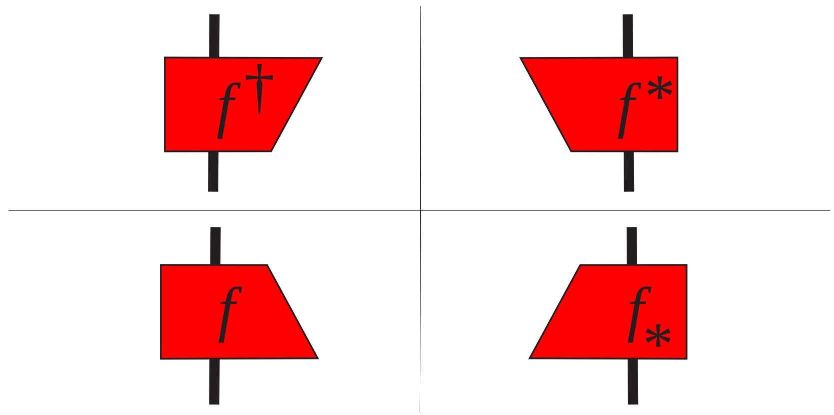
Another covariant involution



$$f_* = (f^\dagger)^* = (f^*)^\dagger$$

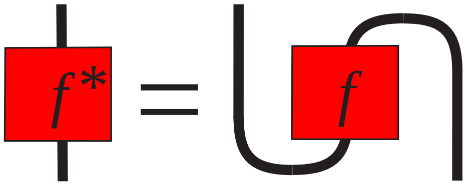
Three intertwined involutions





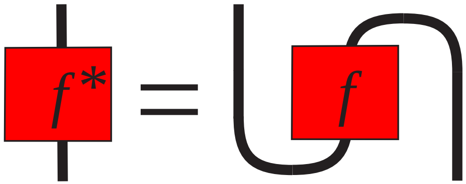
$$f_* = (f^\dagger)^* = (f^*)^\dagger \quad \Rightarrow \quad f^* = (f^\dagger)_* = (f_*)^\dagger$$

Three intertwined involutions



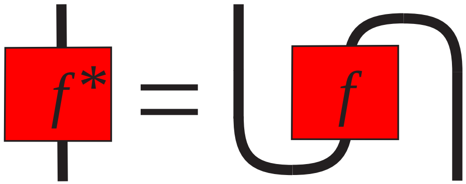
$f^* \sim *$-autonomy

Three intertwined involutions



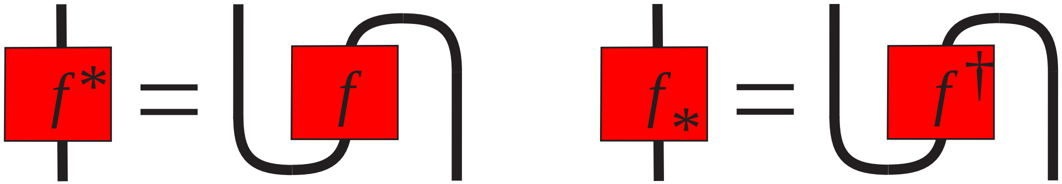
$f^* \sim *$-autonomy with $(A \otimes B)^* \simeq A^* \otimes B^*$

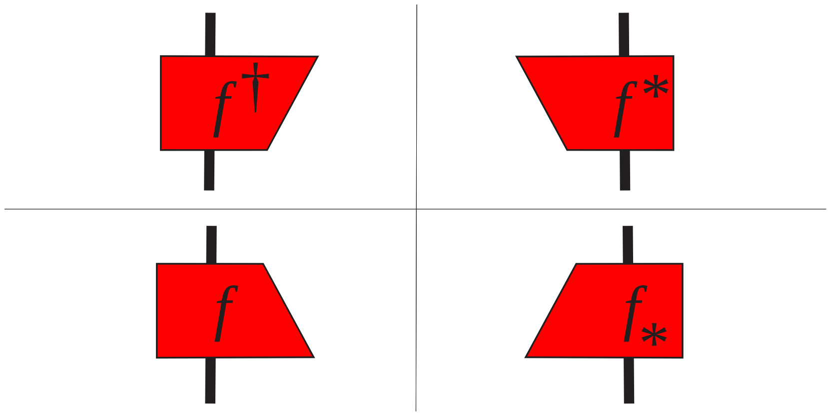
Three intertwined involutions



$f^* \sim$ Max Kelly's compact closure

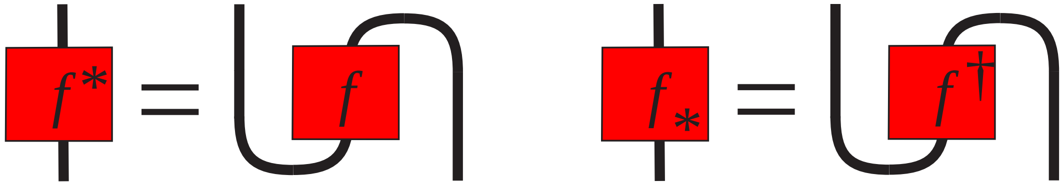
Three intertwined involutions

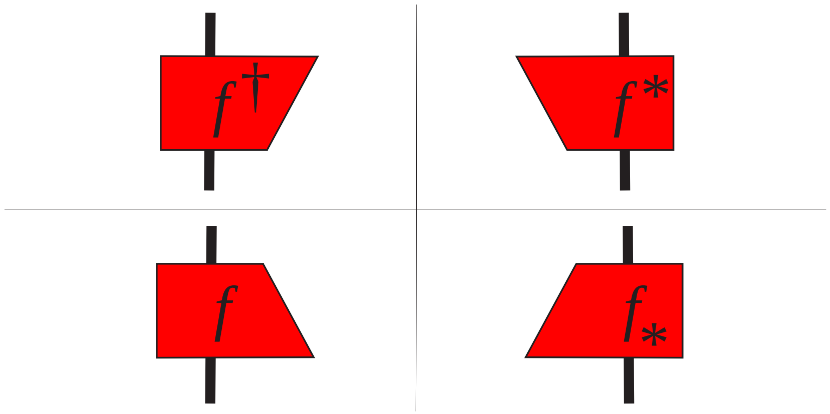




$$(f_*)^* = (f^*)_* = f^\dagger$$

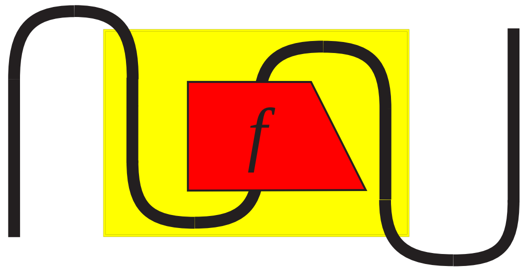
Three intertwined involutions



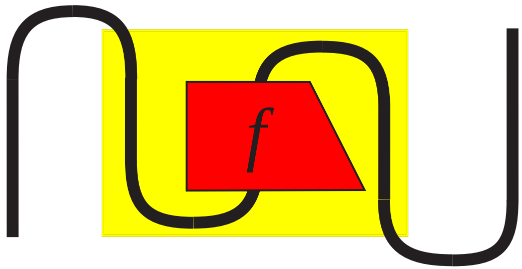


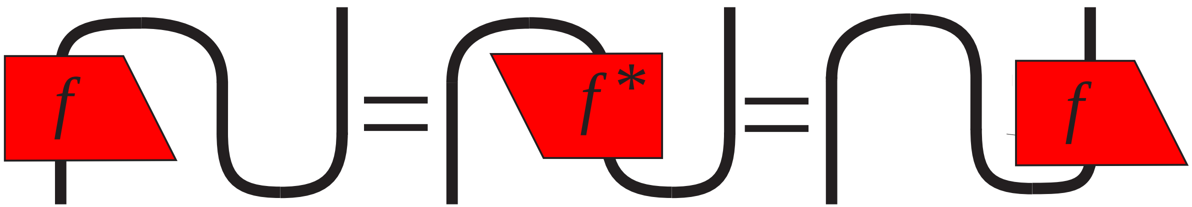
In Hilb: $f^* \sim$ transposed & $f_* \sim$ conjugated

