The Categorification of a Linear Theory is Presentation-Independent

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Category Theory 2007

Outline

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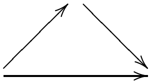
1. Background
   - Categorification
   - Operads
   - Presentations
2. Definition and Results
   - Categorifying a Linear Theory with Presentation
   - Presentation-Independence Theorem
3. Comparison with Other Approaches
   - Blackwell, Kelly and Power

The Problem

- Some cases are well-known (monoids $\mapsto$ monoidal categories, etc.)
- How do we categorify an arbitrary algebraic theory?
- Does it matter how we present the theory we start with?

Operads

- Categories



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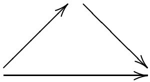

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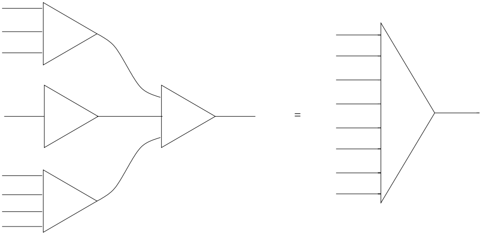

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- Sequences of sets $P_0, P_1, \ldots,$ with P_n being n-ary arrows.

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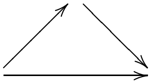

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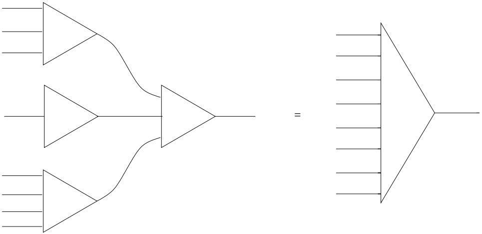

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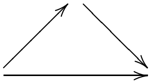

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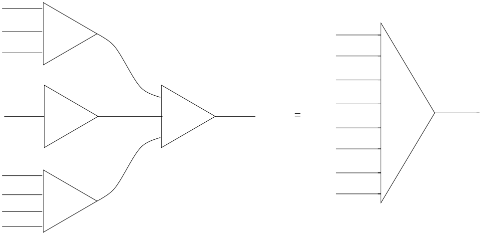

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Symmetric operads

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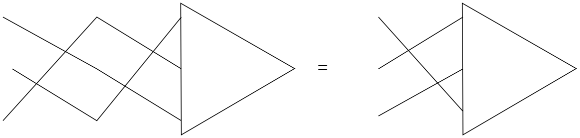

- For each $\sigma \in S_n$ and $a_1 \ldots, a_n, b \in \mathcal{C}$, a map

$$\sigma \cdot - : \mathrm{Hom}_{\mathcal{C}}(a_1, \ldots, a_n; b) \to \mathrm{Hom}_{\mathcal{C}}(a_{\sigma 1}, \ldots, a_{\sigma n}; b)$$

  that plays nicely with composition in S_n and $\mathcal{C}$.
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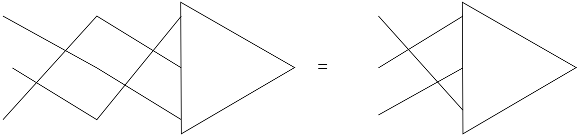

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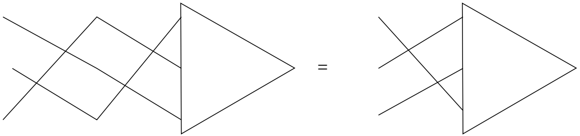

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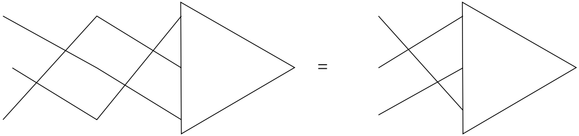

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$$\mathbf{Set}^{\mathbb{N}} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{SymmOperad} \quad (\bot)$$

- U: forget composition and permutation structure
- $F\Phi$: permuted trees
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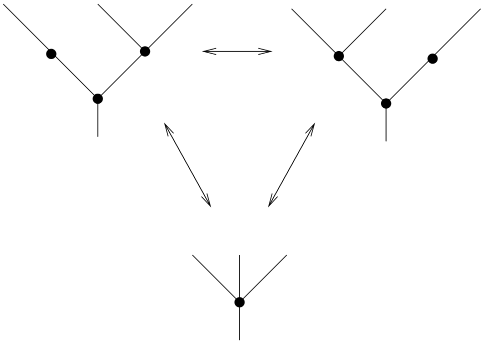
Cat-operads

Instead of sets of arrows, we have categories.



Strongly Regular and Linear theories

Strongly Regular	**“Linear”**
Same variables on each side, exactly once per side, in same order	Same variables on each side, exactly once per side, order may vary
$(a.b).c = a.(b.c)$ is OK	$(a.b).c = a.(b.c)$ is OK
$a.b = b.a$ is not OK	$a.b = b.a$ is OK
$a.(b+c) = a.b+a.c$ is not OK	$a.(b+c) = a.b+a.c$ is not OK
Models are algebras for a plain operad	Models are algebras for a symmetric operad

Presentations

Definition

Let P be a symmetric operad.
A presentation $\langle \Phi | E \rangle$ of P is a coequalizer

$$FE \rightrightarrows F\Phi \xrightarrow{\pi} P$$

where $\Phi, E \in \mathbf{Set}^{\mathbb{N}}$.

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Linear Theories with Presentations

How to categorify $P = \langle \Phi | E \rangle$?

