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Categorical groups for exterior spaces

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1. Introduction

Proper homotopy theory
Classification of non compact surfaces

B. Kerékjártó, *Vorlesungen uber Topologie* , vol.1, Springer-Verlag (1923). *Ideal point*

H. Freudenthal, *Über die Enden topologisher Räume und Gruppen* , Math. Zeith. 53 (1931) 692-713. *End of a space*

L.C. Siebenmann, *The obstruction to finding a boundary for an open manifold of dimension greater than five* , Tesis, 1965.

Proper homotopy invariants at one end represented by a base ray

H.J. Baues, A. Quintero, *Infinite Homotopy Theory*, K-Monographs in Mathematics, 6. Kluwer Publishers, 2001.

Invariants associated at a base tree

One of the main problems of the proper category is that there are few limits and colimits.

Pro-spaces

J.W. Grossman, *A homotopy theory of pro-spaces* , Trans. Amer. Math. Soc.,201 (1975) 161-176.

T. Porter, *Abstract homotopy theory in procategories* , Cahiers de topologie et geometrie differentielle, vol 17 (1976) 113-124.

A. Edwards, H.M. Hastings, *Every weak proper homotopy equivalence is weakly properly homotopic to a proper homotopy equivalence* , Trans. Amer. Math. Soc. 221 (1976), no. 1, 239–248.

Exterior spaces

J. García Calcines, M. García Pinillos, L.J. Hernández Paricio, *A closed model category for proper homotopy and shape theories*, Bull. Aust. Math. Soc. 57 (1998) 221-242.

J. García Calcines, M. García Pinillos, L.J. Hernández Paricio, *Closed Simplicial Model Structures for Exterior and Proper Homotopy Theory*, Applied Categorical Structures, 12, (2004) , pp. 225-243.

J. I. Extremiana, L.J. Hernández, M.T. Rivas , *Postnikov factorizations at infinity*, Top and its Appl. 153 (2005) 370-393.

n-types

J.H.C. Whitehead, *Combinatorial homotopy. I , II* , Bull. Amer. Math. Soc., 55 (1949) 213-245, 453-496.

Crossed complexes and crossed modules

2. Proper maps, exterior spaces and categories of proper and exterior 2-types

A continuous map $f : X \to Y$ is said to be *proper* if for every closed compact subset K of Y, $f^{-1}(K)$ is a compact subset of X.

Top topological spaces and continuous maps

P spaces and proper maps

P does not have enough limits and colimits

Definition 2.1 *Let (X, τ) be a topological space. An* externology *on (X, τ) is a non empty collection ε of open subsets which is closed under finite intersections and such that if $E \in \varepsilon$, $U \in \tau$ and $E \subset U$ then $U \in \varepsilon$. An* exterior space *$(X, \varepsilon \subset \tau)$ consists of a space (X, τ) together with an externology ε. A map $f : (X, \varepsilon \subset \tau) \to (X', \varepsilon' \subset \tau')$ is said to be* exterior *if it is continuous and $f^{-1}(E) \in \varepsilon$, for all $E \in \varepsilon'$.*

The category of exterior spaces and maps will be denoted by $\mathbf{E}$.

$\mathbb{N}$ non negative integers, usual topology, cocompact externology
$\mathbb{R}_+$ $[0, \infty)$, usual topology, cocompact externology
$\mathbf{E}^{\mathbb{N}}$ exterior spaces under $\mathbb{N}$
$\mathbf{E}^{\mathbb{R}_+}$ exterior spaces under $\mathbb{R}_+$
(X, λ) object in $\mathbf{E}^{\mathbb{R}_+}$, $\lambda\colon \mathbb{R}_+ \longrightarrow X$ a *base ray* in X
The natural restriction $\lambda|_{\mathbb{N}}\colon \mathbb{N} \longrightarrow X$ is a *sequence base* in X

$$\mathbf{E}^{\mathbb{R}_+} \longrightarrow \mathbf{E}^{\mathbb{N}} \quad \text{forgetful functor}$$

X, Z exterior spaces, Y topological space
$X \bar{\times} Y$, Z^Y exterior spaces
Z^X topological space (box $\supset$ topology Z^X $\supset$ compact-open)