“Sliding” boxes

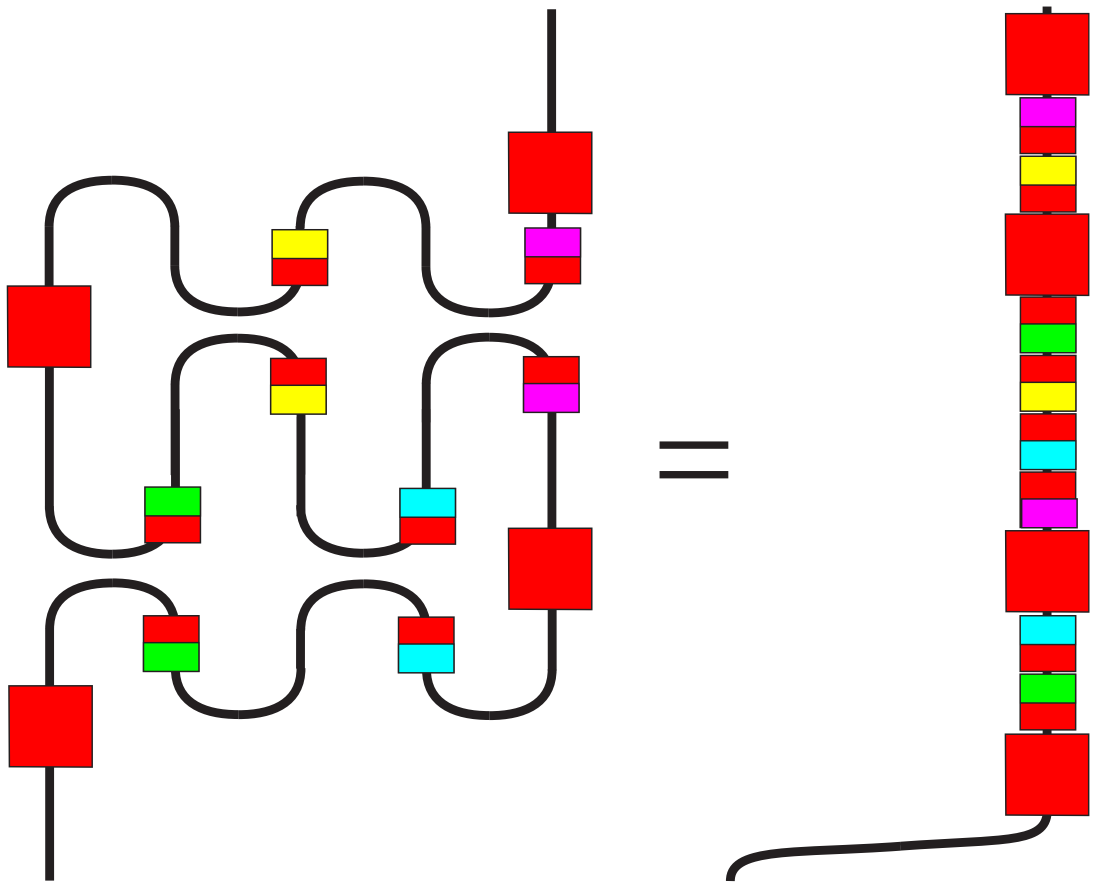


“Sliding” boxes

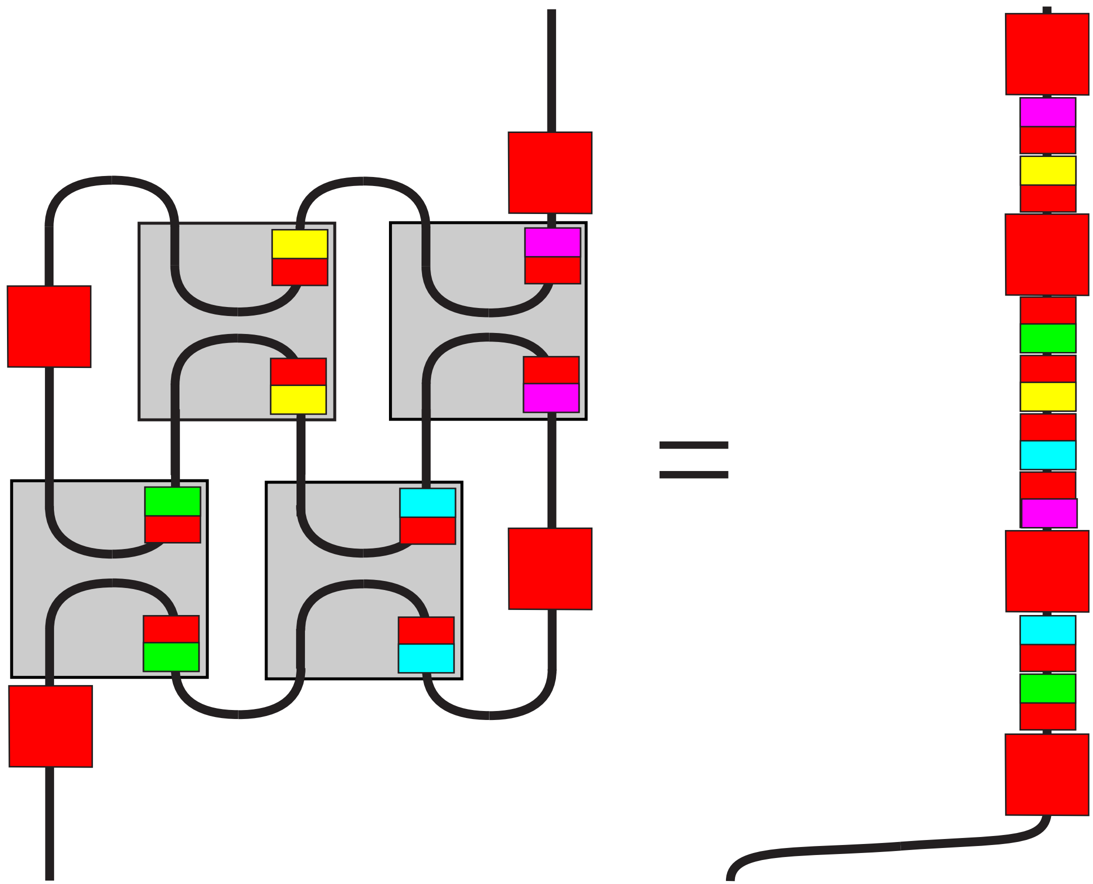




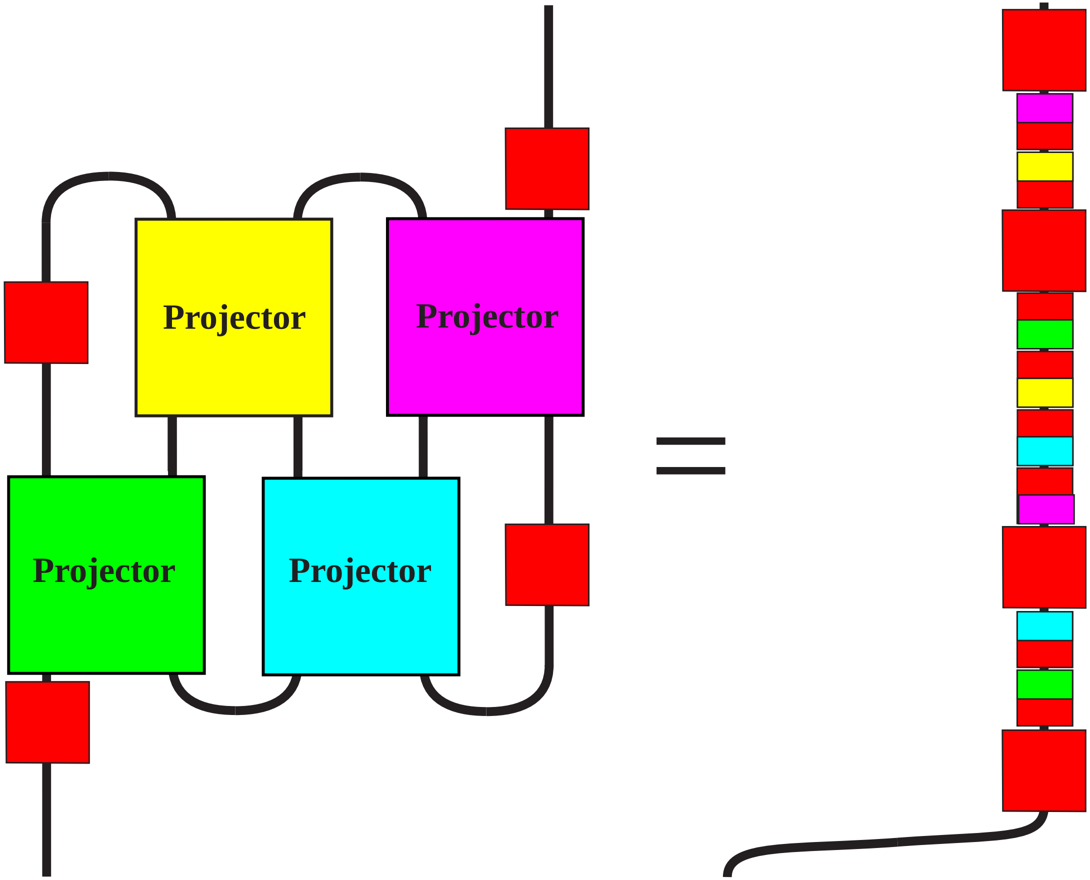
“Decorated” normalization



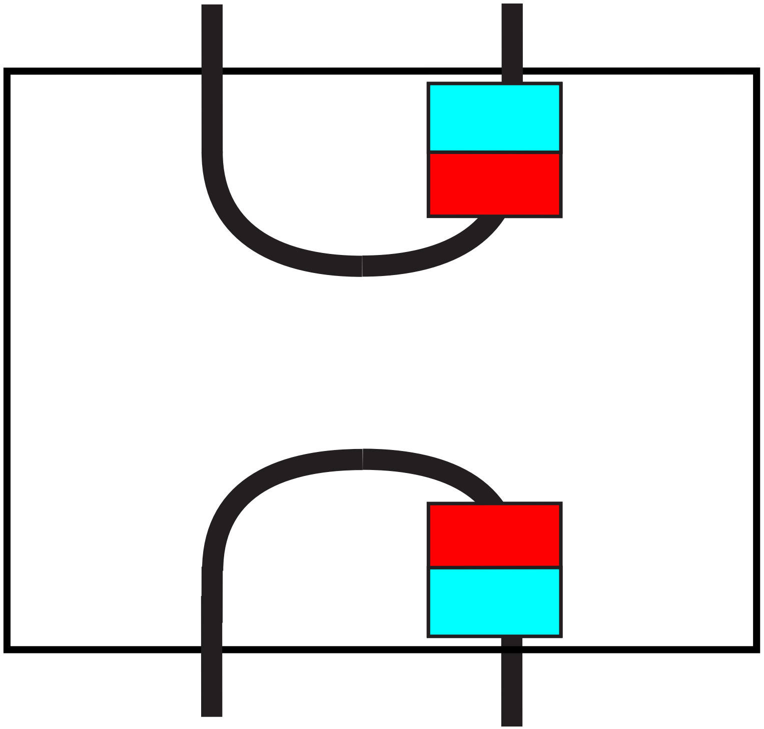
“Decorated” normalization



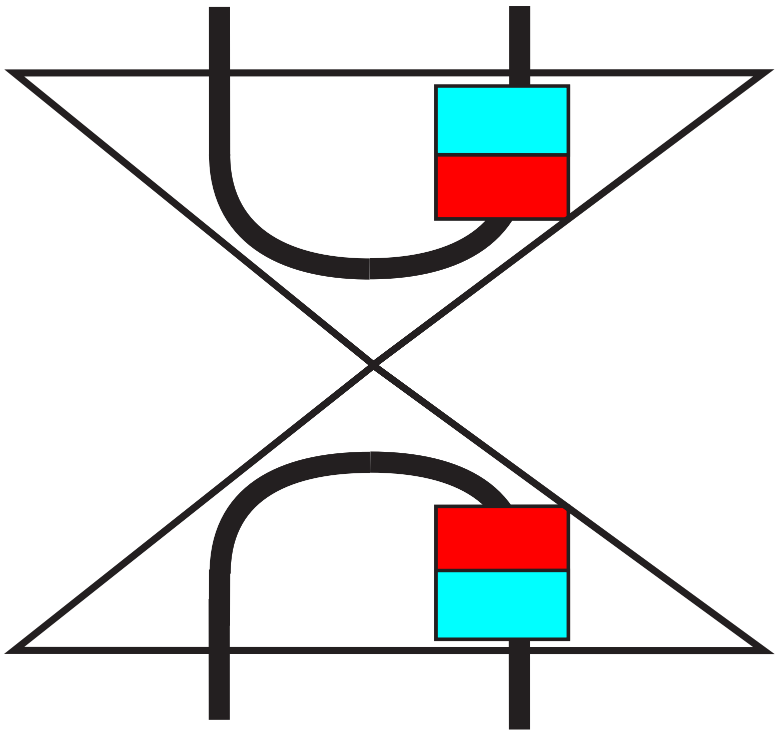
"Decorated" normalization



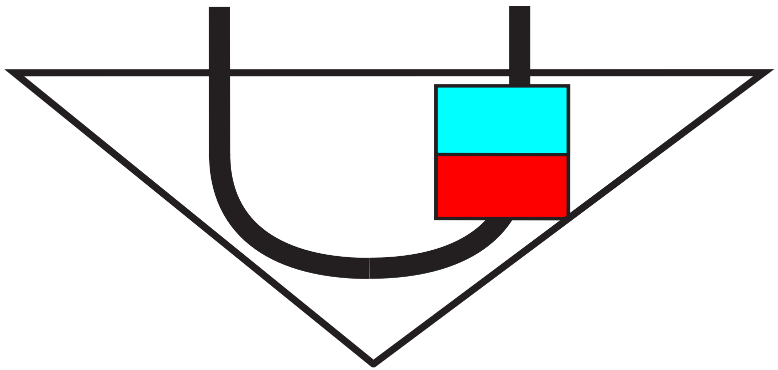
Bipartite projector



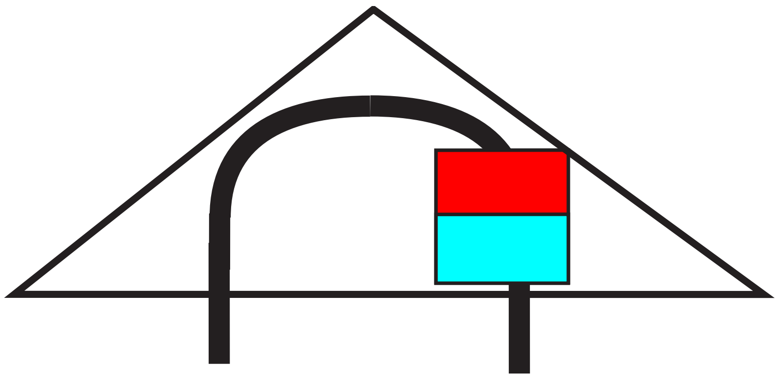
Bipartite projector



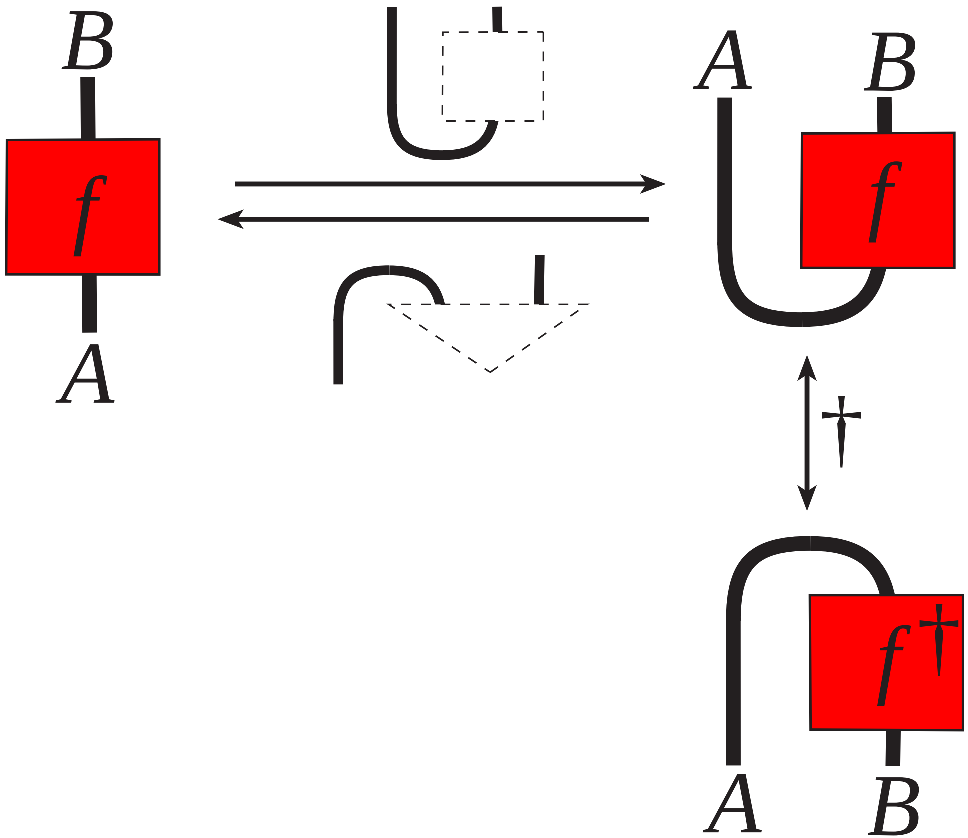
Bipartite state

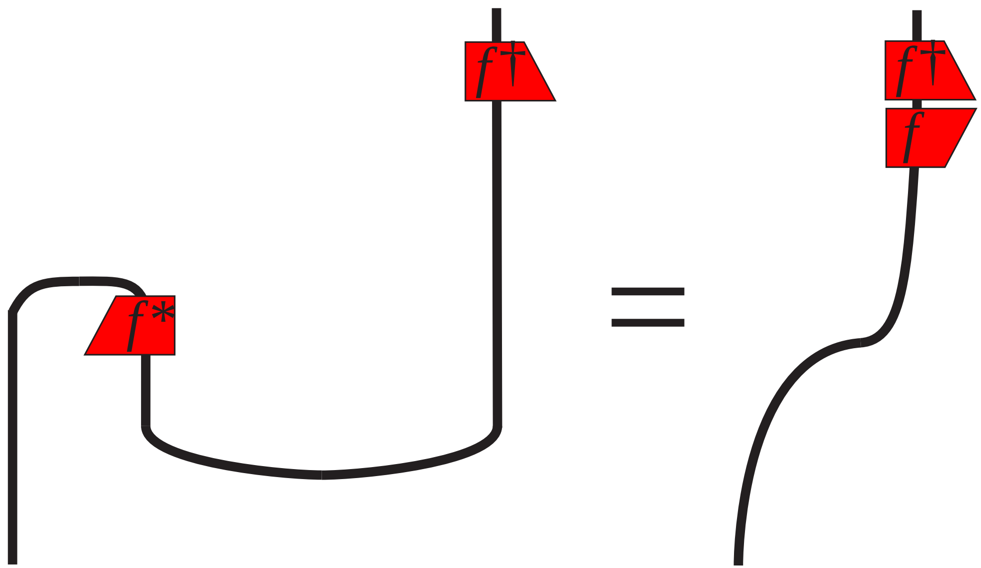


Bipartite costate

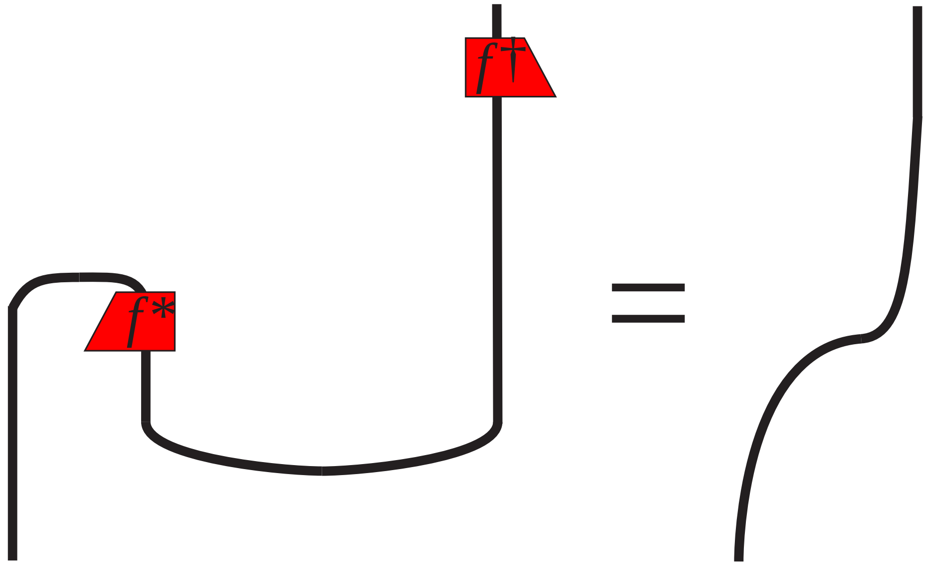


Bipartite (co)states & closedness

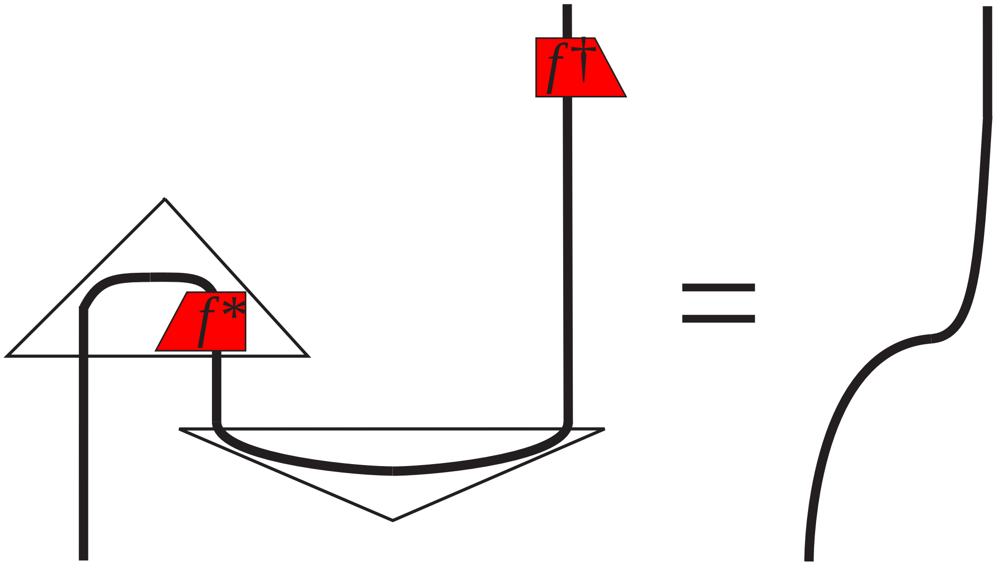


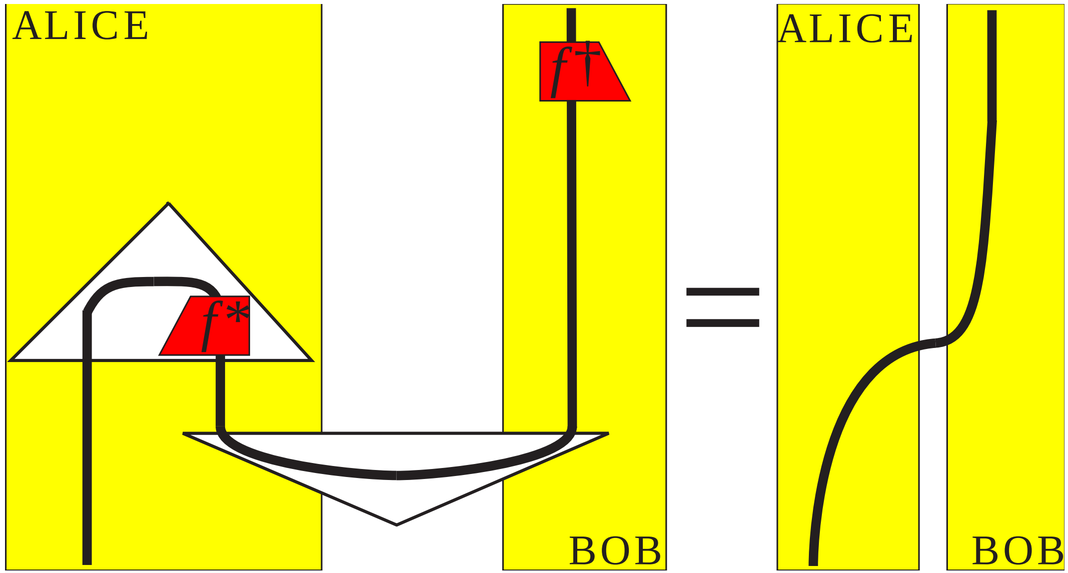


f†
f*
=
f†
f



$f^{\dagger}$
f^{*}





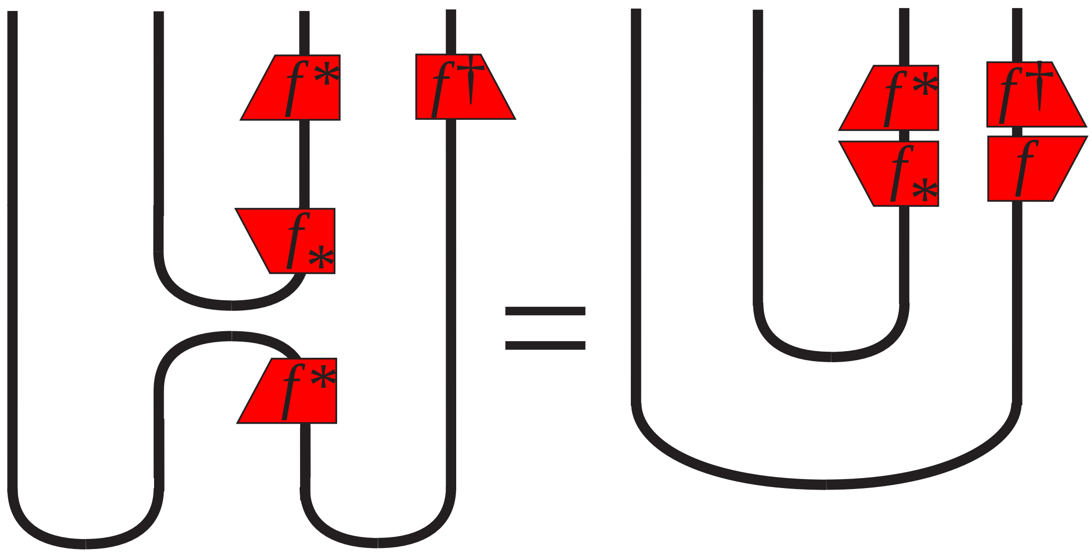
$\Rightarrow$ **Quantum teleportation**

The corresponding TEXTBOOK description (only!)