First thing we think of: a symmetric **Cat**-operad with

- objects: permuted trees of things in Φ
- an arrow $\tau_1 \to \tau_2$ iff $\pi(\tau_1) = \pi(\tau_2)$
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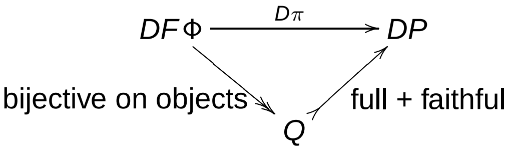
More abstractly

Recall $\pi : F\Phi \to P$ is regular epi.



(D is "levelwise discrete category")

Fact: ({B.O.O arrows}, {f+f arrows}) forms a factorization system on **Cat-SymmOperad**, so Q is unique.

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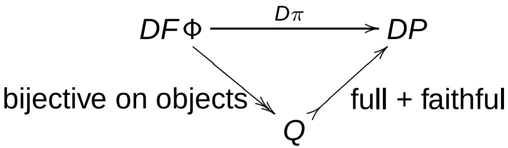

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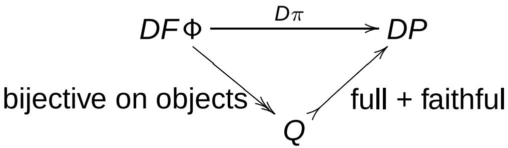

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Unbiased Categorification

- We didn't need a presentation, only a regular epi $\pi : F\Phi \to P$.
- In particular, $\epsilon : FUP \to P$ will do:

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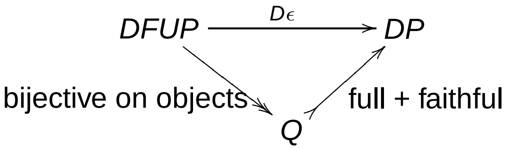

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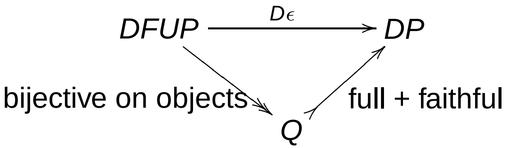

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Presentation-Independence

Theorem

Let P be a symmetric operad.
For all Φ and all regular epis $\pi : F\Phi \longrightarrow P$,

$$Wk(P)_\pi \simeq UWk(P)$$

*as a symmetric **Cat**-operad.*

Corollary

The category of weak P-categories and weak P-functors (w.r.t. π) is equivalent to the category of unbiased weak P-categories and unbiased weak P-functors.

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Pseudo-algebras (Blackwell, Kelly and Power)

- Agrees with our definition in strongly regular case.
- Not so good outside strongly regular case.
- e.g. a pseudo-algebra for the “free commutative monoid” 2-monad on **Cat** is a *strictly* symmetric weak mon. cat.
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Summary

- We can categorify any linear algebraic theory, without making arbitrary choices.
- Up to equivalence, the choice of presentation doesn't matter.
- This gives the Right Thing in cases where other approaches don't.

Further Work

- Most theories aren't linear.
- Finite presentability for categorified theories.
- "Weak thing $\simeq$ strict thing"?

Sketch Proof of Theorem

- **Fact:** regular epis in **SymmOperad** are pointwise surjections.
- So we can choose a section ψ of $U\pi : UF\Phi \to UP$.
- Hence we get $\bar{\psi} : FUP \to F\Phi$.
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$$\begin{array}{ccc} FUP & \xrightarrow{\epsilon} & P \\ \big\downarrow{\scriptstyle \bar{\psi}} & & \big\downarrow{\scriptstyle 1} \\ F\Phi & \xrightarrow{\pi} & P \end{array}$$

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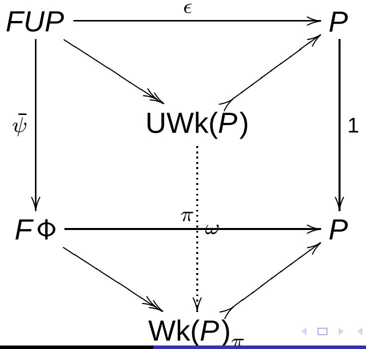
- 

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- Similarly, we have $F\bar{\pi} : F\Phi \to FUP$:
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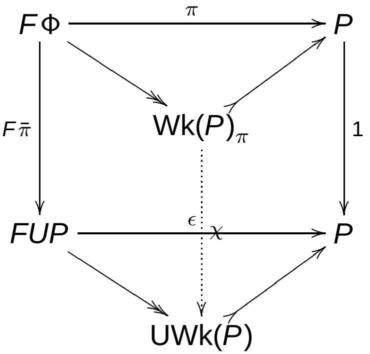
- 

Hence,

$$\begin{array}{ccc} Q & \rightarrowtail & P \\ {\scriptstyle \omega}\downarrow & & \downarrow{\scriptstyle 1} \\ \mathrm{Wk}(P) & \rightarrowtail & P \\ {\scriptstyle \chi}\downarrow & & \downarrow{\scriptstyle 1} \\ Q & \rightarrowtail & P \end{array}$$

commutes. So $Q \underset{\chi\omega}{\overset{1_Q}{\rightrightarrows}} Q \longrightarrow P$ is a fork.

Lemma

*In **Cat-Σ-Operad**, if* $P \overset{\alpha}{\underset{\beta}{\rightrightarrows}} Q \xrightarrow{\gamma} R$ *is a fork, and γ is levelwise full and faithful, then $\alpha \cong \beta$.*

Hence $\chi\omega \cong 1_Q$, and similarly $\omega\chi \cong 1_{\mathrm{Wk}(P)}$. So $Q \simeq \mathrm{Wk}(P)$ as a symmetric **Cat**-operad, as required.