S^q q-dimensional (pointed) sphere:

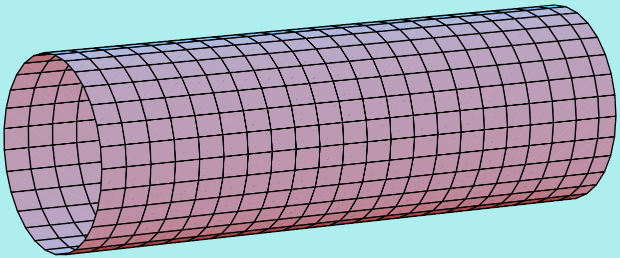
$$Hom_{\mathbf{E}}(\mathbb{N} \bar{\times} S^q, X) \cong Hom_{\mathbf{Top}}(S^q, X^{\mathbb{N}})$$
$$Hom_{\mathbf{E}}(\mathbb{R}_+ \bar{\times} S^q, X) \cong Hom_{\mathbf{Top}}(S^q, X^{\mathbb{R}_+})$$

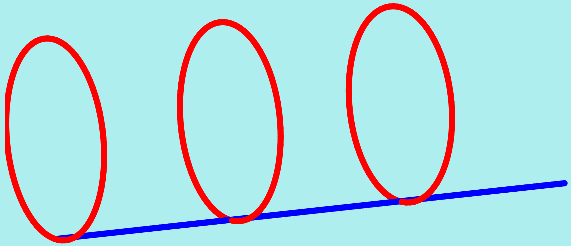
Definition 2.2 *Let* (X,λ) *be in* $\mathbf{E}^{\mathbb{R}_+}$ *and an integer* $q \geq 0$ *.*
The q-th $\mathbb{R}_+$-exterior homotopy group *of* (X,λ)*:*

$$\pi_q^{\mathbb{R}_+}(X,\lambda) = \pi_q(X^{\mathbb{R}_+},\lambda)$$

The q-th $\mathbb{N}$-exterior homotopy group *of* (X,λ)*:*

$$\pi_q^{\mathbb{N}}(X,\lambda|_{\mathbb{N}}) = \pi_q(X^{\mathbb{N}},\lambda|_{\mathbb{N}})$$





Definition 2.3 *An exterior map* $f\colon (X,\lambda) \to (X',\lambda')$ *is said to be a weak* $[1,2]$-$\mathbb{R}_+$-*equivalence (weak* $[1,2]$-$\mathbb{N}$-*equivalence) if* $\pi_1^{\mathbb{R}_+}(f), \pi_2^{\mathbb{R}_+}(f)$ *(*$\pi_1^{\mathbb{N}}(f), \pi_2^{\mathbb{N}}(f)$ *) are isomorphisms.*

$\Sigma_{\mathbb{R}_+}$ class of weak $[1,2]$-$\mathbb{R}_+$-equivalences
$\Sigma_{\mathbb{N}}$ class of weak $[1,2]$-$\mathbb{N}$-equivalences

The category of *exterior* $\mathbb{R}_+$*-2-types* is the category of fractions

$$\mathbf{E}^{\mathbb{R}_+}[\Sigma_{\mathbb{R}_+}]^{-1},$$

the category of *exterior* $\mathbb{N}$*-2-types*

$$\mathbf{E}^{\mathbb{R}_+}[\Sigma_{\mathbb{N}}]^{-1}$$

and the corresponding subcategories of *proper 2-types*

$$\mathbf{P}^{\mathbb{R}_+}[\Sigma_{\mathbb{R}_+}]^{-1}, \quad \mathbf{P}^{\mathbb{R}_+}[\Sigma_{\mathbb{N}}]^{-1}.$$

Two objects X, Y have the same type if they are isomorphic in the corresponding category of fractions

$$\text{type}(X) = \text{type}(Y) \; .$$

Example 2.1 $X = \mathbb{R}^2$, $Y = \mathbb{R}^3$:

$$\textit{2-type}(X) = \textit{2-type}\ (Y)$$

$$\mathbb{N}\textit{-2-type}(X) \neq \mathbb{N}\textit{-2-type}(Y), \quad \mathbb{R}_+\textit{-2-type}(X) \neq \mathbb{R}_+\textit{-2-type}(Y)$$