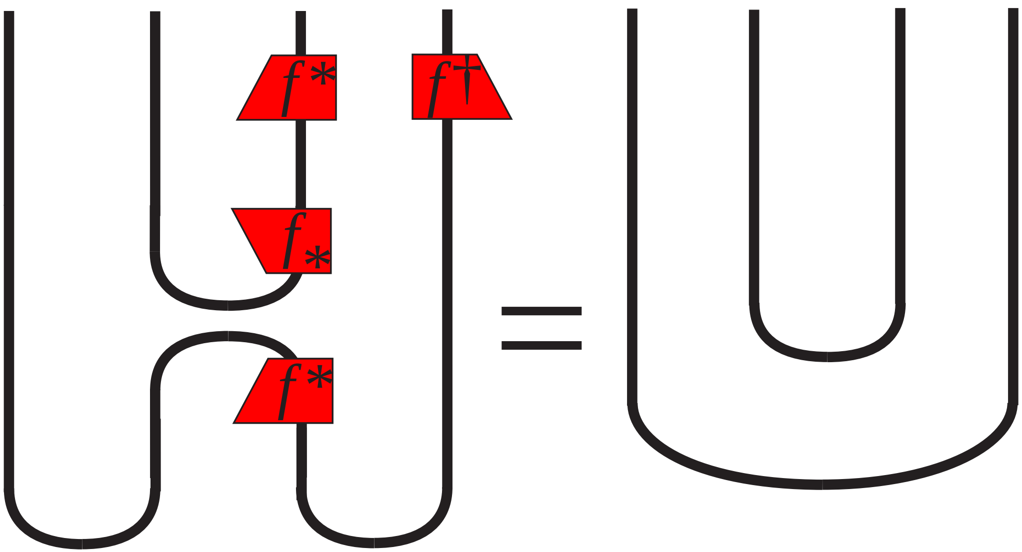
Alice has an 'unknown' qubit $|\phi\rangle$ and wants to send it to Bob. They have the ability to communicate classical bits, and they share an entangled pair in the EPR-state, that is $\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$, which Alice produced by first applying a Hadamard-gate $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ to the first qubit of a qubit pair in the ground state $|00\rangle$, and by then applying a CNOT-gate, that is $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, then she sends the first qubit of the pair to Bob. To teleport her qubit, Alice first performs a bipartite measurement on the unknown qubit and her half of the entangled pair in the Bell-base, that is

$$\{|0x\rangle + (-1)^z \mid 1(1-x)\rangle \mid x, z \in \{0,1\}\},$$

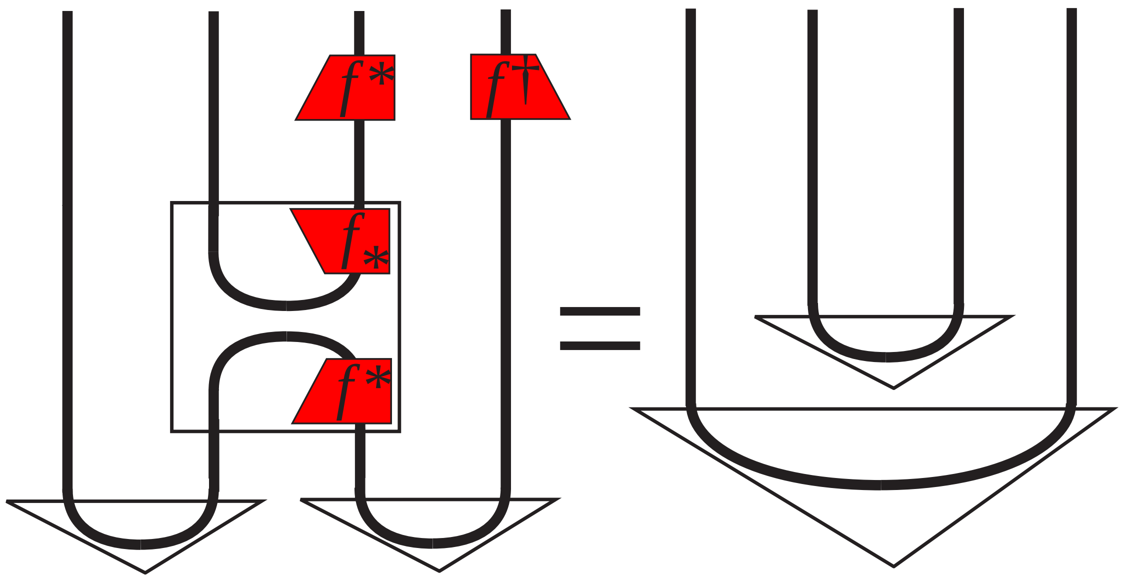
where we denote the four possible outcomes of the measurement by xz. Then **she sends the 2-bit outcome xz to Bob using the classical channel**. Then, if $x = 1$, Bob performs the unitary operation $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on its half of the shared entangled pair, and he also performs a unitary operation $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on it if $z = 1$. Now Bob's half of the initially entangled pair is in state $|\phi\rangle$.



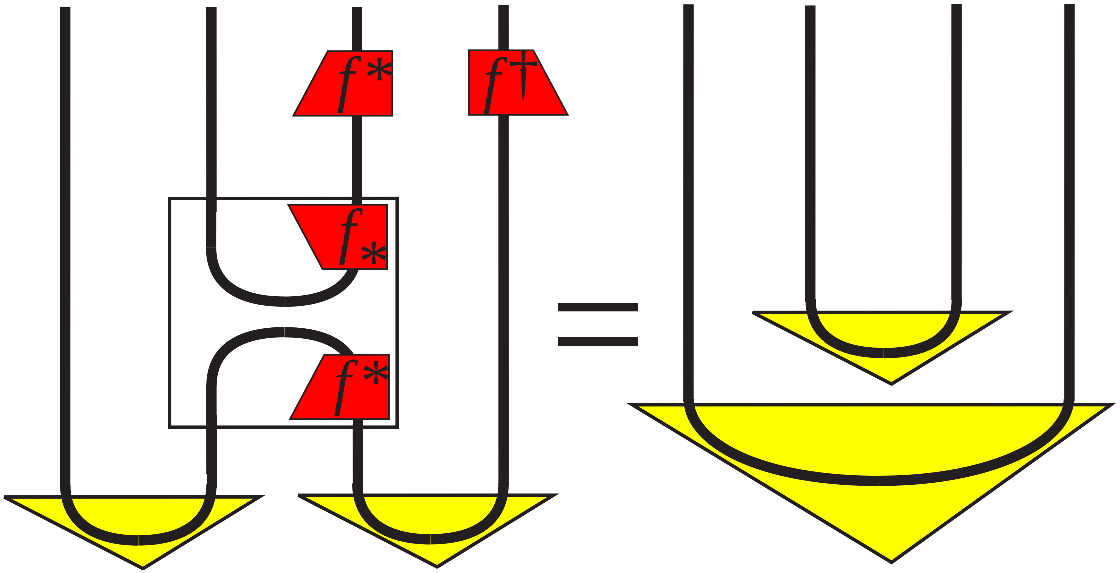
f^*
$f^\dagger$
f_*
f^*
=
f^*
$f^\dagger$
f_*
f



f^*
$f^\dagger$
f_*
f^*
=

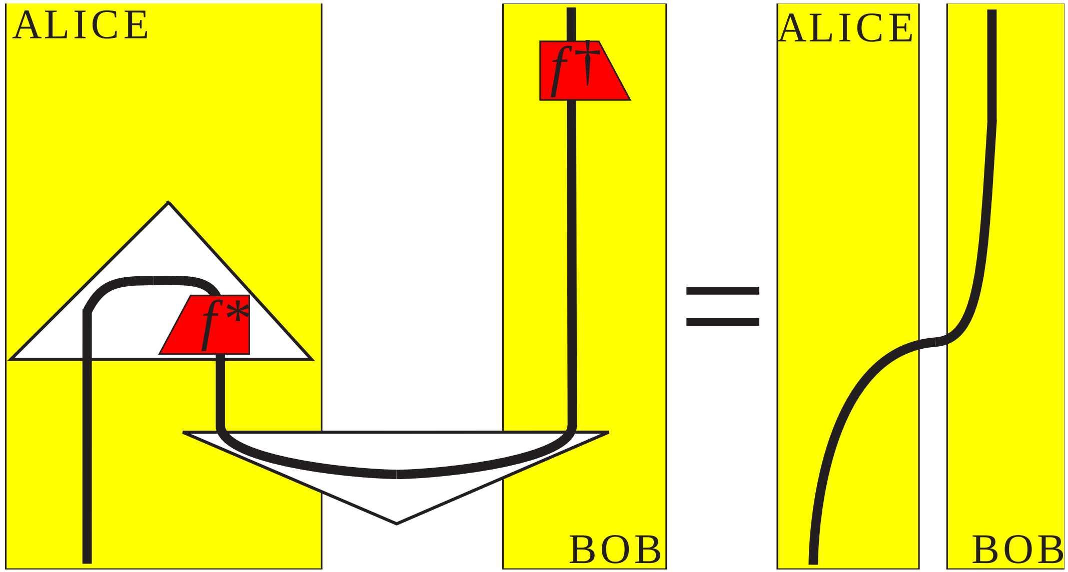


f^*
$f^\dagger$
f_*
f^*
=

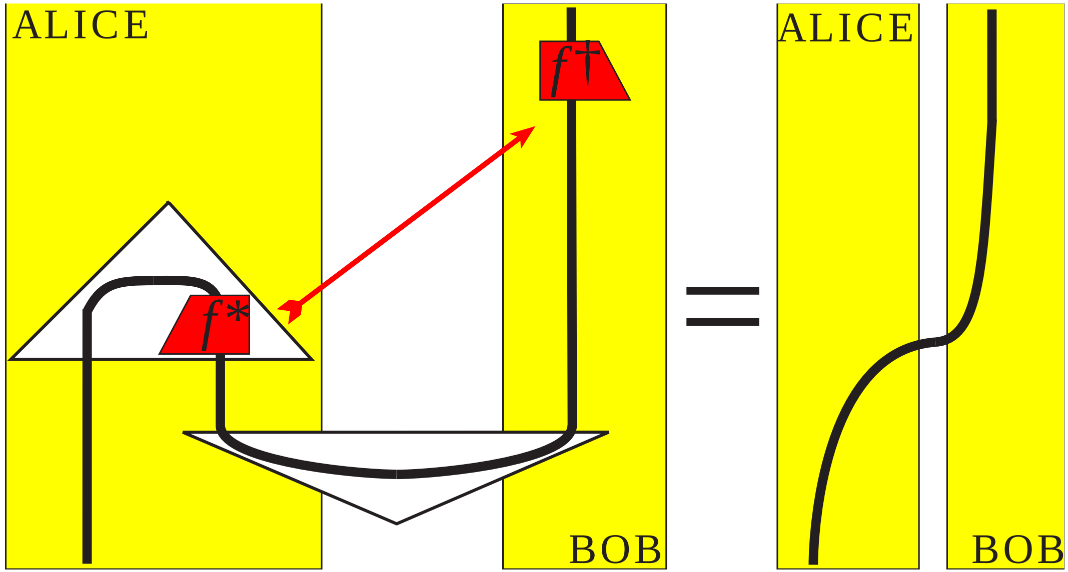


$\Rightarrow$ **Entanglement swapping**

Classical data flow?



Classical data flow?



CLASSICAL STRUCTURE

Coecke-Pavlovic (2006) quant-ph/0608035v1

Carboni-Walters (1986) *Cartesian bicategories* I.

quantum data cannot be cloned nor deleted

quantum data cannot be cloned nor deleted

classical data CAN be cloned and deleted

NON-FEATURE:

quantum data cannot be cloned nor deleted

FEATURE:

classical data CAN be cloned and deleted

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FEATURE:

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Classical data comes with cloning and deleting:

$$(X\,,\delta : X \to X \otimes X\,,\epsilon : X \to \mathrm{I})$$

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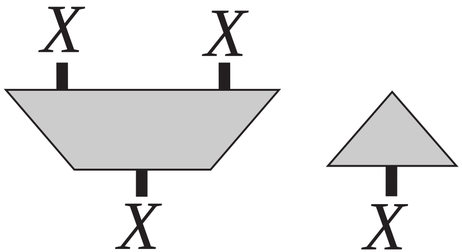
Classical data comes with cloning and deleting:

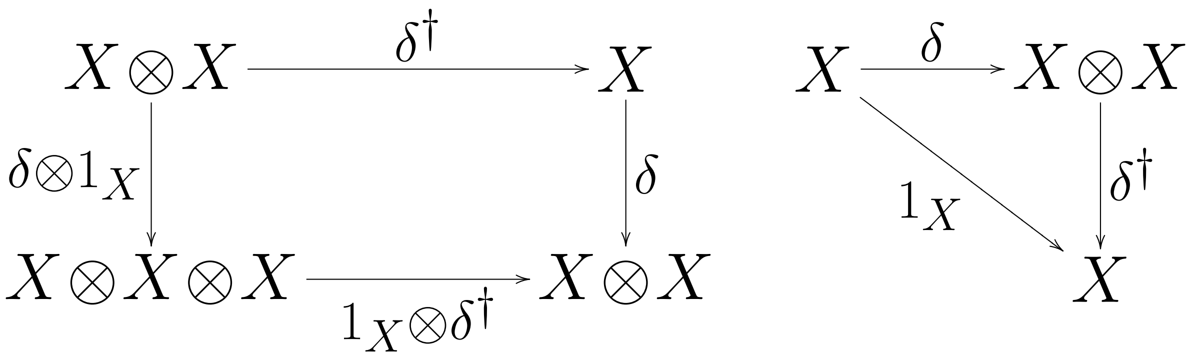


Object with classical structure

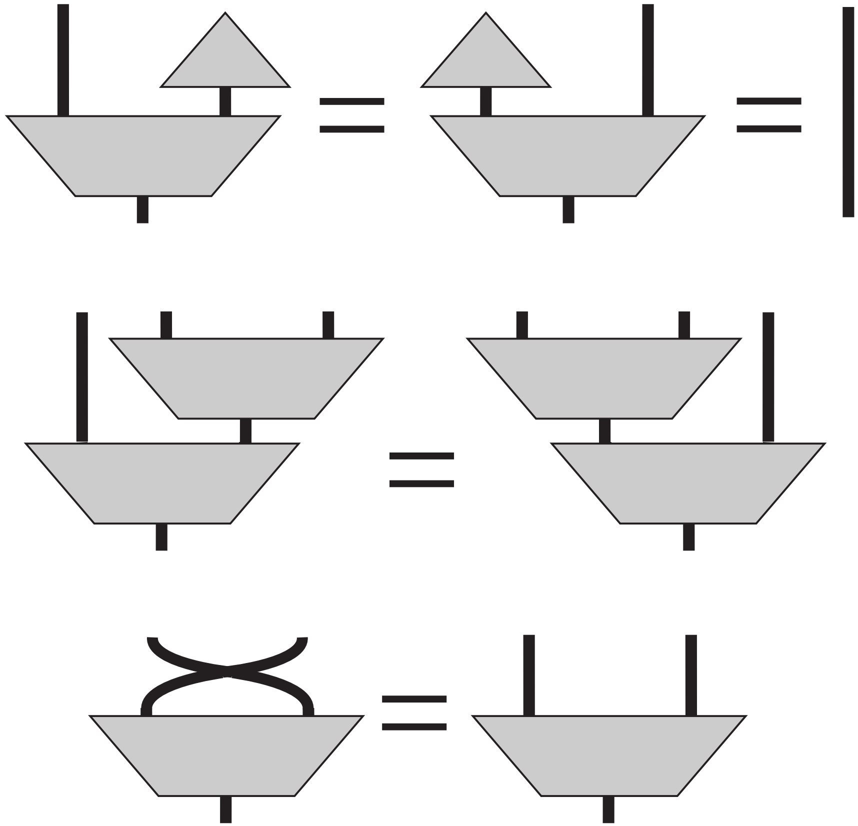
A commutative comonoid

$$(X\,,\delta : X \to X \otimes X\,,\epsilon : X \to \mathrm{I})$$

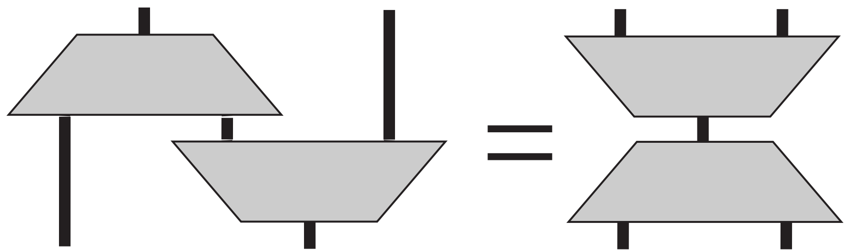
such that



Object with classical structure

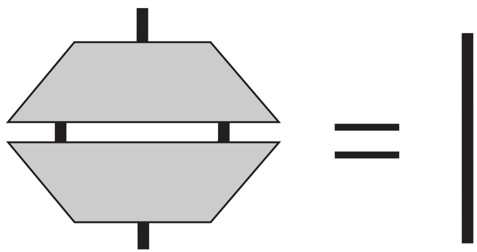


Object with classical structure

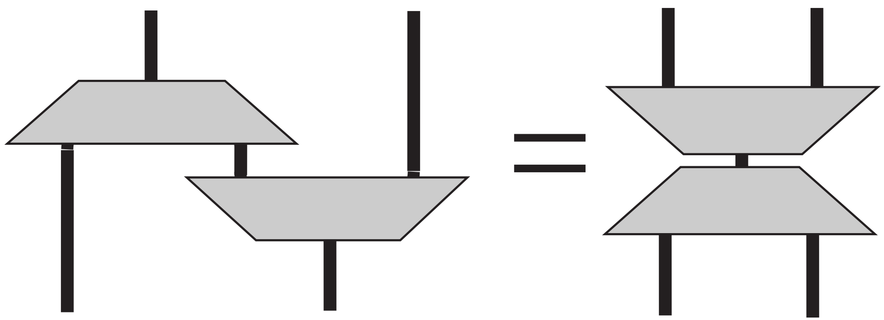


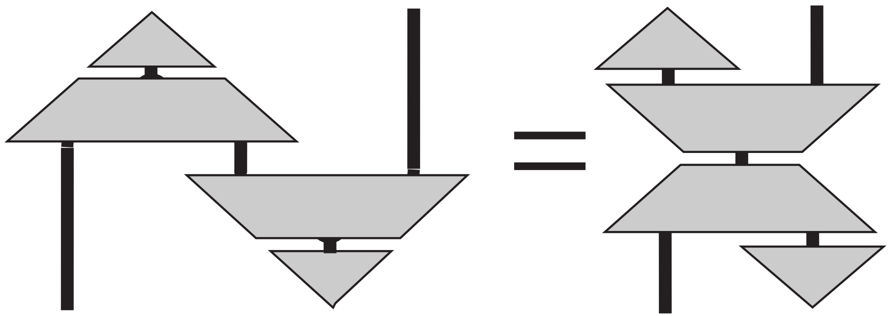
"Frobenius"

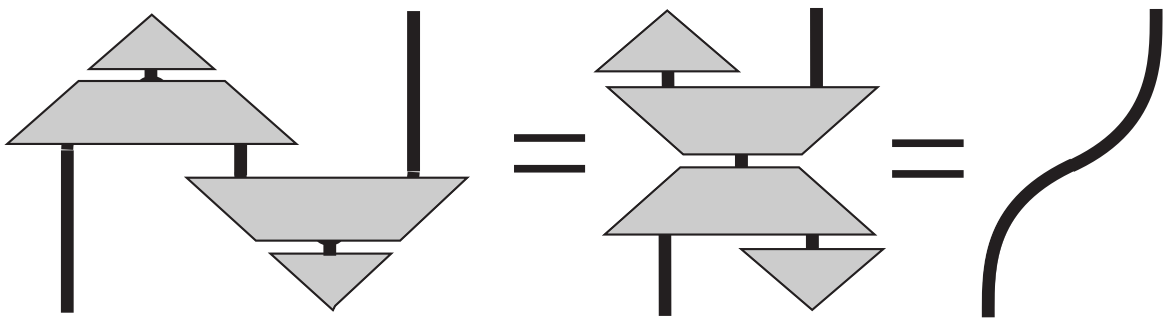
(Carboni-Walters 1987 *Cartesian bicategories* I)



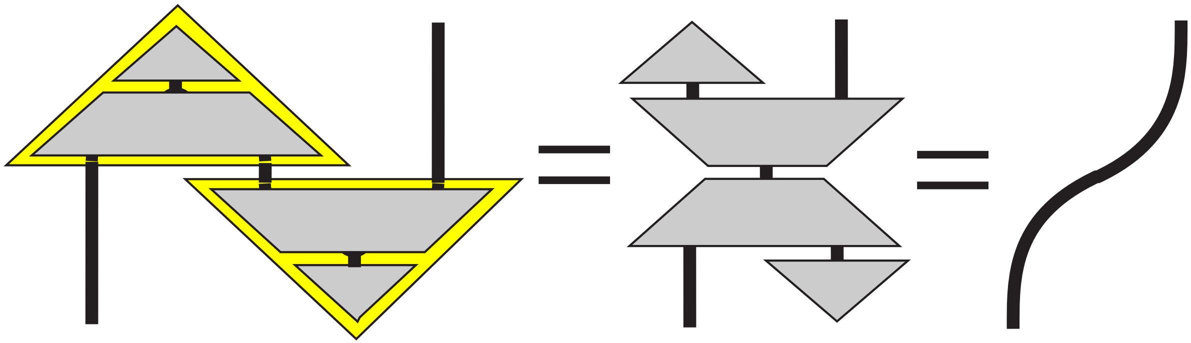
"unitarity"



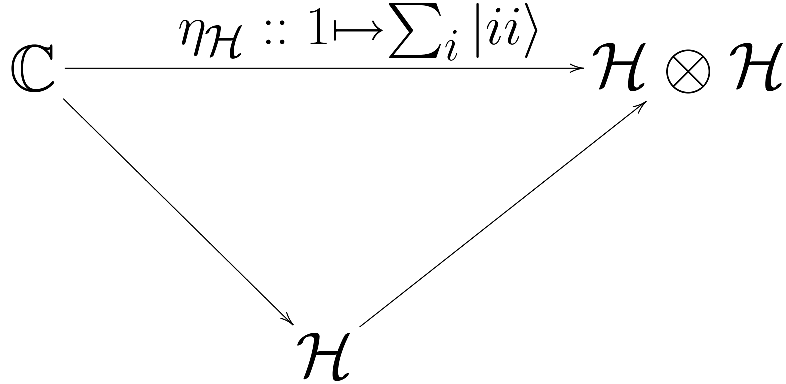




Classical structure ⇒ quantum structure



In FdHilb we have commutation of:



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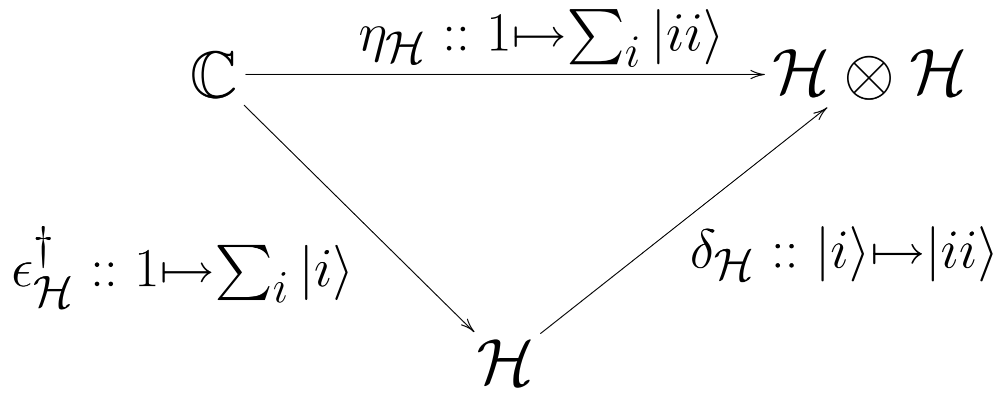

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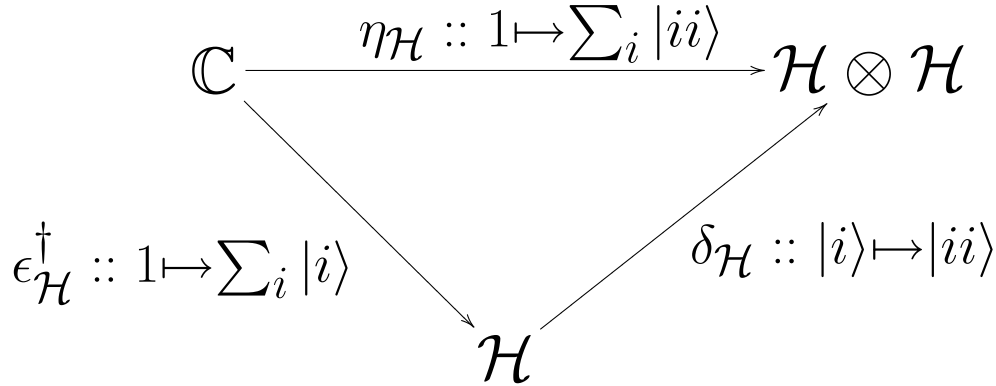

The only states $|\psi\rangle$ which are such that

$$\delta_{\mathcal{H}} \circ |\psi\rangle = |\psi\rangle \otimes |\psi\rangle$$

are the base vectors $\{|i\rangle\}_i$.

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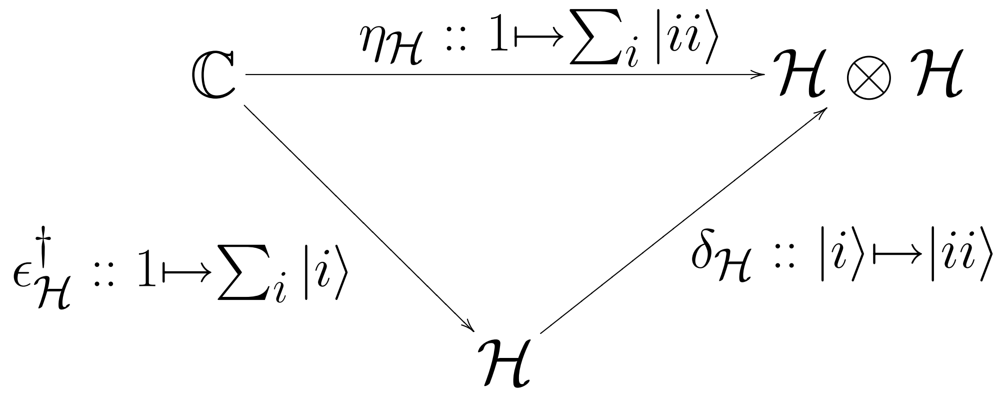
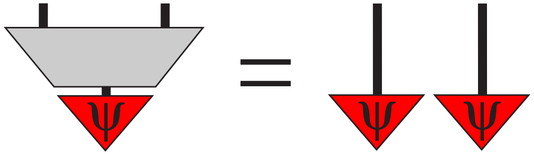


The only states $|\psi\rangle$ which are such that

$$\delta_{\mathcal{H}} \circ |\psi\rangle = |\psi\rangle \otimes |\psi\rangle$$

are the base vectors $\{|i\rangle\}_i \Rightarrow \delta_{\mathcal{H}}$ is base capturing!

An element $\psi : \mathrm{I} \to X$ is a *base vector* iff:



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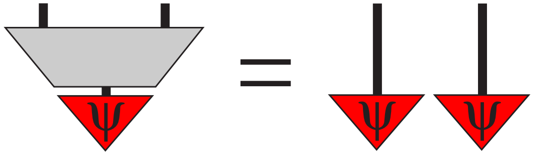

A set of elements $\{\psi_i : \mathrm{I} \to X\}_i$ is *orthonormal* iff $\langle\psi_i|\psi_j\rangle = \psi_i^\dagger \circ \psi_j$ is idempotent for all i, j.

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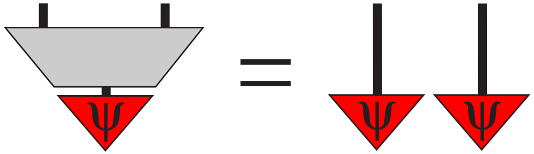

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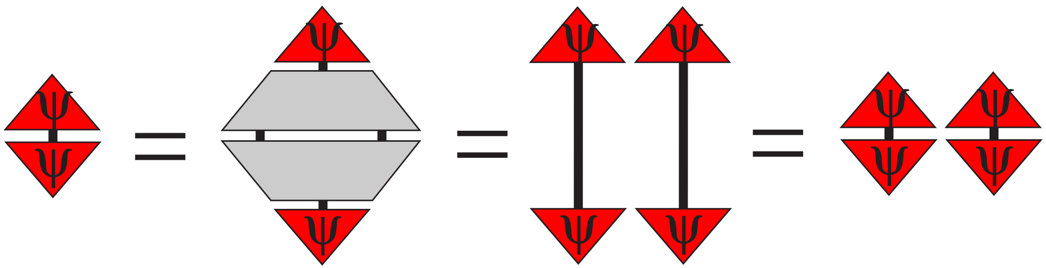
The base vectors constitute an orthonormal set:



“What’s inside the box?”

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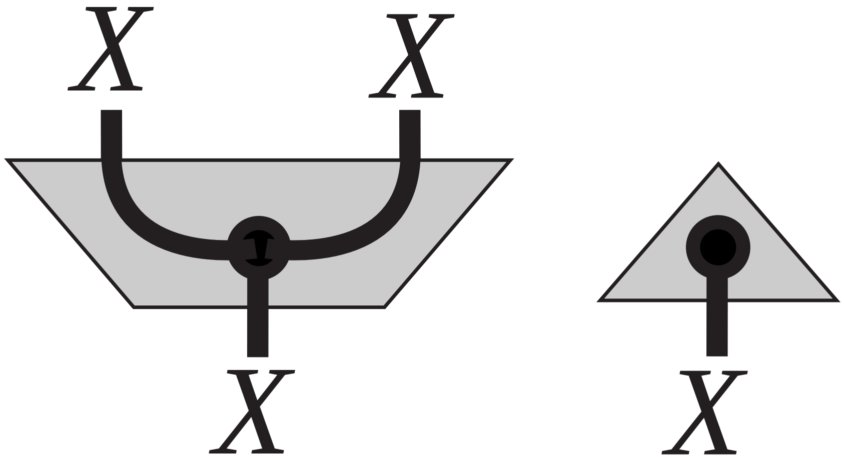
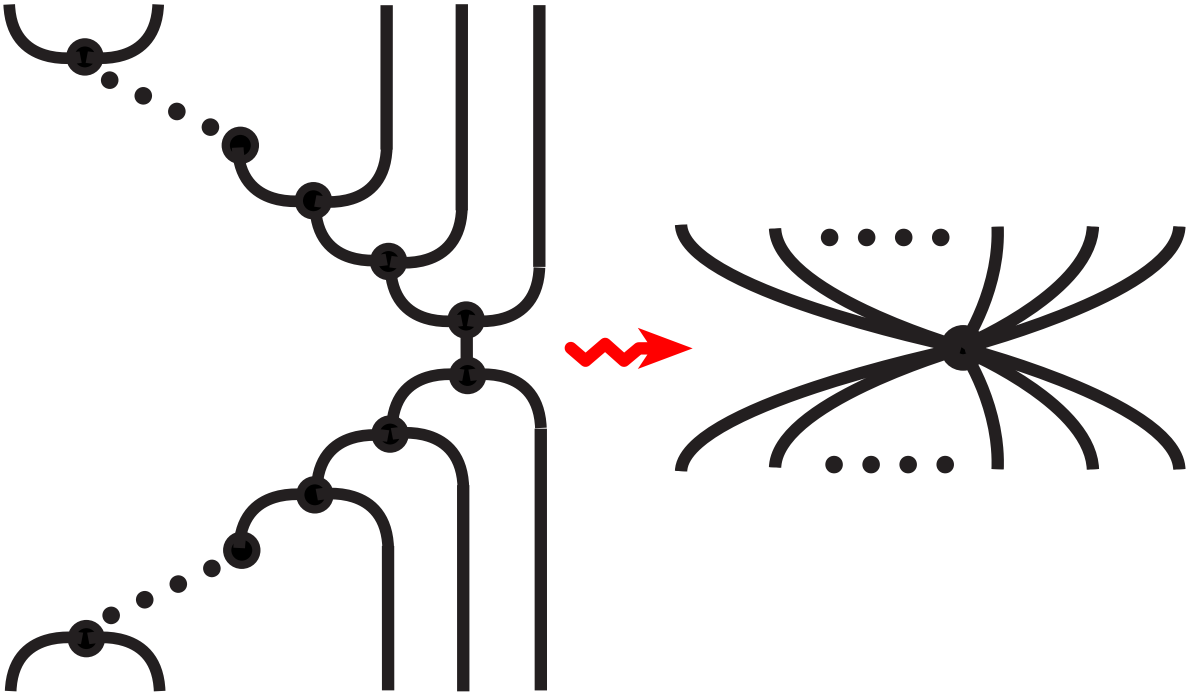
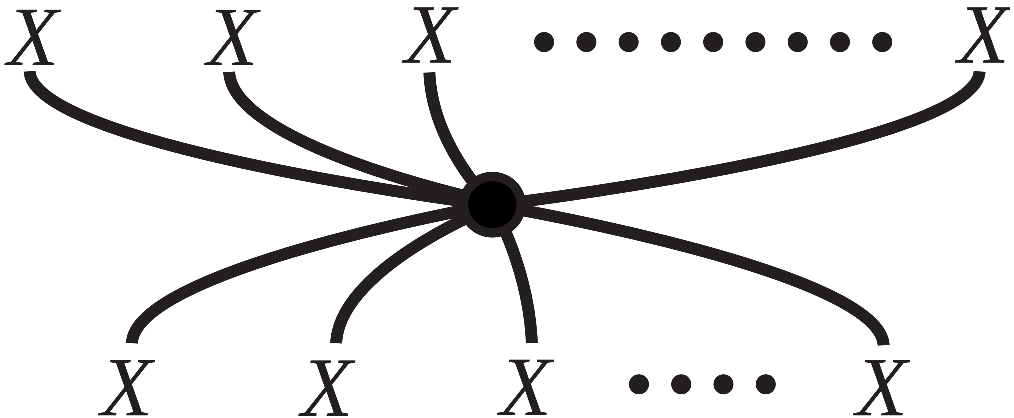


Notational convention:



Normalisation theorem: A "connected" network build from δ, $\delta^\dagger$, ϵ, $\epsilon^\dagger$ admits a 'spider-like' normal form:



Kock, J. (2003) ***Frobenius algebras and 2D TQFTs*****.**
Coecke-Paquette (2006) ***POVMs & Naimark's thm without sums*****.**

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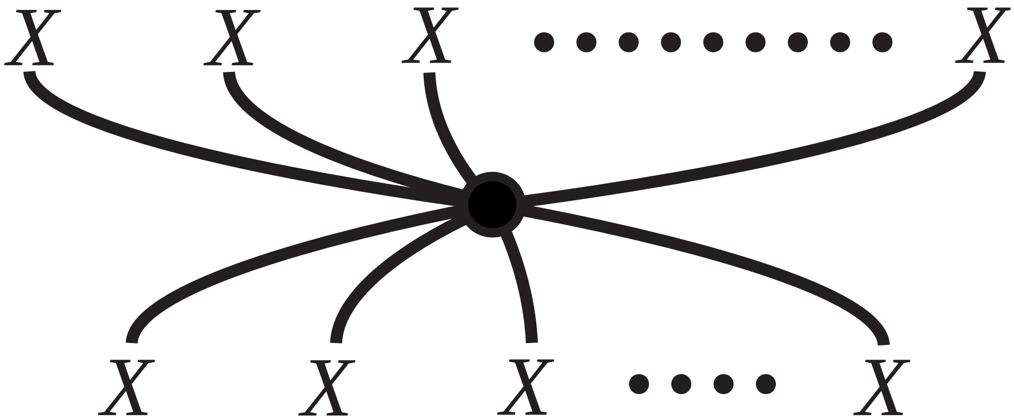
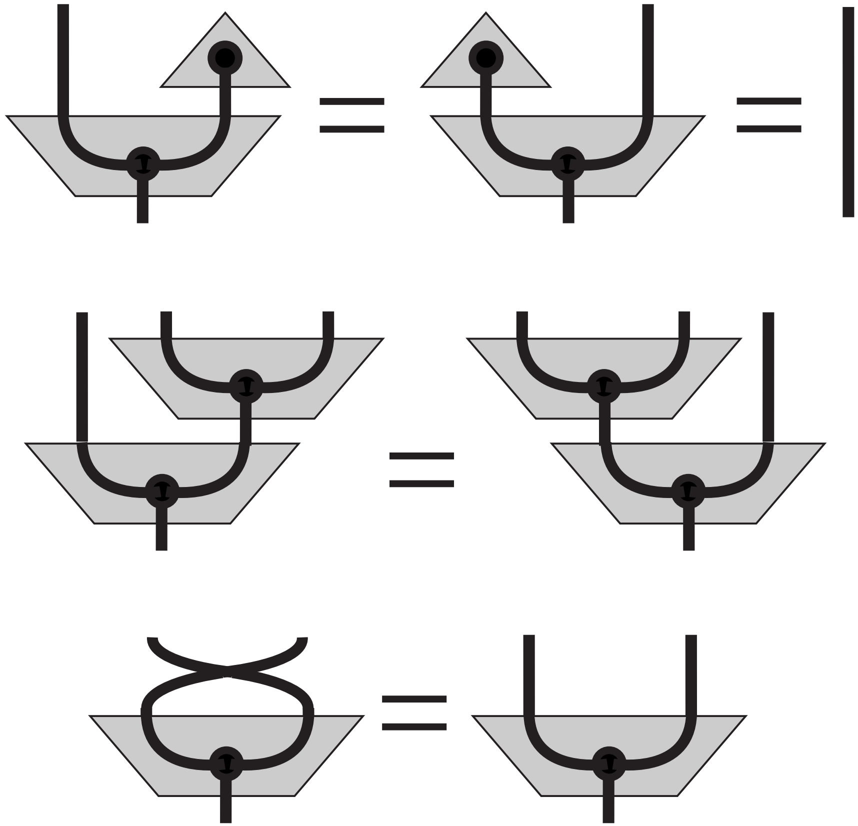
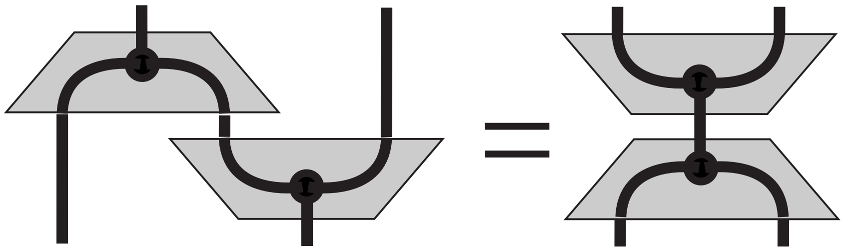


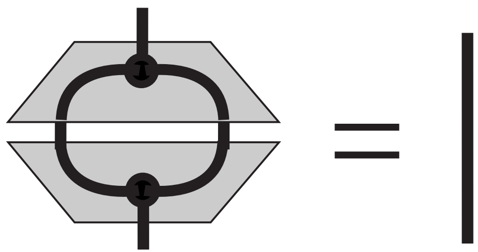
proof $\sim$ "fusion" of dots $\Rightarrow$ graphical rewrite system

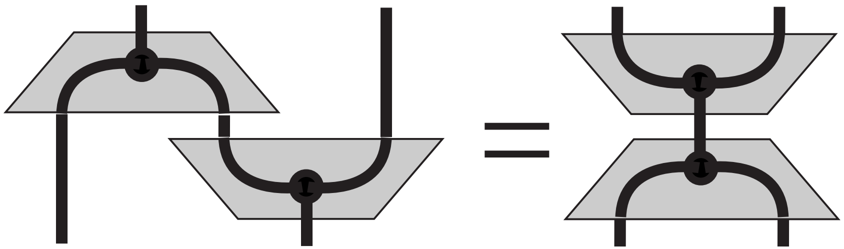
Kock, J. (2003) ***Frobenius algebras and 2D TQFTs*****.**

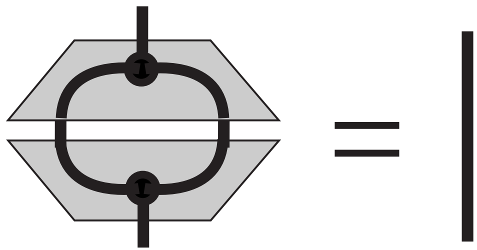
Coecke-Paquette (2006) ***POVMs & Naimark's thm without sums*****.**





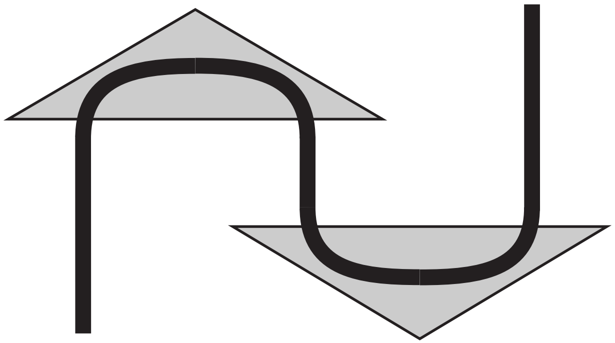






All five axioms follow from spider-normal-form.

Summary: refining quantum structure



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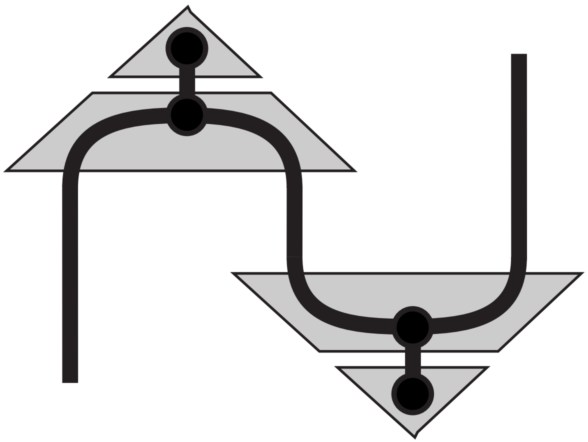
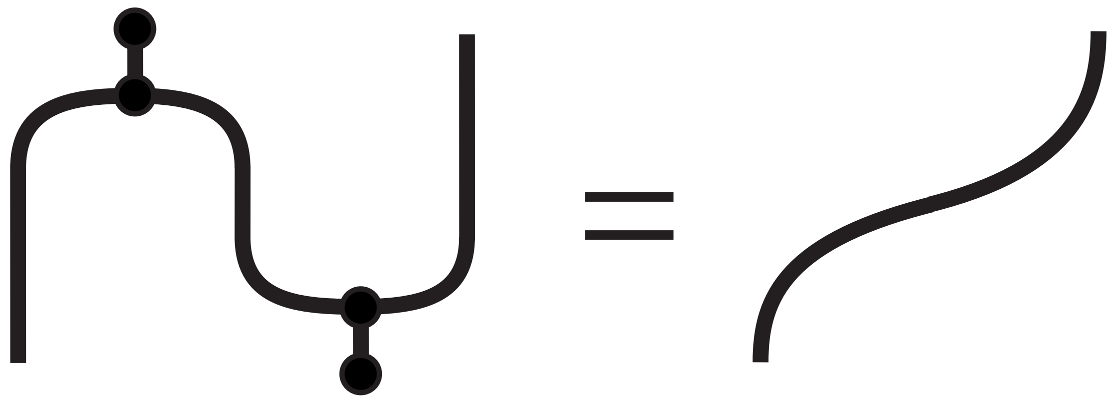


Summary: refining quantum structure



Quantum measurement:

$$\mathcal{M} : A \to X \otimes A$$

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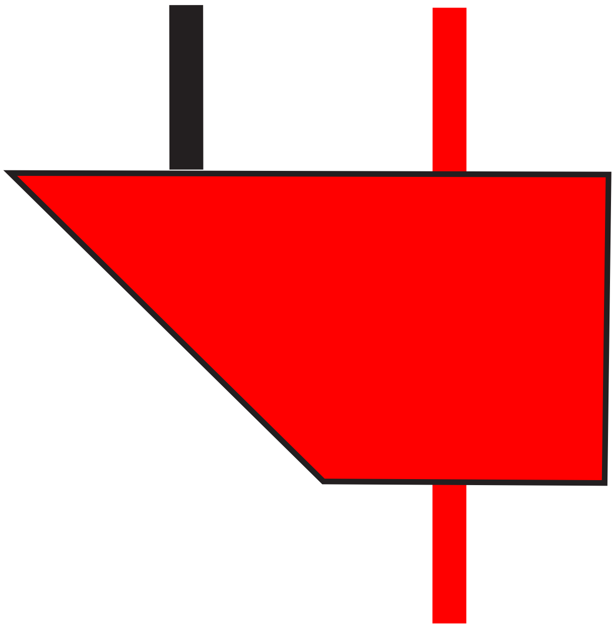

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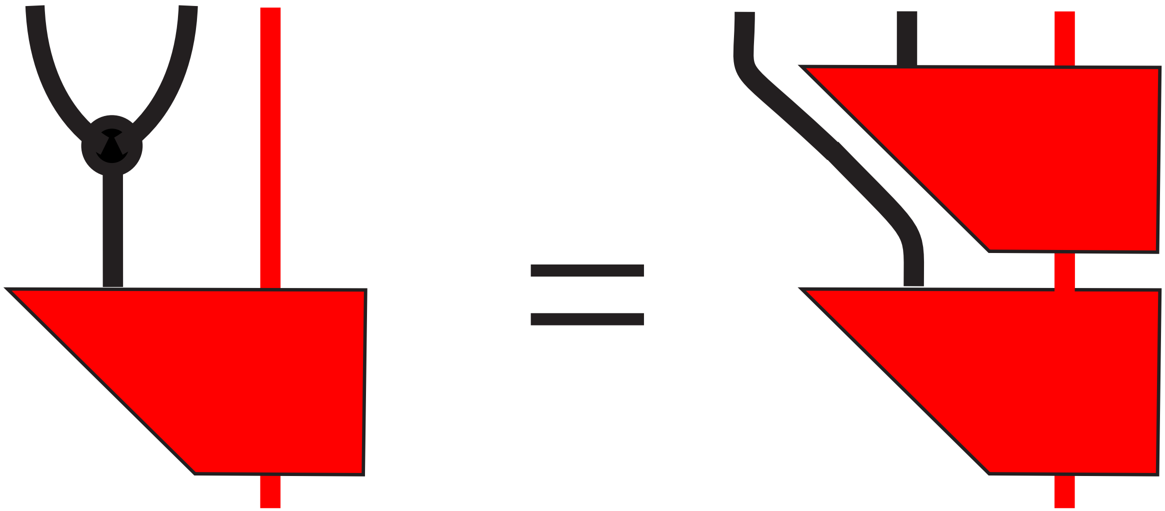

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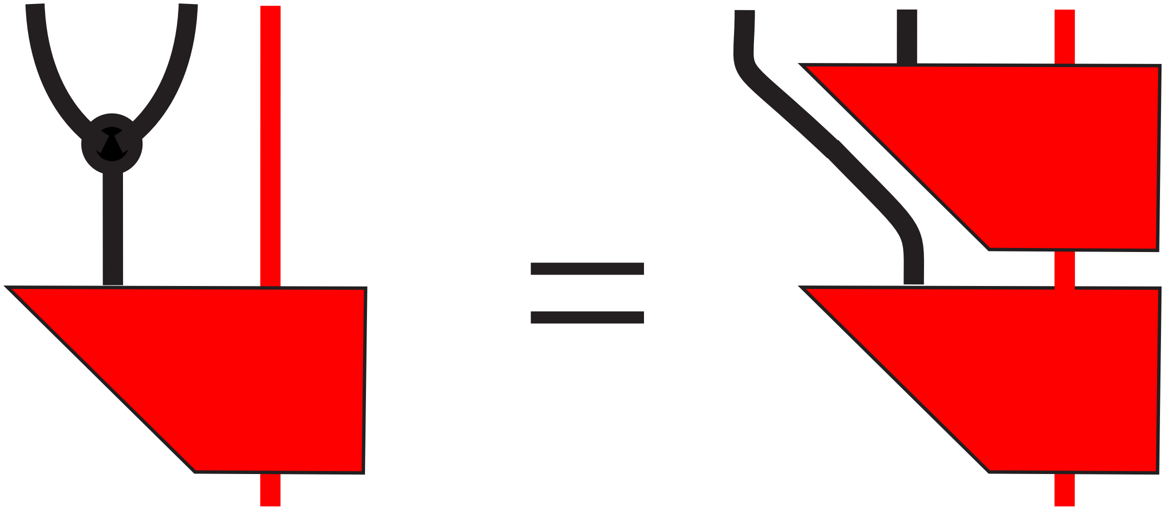

$\Rightarrow$ Quantum measurements turn out to be Eilenberg-Moore coalgebras for the comonad $(X \otimes -) : \mathbf{C} \rightarrow \mathbf{C}$.