Example 2.2 $X = \mathbb{R}_+ \sqcup (\sqcup_n S^3))/n \sim *_n$, $Y = \mathbb{R}_+$:

$$\textit{2-type}(X) = \textit{2-type}\ (Y)$$

$$\mathbb{N}\textit{-2-type}(X) = \mathbb{N}\textit{-2-type}(Y), \quad \mathbb{R}_+\textit{-2-type}(X) \neq \mathbb{R}_+\textit{-2-type}(Y)$$

Example 2.3 $X = \mathbb{R}_+ \sqcup (\sqcup_n S^1))/n \sim *_n$, $Y = \mathbb{R}_+$:

$$\textit{2-type}(X) \neq \textit{2-type}\ (Y)$$

$$\mathbb{N}\textit{-2-type}(X) \neq \mathbb{N}\textit{-2-type}(Y), \quad \mathbb{R}_+\textit{-2-type}(X) = \mathbb{R}_+\textit{-2-type}(Y)$$

3. Categorical groups

A *monoidal category* $\mathbb{G} = (\mathbb{G}, \otimes, a, I, l, r)$ consists of a category $\mathbb{G}$, a functor (tensor product) $\otimes : \mathbb{G} \times \mathbb{G} \to \mathbb{G}$, an object I (unit) and natural isomorphisms called, respectively, the associativity, left-unit and right-unit constraints

$$a = a_{\alpha,\beta,\omega} : (\alpha \otimes \beta) \otimes \omega \xrightarrow{\sim} \alpha \otimes (\beta \otimes \omega) \ ,$$

$$l = l_{\alpha} : I \otimes \alpha \xrightarrow{\sim} \alpha \quad , \quad r = r_{\alpha} : \alpha \otimes I \xrightarrow{\sim} \alpha \ ,$$

which satisfy that the following diagrams are commutative

$$\begin{array}{ccc} ((\alpha \otimes \beta) \otimes \omega) \otimes \tau & \xrightarrow{a \otimes 1} & (\alpha \otimes (\beta \otimes \omega)) \otimes \tau \\ \downarrow a & & \downarrow a \\ (\alpha \otimes \beta) \otimes (\omega \otimes \tau) & & \alpha \otimes ((\beta \otimes \omega) \otimes \tau) \\ & {}_{a}\searrow \quad \swarrow_{1 \otimes a} & \\ & \alpha \otimes (\beta \otimes (\omega \otimes \tau)) & \end{array} \quad ,$$

$$
\begin{array}{ccc}
(\alpha \otimes I) \otimes \beta & \xrightarrow{a} & \alpha \otimes (I \otimes \beta) \\
& {\scriptstyle r \otimes 1} \searrow \quad \swarrow {\scriptstyle 1 \otimes l} & \\
& \alpha \otimes \beta \ . &
\end{array}
$$

A *categorical group* is a monoidal groupoid, where every object has an inverse with respect to the tensor product in the following sense:
For each object α there is an inverse object α^* and canonical isomorphisms

$$(\gamma_r)_\alpha \colon \alpha \otimes \alpha^* \longrightarrow I$$

$$(\gamma_l)_\alpha \colon \alpha^* \otimes \alpha \longrightarrow I$$

CG categorical groups

4. The small category $E(E(\bar{4}) \times EC(\Delta/2))$. Realization and categorical group of a presheaf

Objective: To give a more geometric version of the well known equivalence between 2-types and categorical groups up to weak equivalences, which can be adapted to exterior 2-types.

Find a small category S and the induced presheaf notion

(pointed spaces) adjunction (presheaves) adjuntion (categorical groups)

4.1. The small category

$\Delta/2$ is the 2-truncation of the usual category Δ whose objects are ordered sets $[q] = \{0 < 1 \cdots < q\}$ and monotone maps.

Now we can construct the pushouts

$$\begin{array}{ccc} [0] & \xrightarrow{\delta_1} & [1] \\ {\scriptstyle \delta_0}\downarrow & & \downarrow{\scriptstyle \text{in}_r} \\ [1] & \xrightarrow[\text{in}_l]{} & [1] +_{[0]} [1] \end{array} \qquad \begin{array}{ccc} [1] & \xrightarrow{\text{in}_l} & [1] +_{[0]} [1] \\ {\scriptstyle \text{in}_r}\downarrow & & \downarrow \\ [1] +_{[0]} [1] & \longrightarrow & [1] +_{[0]} [1] +_{[0]} [1] \end{array}$$

$C(\Delta/2)$ is the extension of the category $\Delta/2$ given by the objects

$$[1] +_{[0]} [1], \quad [1] +_{[0]} [1] +_{[0]} [1]$$

and all the natural maps induced by these pushouts.

In order to have vertical composition and inverses up to homotopy we extend this category with some additional maps and relations:
$V\colon [2] \longrightarrow [1]$, $V\delta_2 = \mathrm{id}$, $V\delta_1 = \delta_1\epsilon_0$, $(V\delta_0)^2 = \mathrm{id}$,
$K\colon [2] \longrightarrow [1] +_{[0]} [1]$, $K\delta_2 = \mathsf{in}_l$, $K\delta_0 = \mathsf{in}_r$,
$A\colon [2] \longrightarrow [1] +_{[0]} [1] +_{[0]} [1]$, $A\delta_2 = (K\delta_1 + \mathrm{id})K\delta_1$, $A\delta_1 = (\mathrm{id} + K\delta_1)K\delta_1$,
$A\delta_0 = A\delta_1\delta_0\epsilon_0$.
The new extended category will be denoted by $EC(\Delta/2)$.

With the objective of obtaining a tensor product with a unit object and inverses, we take the small category $\bar{4}$ generated by the object 1 and the induced coproducts $0, 1, 2, 3, 4$, all the natural maps induced by coproducts and three additonal maps:
$e_0\colon 1 \longrightarrow 0$, $\nu\colon 1 \longrightarrow 1$ and $\mu\colon 1 \longrightarrow 2$.
This gives a category $E(\bar{4})$.

Consider the product category $E(\bar{4}) \times EC(\Delta/2)$.

The object $(i,[j])$, and morphisms $\mathrm{id}_i \times g$, $f \times \mathrm{id}_{[j]}$ will be denoted by $i[j]$ and g , f, respectively.

We extend again this category by adding new maps:
$a\colon 1[1] \longrightarrow 3[0]$, $r\colon 1[1] \longrightarrow 1[0]$, $l\colon 1[1] \longrightarrow 1[0]$, $\gamma_r\colon 1[1] \longrightarrow 1[0]$, $\gamma_l\colon 1[1] \longrightarrow 1[0]$, $t\colon 1[2] \longrightarrow 2[0]$, $p\colon 1[2] \longrightarrow 4[0]$,
satisfying adequate relations to induce asociativity, identity and inverse isomorphisms for the associated categorical group structure. The commutativity of the pentagonal and triangular diagrams of a categorical group will be a consequence of the maps and properties of p and t.
The new extended category will be denoted by

$$\mathbf{E}(\mathbf{E}(\bar{\mathbf{4}}) \times \mathbf{EC}(\boldsymbol{\Delta}/\mathbf{2}))$$

4.2. The functor $S \wedge \Delta^{+}\colon \mathbf{E}(\mathbf{E}(\bar{4}) \times \mathbf{EC}(\boldsymbol{\Delta}/\mathbf{2})) \to \mathbf{Top}^{*}$