Quantum measurement:

$$\begin{array}{ccc} A & \xrightarrow{\mathcal{M}} & X \otimes A \\ {\scriptstyle \mathcal{M}}\downarrow & & \downarrow{\scriptstyle 1_X \otimes \mathcal{M}} \\ X \otimes A & \xrightarrow[\delta \otimes 1_A]{} & X \otimes X \otimes A \end{array}$$

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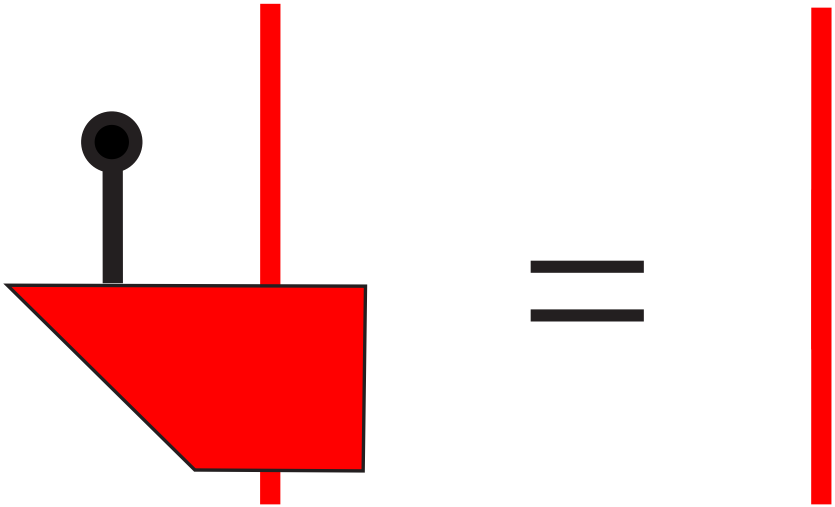

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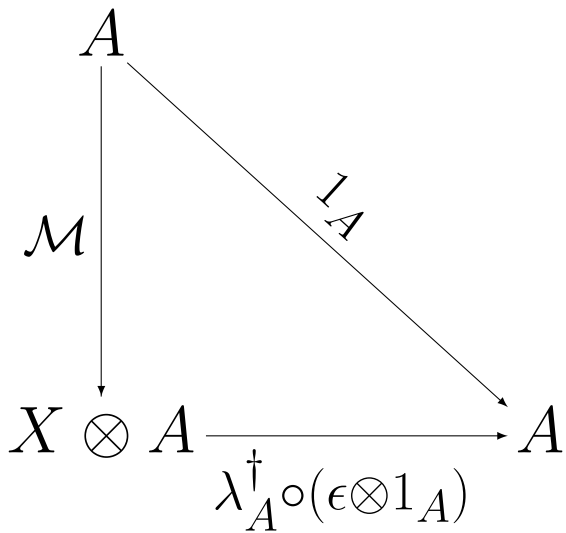

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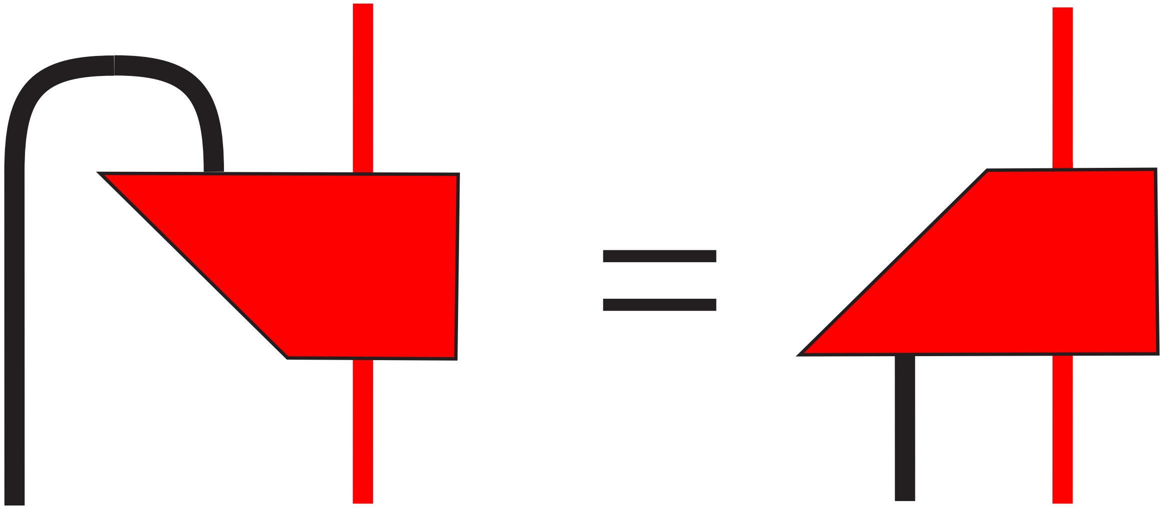
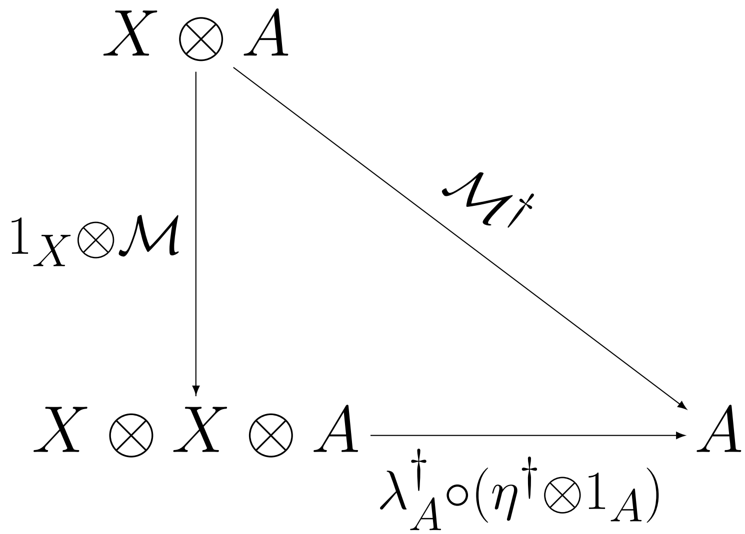


$\Rightarrow$ self-adjointness.

Quantum measurement:



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Thm. Self-adjoint Eilenberg-Moore coalgebras for

$$\mathcal{H} \otimes - : \mathbf{FdHilb} \to \mathbf{FdHilb}$$

are exactly $\mathsf{dim}\mathcal{H}$**-outcome quantum measurements**.

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$$\mathcal{H} \otimes - : \mathbf{FdHilb} \to \mathbf{FdHilb}$$

are exactly $\mathsf{dim}\mathcal{H}$**-outcome quantum measurements**.

Coalg-square $\Rightarrow$

idempotence $\mathrm{P}_i^2 = \mathrm{P}_i$

mutual orthogonality $\mathrm{P}_i \circ \mathrm{P}_{j \neq i} = \mathbf{0}$

Coalg-triangle $\Rightarrow$

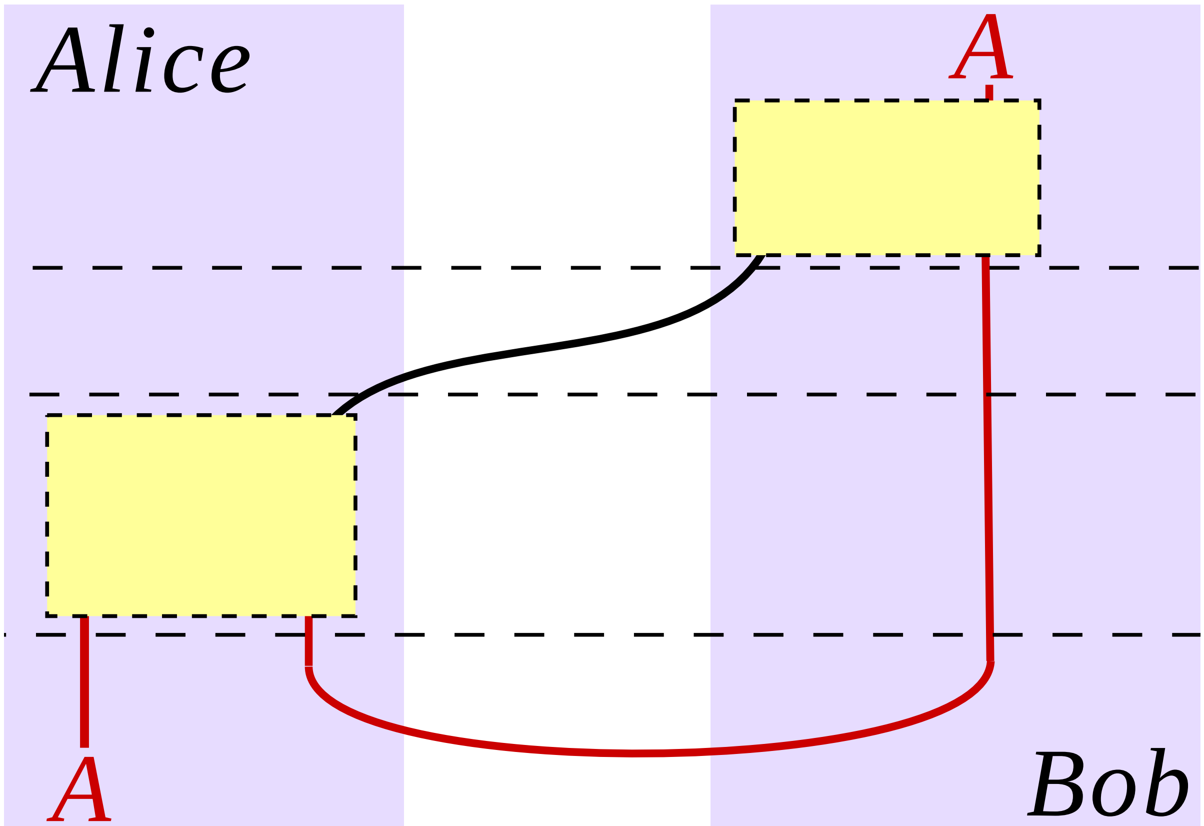
Completeness of spectrum $\sum_i \mathrm{P}_i = 1_{\mathcal{H}}$

Self-adjointness $\Rightarrow$

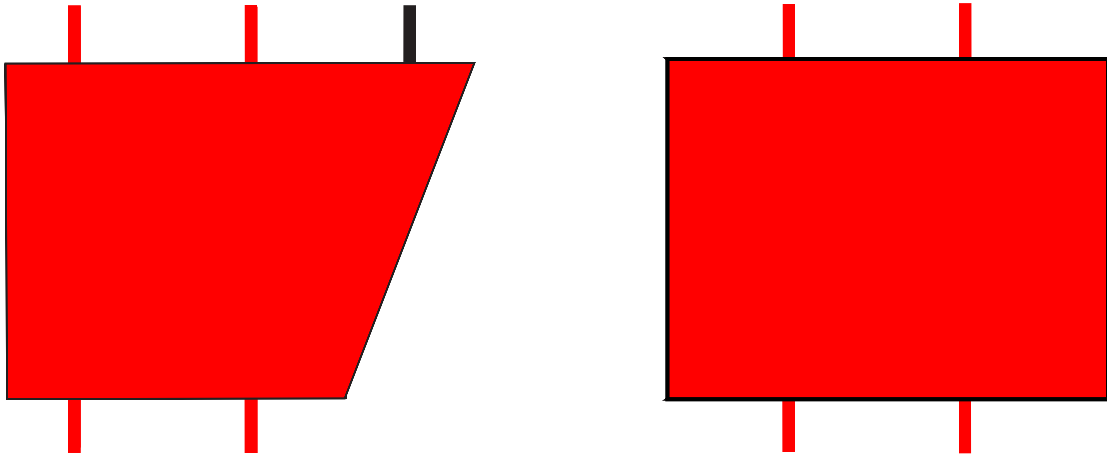
Orthogonality of projectors $\mathrm{P}_i^\dagger = \mathrm{P}_i$

PROJECTOR SPECTRUM

Teleportation:



Bipartite quantum measurement:



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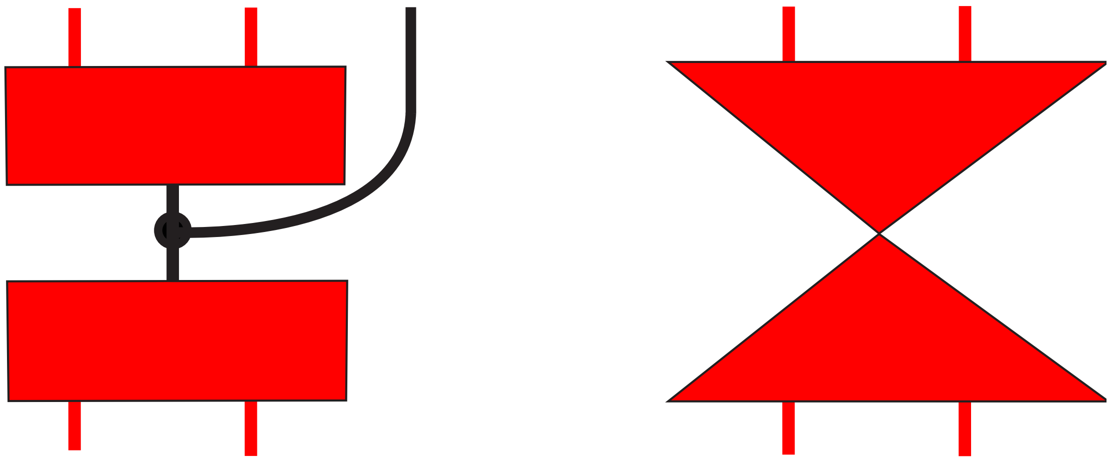

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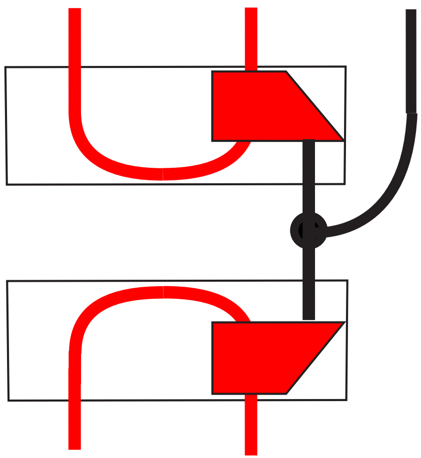

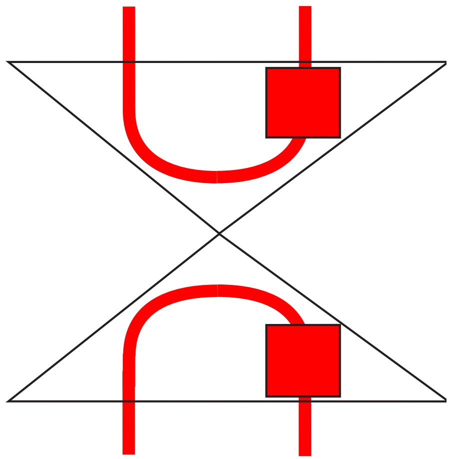

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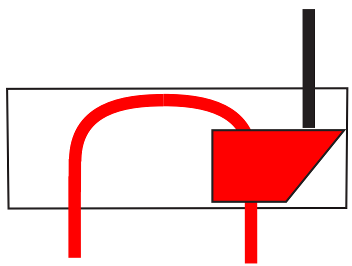
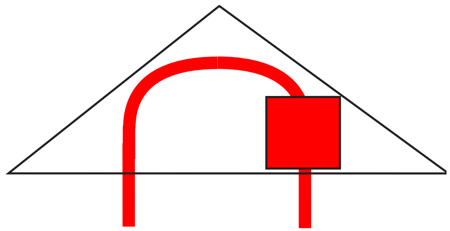
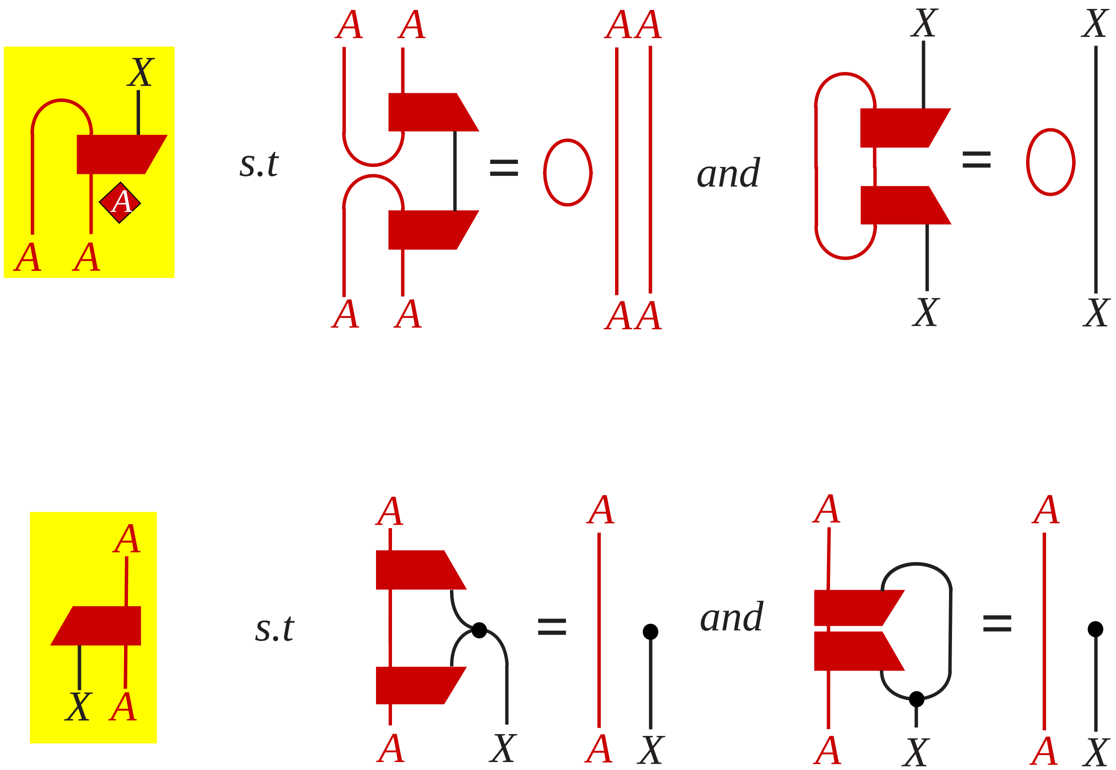