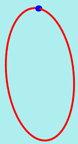
Now we take the covariant functors:
$S\colon E(\bar{4}) \to \mathbf{Top}^{*}$, preserving coproducts and such that $S(1) = S^{1}$, $S(\mu)\colon S^{1} \to S^{1} \vee S^{1}$ is the co-multiplication and $S(\nu)\colon S^{1} \to S^{1}$ gives the inverse loop.
$\Delta\colon \Delta/2 \to \mathbf{Top}$ is given by $\Delta[p] = \Delta_{p}$ and extends to $C(\Delta/2)$ preserving pushouts, $\Delta([1] +_{[0]} [1]) = \Delta_{1} \cup_{\Delta_{0}} \Delta_{1}$, et cetera.
We also consider adequate maps: $\Delta(V)$, $\Delta(K)$, $\Delta(A)$ that will give vertical inverses, vertical composition and associativity properties. Then, one has an induced functor $\Delta\colon EC(\Delta/2) \to \mathbf{Top}$.
Taking the functors $()^{+}\colon \mathbf{Top} \to \mathbf{Top}^{*}$, $X^{+} = X \sqcup \{*\}$, and the smash $\wedge\colon \mathbf{Top}^{*} \times \mathbf{Top}^{*} \to \mathbf{Top}^{*}$, we construct an induced functor

$$S \wedge \Delta^{+}\colon E(\bar{4}) \times EC(\Delta/2)) \to \mathbf{Top}^{*}.$$

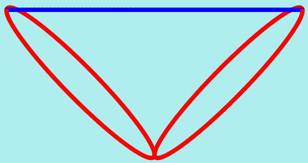
Finally, we can give maps $(S \wedge \Delta^+)(a)$, $(S \wedge \Delta^+)(r)$, $(S \wedge \Delta^+)(l)$, $(S \wedge \Delta^+)(\gamma_r)$, $(S \wedge \Delta^+)(\gamma_l)$, $(S \wedge \Delta^+)(p)$, $(S \wedge \Delta^+)(t)$ to obtain the desired functor

$$S \wedge \Delta^+ \colon \mathbf{E}(\mathbf{E}(\bar{\mathbf{4}}) \times \mathbf{EC}(\boldsymbol{\Delta}/\mathbf{2})) \to \mathbf{Top}^*.$$

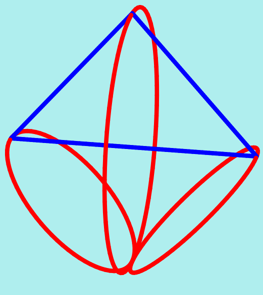
$S \wedge \Delta^+(1[0])$



$S \wedge \Delta^+(1[1])$



$S \wedge \Delta^+(1[2])$



4.3. Singular and realization functors. The categorical group of a presheaf

$S \wedge \Delta^+ \colon \mathbf{E}(\mathbf{E}(\bar{\mathbf{4}}) \times \mathbf{EC}(\boldsymbol{\Delta}/\mathbf{2})) \longrightarrow \mathbf{Top}^*$ induces a pair of adjoint functors

$$\text{Sing} \colon \mathbf{Top}^* \longrightarrow \mathbf{Set}^{E(E(\bar{4}) \times EC(\Delta/2)))^{op}}$$

$$| \cdot | \colon \mathbf{Set}^{E(E(\bar{4}) \times EC(\Delta/2)))^{op}} \longrightarrow \mathbf{Top}^*$$

We will denote by

$$\mathbf{Set}_{pp}^{E(E(\bar{4}) \times EC(\Delta/2)))^{op}}$$

the category of presheaves $X \colon (E(E(\bar{4}) \times EC(\Delta/2)))^{op} \longrightarrow \mathbf{Set}$ such that $X(i,-)$ transforms the pushouts of $C(\Delta/2)$ in pullbacks and $X(-,[j])$ transforms the coproducts of $\bar{4}$ in products.

Given a presheaf X in $\mathbf{Set}_{pp}^{E(E(\bar{4})\times EC(\Delta/2)))^{op}}$ one can define its fundamental categorical group $G(X)$ as a quotient object. This gives a functor

$$G\colon \mathbf{Set}_{pp}^{E(E(\bar{4})\times EC(\Delta/2)))^{op}} \longrightarrow \mathbf{CG}$$

Proposition 4.1 *The functor* $G\colon \mathbf{Set}_{pp}^{E(E(\bar{4})\times EC(\Delta/2)))^{op}} \longrightarrow \mathbf{CG}$ *is left adjoint to the forgetful functor* $U\colon \mathbf{CG} \to \mathbf{Set}_{pp}^{E(E(\bar{4})\times EC(\Delta/2)))^{op}}$ *.*

The composites $\rho_2 = G\,\mathrm{Sing}$, $B = |\cdot|\,U$

$$\rho_2\colon \mathbf{Top}^* \to \mathbf{CG}$$

$$B\colon \mathbf{CG} \to \mathbf{Top}^*$$

will be called the *fundamental categorical group* and *classifying* functors.