Teleportation enabling measurement:



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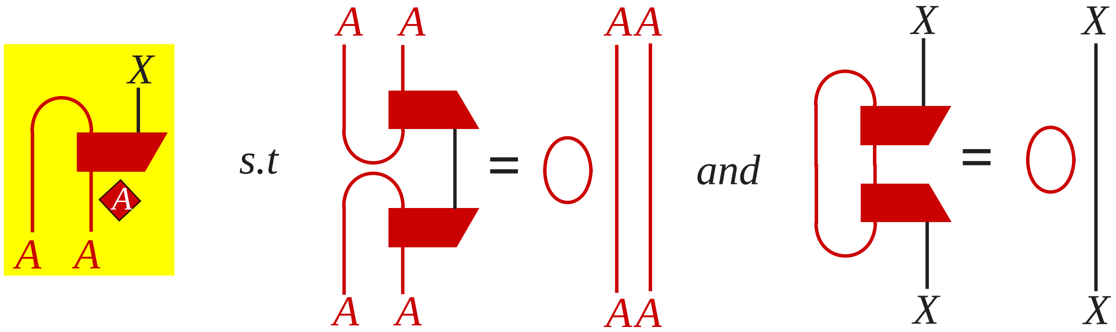
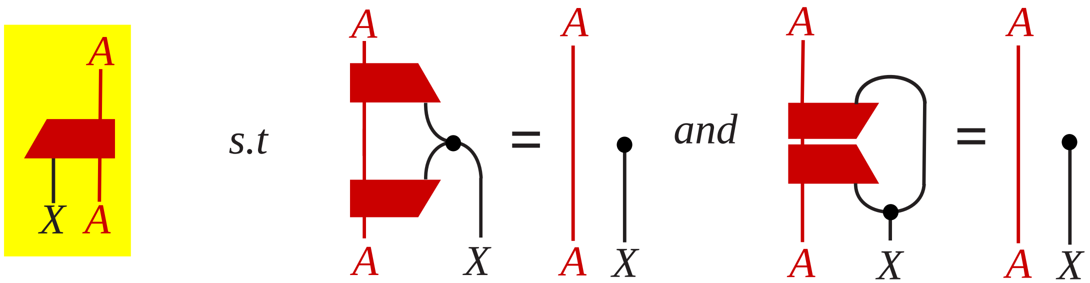
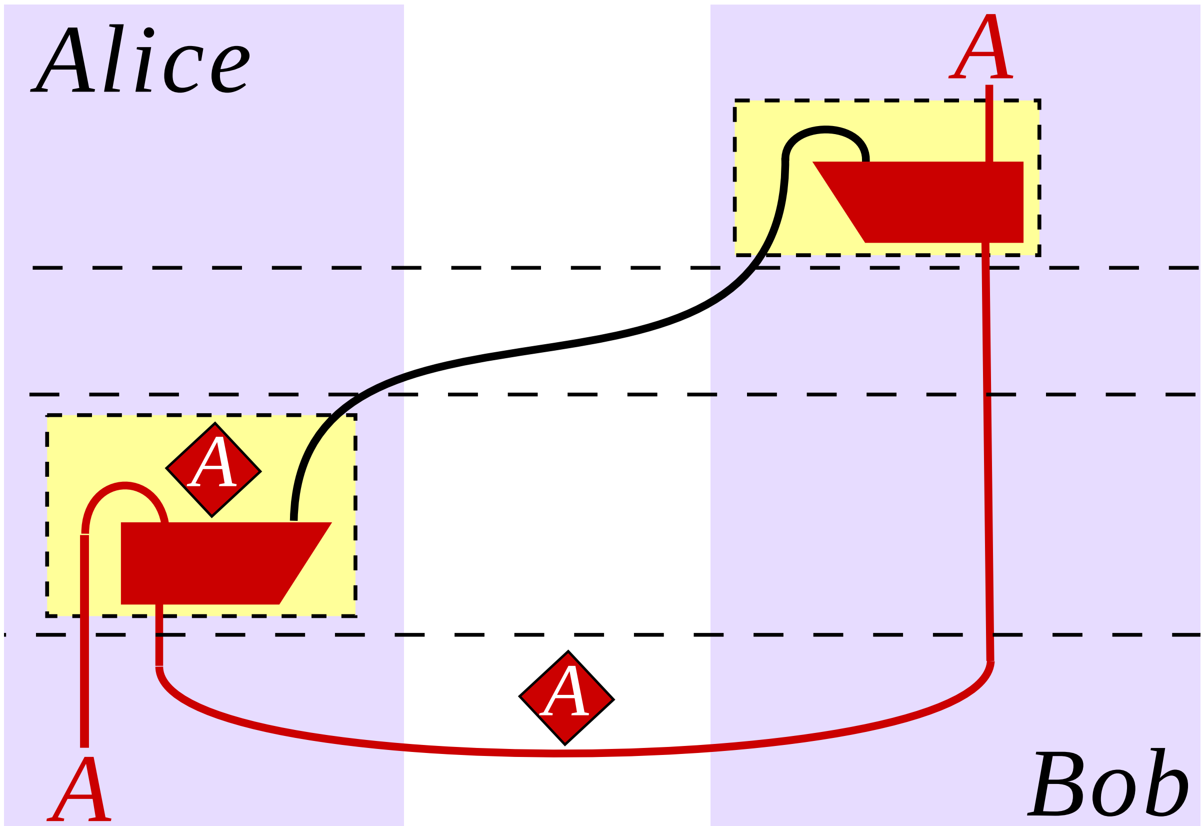


abstracts $\dim(X) \geq (\dim(A))^2$ and $\mathrm{Tr}(U_x \circ U_y^\dagger) = \delta_{xy}$.

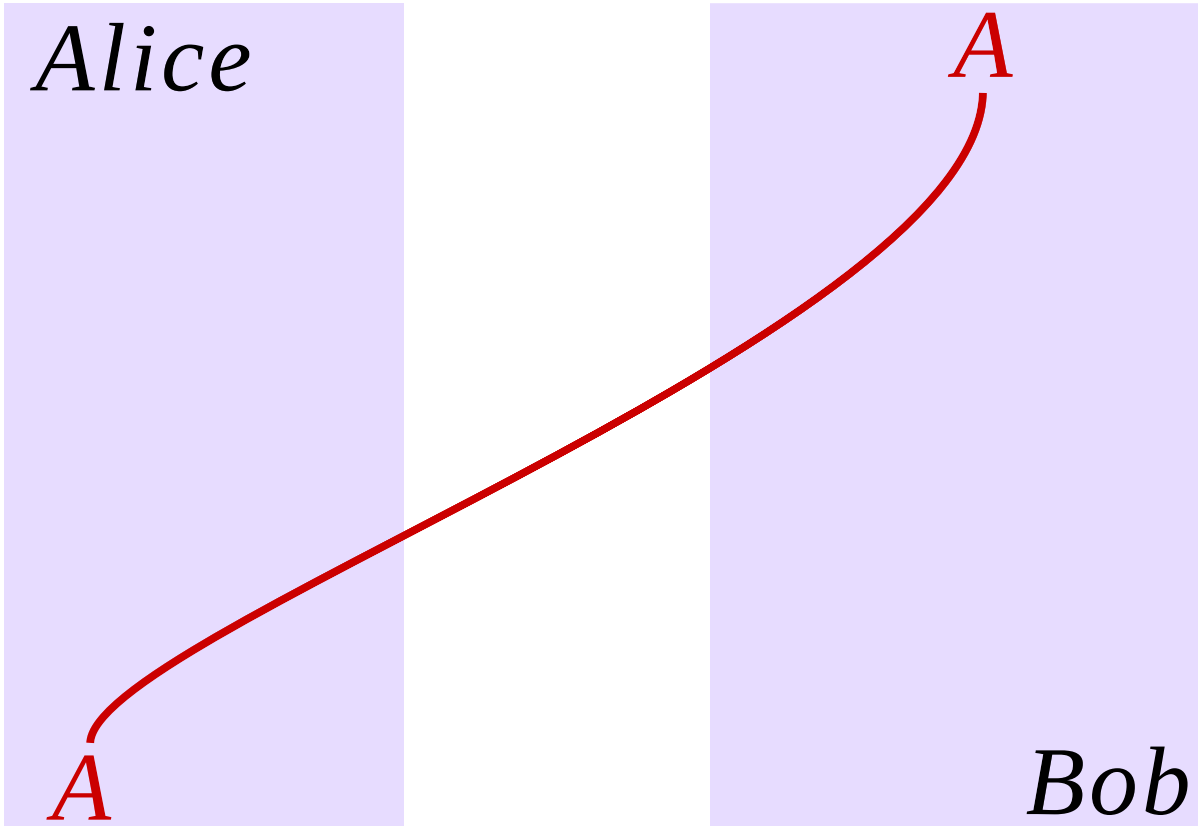


abstracts unitarity of $\{U_x\}_x$ i.e. $U_x^\dagger \circ U_x = U_x \circ U_x^\dagger = 1_A$.

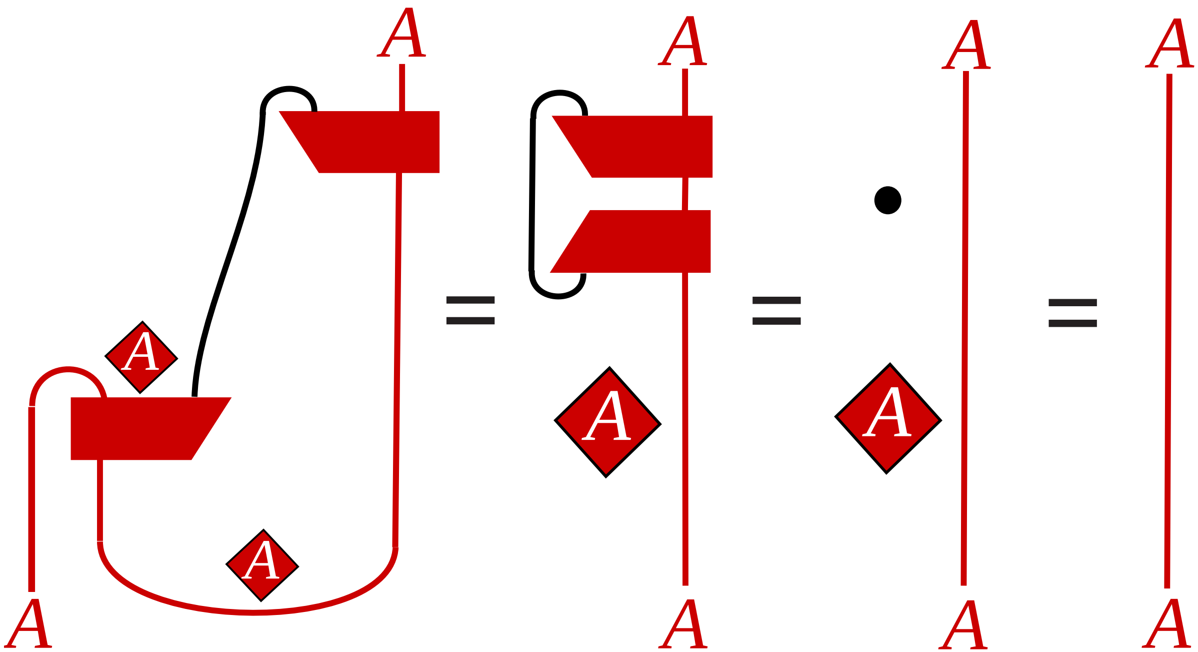
Teleportation:



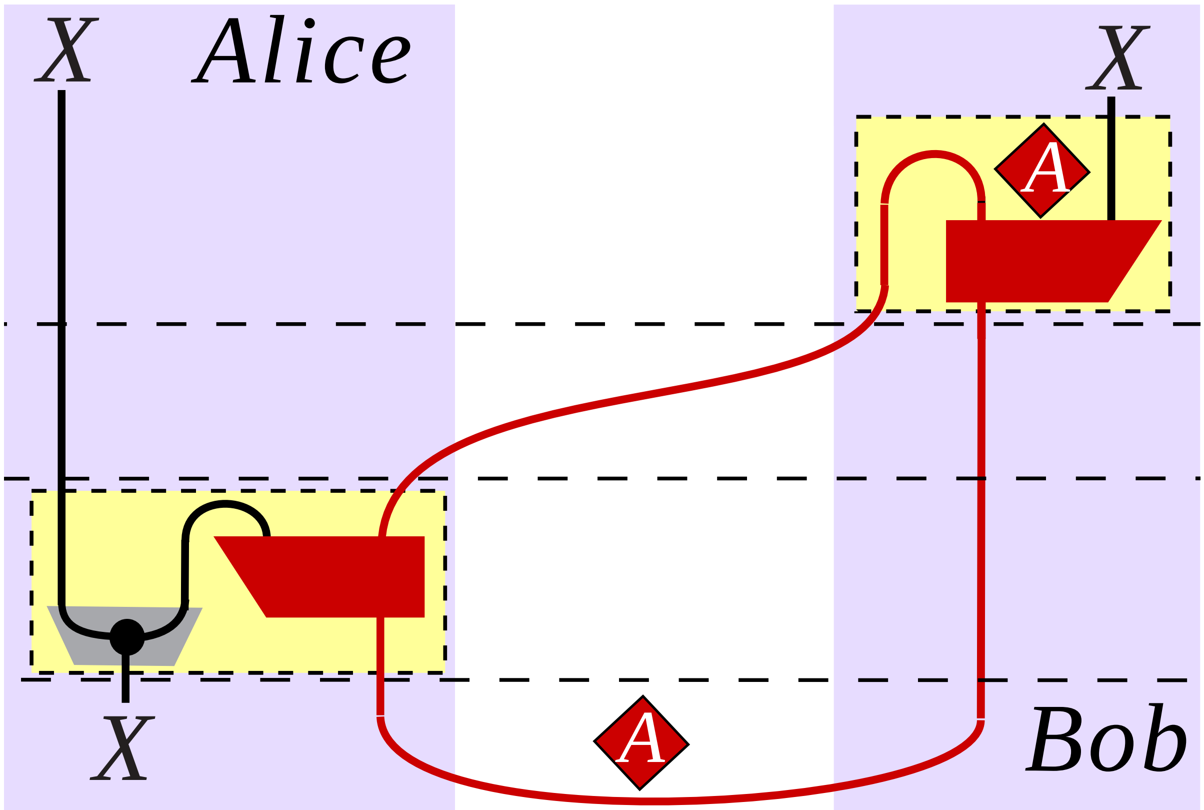
Intended behavior:



Proof:



Dense coding:



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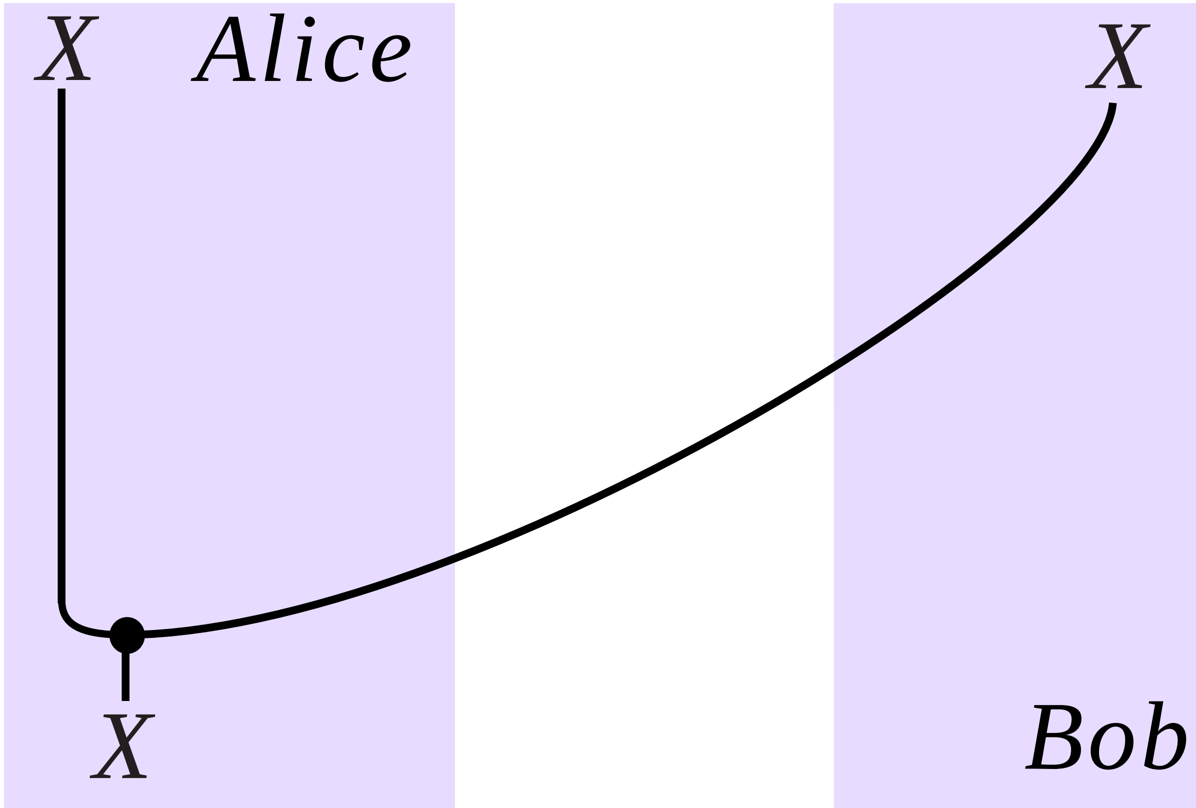

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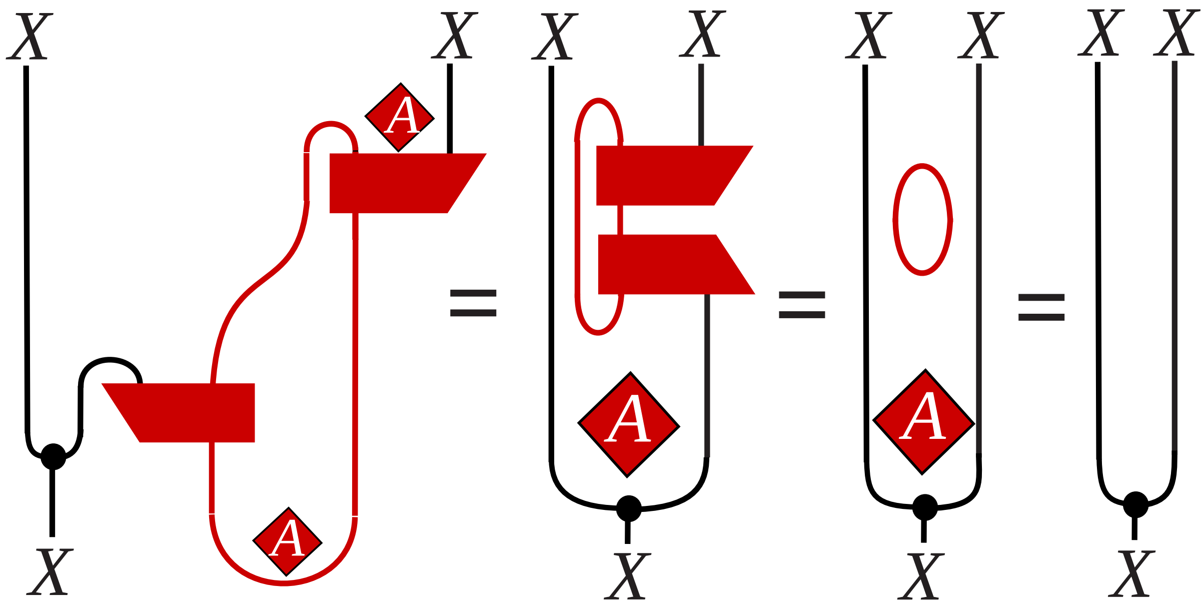


CLASSICAL MAPS

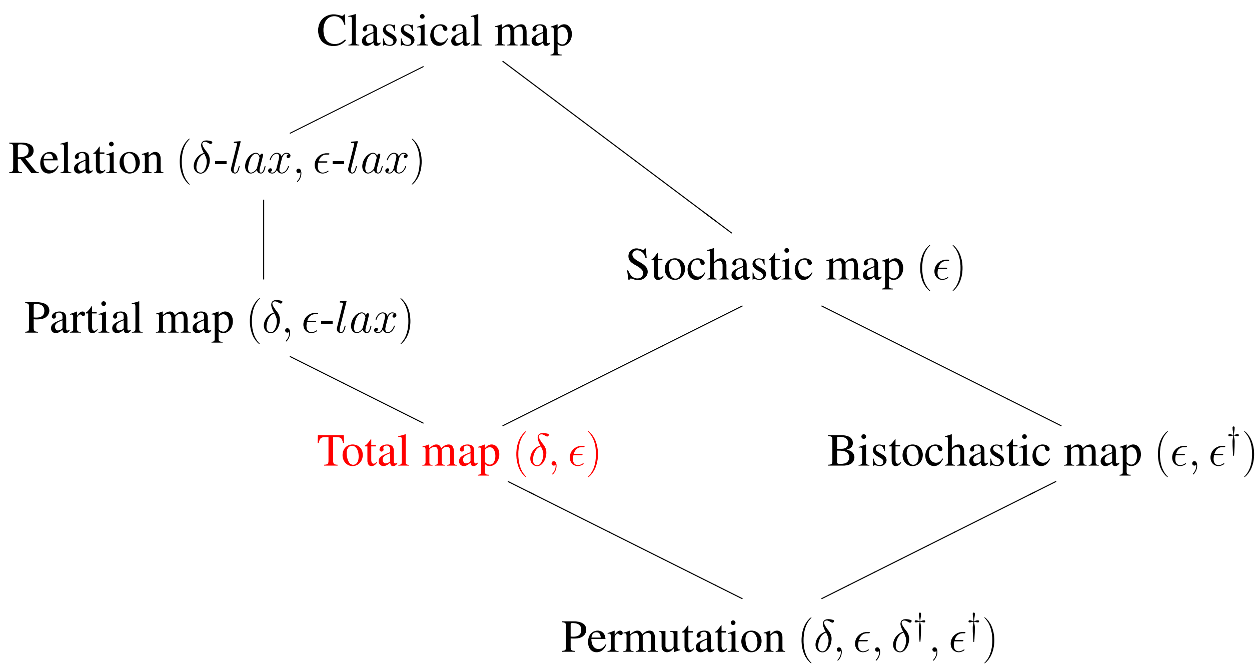
(Coecke-Paquette-Pavlovic 2007)

Cartesian structure as a limit

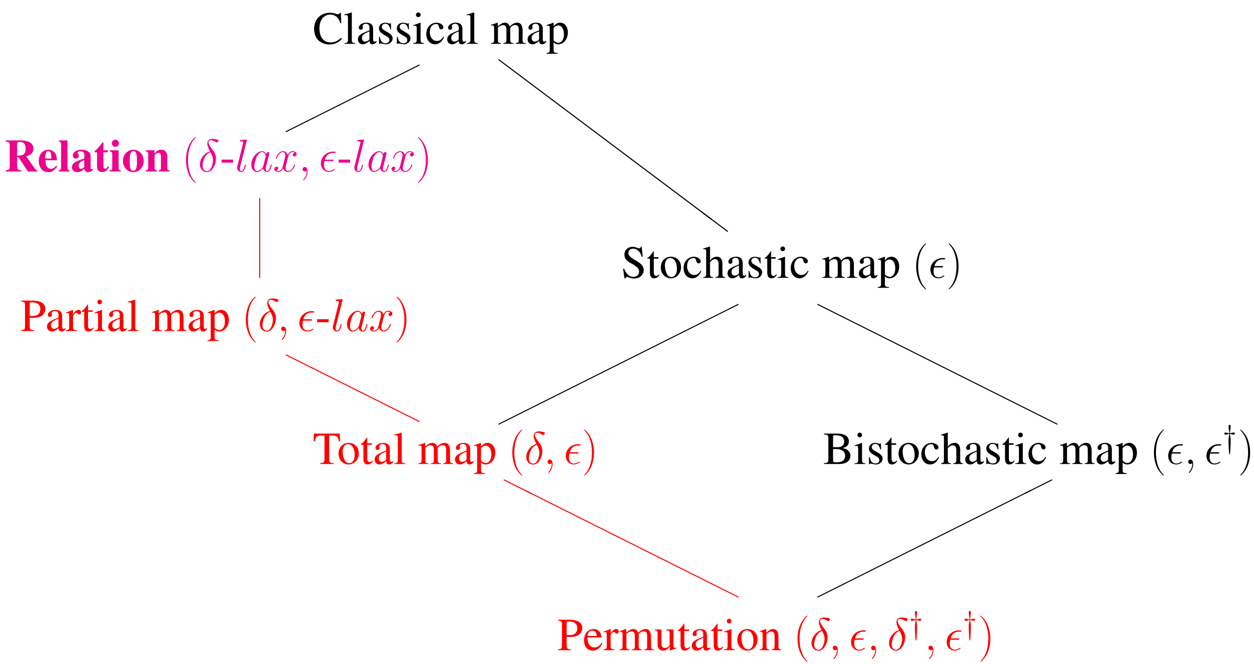
Theorem. **[Fox 1976]** The category $\mathbf{C}_\times$ of commutative comonoids and corresponding morphisms of a symmetric monoidal category with the forgetful functor $\mathbf{C}_\times \to \mathbf{C}$, is final among all **cartesian categories** with a monoidal functor to $\mathbf{C}$, mapping the cartesian product to the monoidal tensor.

- Deterministic classical states = clone-able ones
- Deterministic classical operations = clone-able ones
- $\mathbf{FdHilb}_\times := \mathbf{FSet}$

Classical genera:

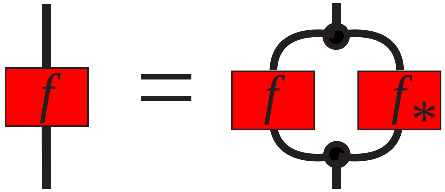


Classical genera:

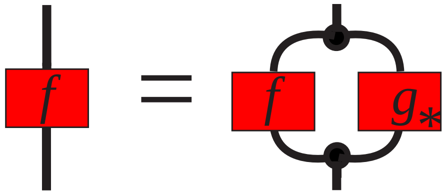


Carboni-Walters (1987) *Cartesian Bicategories* I.

Proposition. Morphisms satisfying



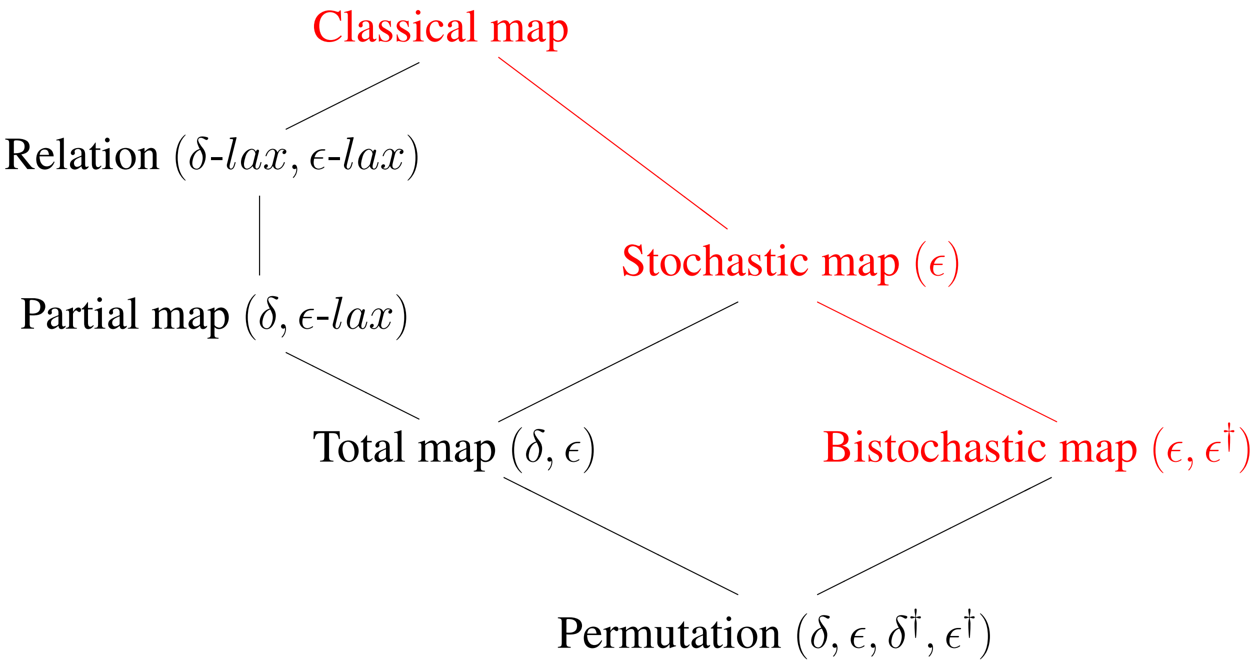
subject to the local partial order $f \leq g$ *iff*



constitute a *bicategory of relations* $\mathbf{C}_r$ in the sense of Carboni-Walters (1987).[‡] In particular, relations are lax comonoid homomorphisms w.r.t. $\leq$ and $\circ_r \neq \circ$.

[‡] There is an issue with finiteness of comonoid structures.

Classical genera:



Let $\Omega(\mathcal{H})$ be density matrices $\rho : \mathcal{H} \to \mathcal{H}$ with trace 1.

A completely positive map $\delta : \Omega(\mathcal{H}) \longrightarrow \Omega(\mathcal{H} \otimes \mathcal{H})$ is a **cloning operation** if for all $\rho \in \Omega(\mathcal{H})$:

$$\delta(\rho) = \rho \otimes \rho\,.$$

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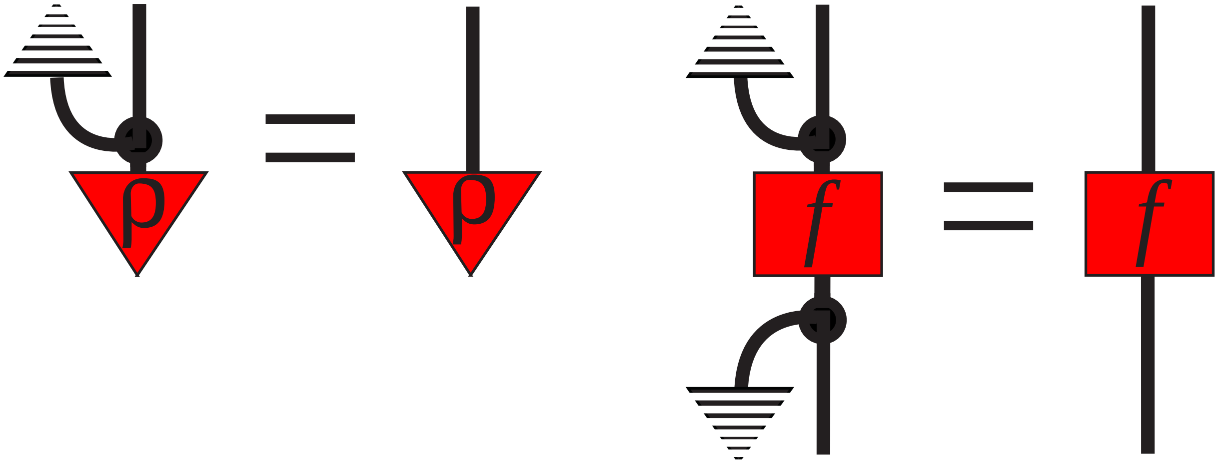
It is a **broadcasting operation** if for all $\rho \in \Omega(\mathcal{H})$:

$$\mathrm{Tr}_1(\delta(\rho)) = \mathrm{Tr}_2(\delta(\rho)) = \rho\,.$$

Existence of a cloning/broadcasting operation for restricted sets of density operators relative to a fixed base:

	cloning	**broadcasting**
bases vectors	yes	yes
diagonal density operators	→ no ←	→ yes ←
pure density operators	no	no
arbitrary density operators	no	no

Classical maps are broadcast-able maps



What's next:

- **More structural resources for quantum things.**
- **Quantum Computer Science.**
- **Real physics problems involving 'energy' etc.**
- **Interaction with other instances of physics.**
- **What is true quantumness?**