5. The categorical groups $\rho_2, \rho_2^{\mathbb{N}}, \rho_2^{\mathbb{R}_+}$ and long exact sequences

For a given pointed topological space X, we can consider its fundamental categorical group

$$\rho_2(X) = G\,\mathsf{Sing}(X)$$

and its higher dimensional analogues $\rho_{q+2}(X) = G\,\mathsf{Sing}\,\Omega^q(X)$, where Ω is the loop functor.
Given an object (X, λ) in the category $\mathbf{E}^{\mathbb{R}_+}$, one has the pointed spaces $(X^{\mathbb{R}_+}, \lambda)$, $(X^{\mathbb{N}}, \lambda|_{\mathbb{N}})$ and the restriction fibration $\mathrm{res}\colon \mathrm{X}^{\mathbb{R}_+} \to \mathrm{X}^{\mathbb{N}}$, $\mathrm{res}(\mu) = \mu|_{\mathbb{N}}$. The fibre is the space

$$F_{\mathrm{res}} = \{\mu \in X^{\mathbb{R}_+} |\ \mu|_{\mathbb{N}} = \lambda|_{\mathbb{N}}\}$$

Denote $\mu_i = \mu|_{[i,i+1]}$. The maps $\varphi\colon (F_{\mathrm{res}}, \lambda) \to \Omega(X^{\mathbb{N}}, \lambda)$, $\phi\colon \Omega(X^{\mathbb{N}}, \lambda) \to (F_{\mathrm{res}}, \lambda)$, given by $\varphi(\mu) = (\mu_0\lambda_0^{-1}, \mu_1\lambda_1^{-1}, \cdots)$ for $\mu \in F_{\mathrm{res}}$ and $\phi(\alpha) = (\alpha_0\lambda_0, \alpha_1\lambda_1, \cdots)$ for $\alpha \in \Omega(X^{\mathbb{N}}, \lambda)$, determine a pointed homotopy equivalence.

Therefore, the pointed map $\mathrm{res}\colon \mathrm{X}^{\mathbb{R}_+} \to \mathrm{X}^{\mathbb{N}}$ induces the fibre sequence

$$\cdots \to \Omega^2(X^{\mathbb{N}}) \to \Omega^2(X^{\mathbb{N}}) \to \Omega(X^{\mathbb{R}_+}) \to \Omega(X^{\mathbb{N}}) \to \Omega(X^{\mathbb{N}}) \to X^{\mathbb{R}_+} \to X^{\mathbb{N}}$$

We define the $\mathbb{R}_+$-*fundamental exterior categorical group* by

$$\rho_2^{\mathbb{R}_+}(X) = \rho_2(X^{\mathbb{R}_+})$$

and the $\mathbb{N}$-*fundamental exterior categorical group* by

$$\rho_2^{\mathbb{N}}(X) = \rho_2(X^{\mathbb{N}}) \ .$$

In the obvious way we have the higher analogues and we can consider fundamental groupoids for the one dimensional cases

$$\rho_1^{\mathbb{R}_+}(X) = \rho_1(X^{\mathbb{R}_+}), \quad \rho_1^{\mathbb{R}_+}(X) = \rho_1(X^{\mathbb{R}_+}).$$

All these exterior homotopy invariants are related as follows:

Theorem 5.1 *Given an exterior space X with a base ray $\lambda\colon \mathbb{R}_+ \to X$ there is a long exact sequence*

$$\cdots \to \rho_q^{\mathbb{R}_+}(X) \to \rho_q^{\mathbb{N}}(X) \to \rho_q^{\mathbb{N}}(X) \to \rho_{q-1}^{\mathbb{R}_+}(X) \to$$

$$\cdots \to \rho_3^{\mathbb{R}_+}(X) \to \rho_3^{\mathbb{N}}(X) \to \rho_3^{\mathbb{N}}(X) \to \rho_2^{\mathbb{R}_+}(X) \to \rho_2^{\mathbb{N}}(X) \to \rho_2^{\mathbb{N}}(X) \to$$
$$\rho_1^{\mathbb{R}_+}(X) \to \rho_1^{\mathbb{N}}(X)$$

which satifies the following properties:

1. $\rho_1^{\mathbb{N}}(X)$, $\rho_1^{\mathbb{R}_+}(X)$ *have the structure of a groupoid.*
2. $\rho_2^{\mathbb{N}}(X)$, $\rho_2^{\mathbb{R}_+}(X)$ *have the structure of a categorical group.*
3. $\rho_3^{\mathbb{N}}(X)$, $\rho_3^{\mathbb{R}_+}(X)$ *have the structure of a braided categorical group.*
4. $\rho_q^{\mathbb{N}}(X)$, $\rho_q^{\mathbb{R}_+}(X)$ *have the structure of a symmetric categorical group for* $q \geq 4$ *.*

6. Exterior $\mathbb{N}$-2-types and global towers of categorical groups

$\mathcal{C}$

$\mathrm{pro}\,\mathcal{C}$ pro-objects X in $\mathcal{C}$ ($X\colon J \to \mathcal{C}$ functor, J left-filtering small category)

$\mathrm{pro}^{+}\mathcal{C}$ global pro-objects Y in $\mathcal{C}$ ($Y\colon K \to \mathcal{C}$ functor, K left-filtering small category with final object, pro-morphisms compatible with the final object)

$\mathrm{tow}\,\mathcal{C}$ towers X in $\mathcal{C}$ ($X\colon \mathbf{N} \to \mathcal{C}$ functor, $\mathbf{N}$ natural numbers)

$\mathrm{tow}^{+}\mathcal{C}$ global towers Y in $\mathcal{C}$ ($Y\colon \mathbf{N} \to \mathcal{C}$ functor, $\mathbf{N}$ natural numbers with the final object 0)

For $\mathbf{Top}^{*}$ and $\mathbf{CG}$, we have

$$\mathrm{pro}^{+}\mathbf{Top}^{*}, \quad \mathrm{pro}^{+}\mathbf{CG}, \quad \mathrm{tow}^{+}\mathbf{Top}^{*}, \quad \mathrm{tow}^{+}\mathbf{CG}$$

The fundamental categorical group and classifying functors

$$\rho_2\colon \mathbf{Top}^* \to \mathbf{CG}, \quad B\colon \mathbf{CG} \to \mathbf{Top}^*$$

induce

$$\mathrm{pro}^+\rho_2\colon \mathrm{pro}^+\mathbf{Top}^* \to \mathrm{pro}^+\mathbf{CG}, \quad \mathrm{tow}^+\rho_2\colon \mathrm{tow}^+\mathbf{Top}^* \to \mathrm{tow}^+\mathbf{CG}$$

$$\mathrm{pro}^+\mathrm{B}\colon \mathrm{pro}^+\mathbf{CG} \to \mathrm{pro}^+\mathbf{Top}^*, \quad \mathrm{tow}^+\mathrm{B}\colon \mathrm{tow}^+\mathbf{CG} \to \mathrm{tow}^+\mathbf{Top}^*$$

Given a exterior space $(X, \lambda) \in \mathbf{E}^{\mathbb{R}+}$ the externology ε_X can be seen as a left-filtering category with a final object and we can consider the functor

$$\varepsilon(X)\colon \varepsilon_X \to \mathbf{Top}^*, \varepsilon(X)(E) = (E \cup [0, \infty)/t \sim \lambda(t), 0),\ t \in \lambda^{-1}(E)$$

This induces a full embedding

$$\varepsilon\colon \mathbf{E}^{\mathbb{R}+} \to \mathrm{pro}^+\mathbf{Top}^*$$

An exterior space is said to be first countable at infinity if there is a countable base of the externology

$$X = E_0 \supset E_1 \supset E_2 \supset \cdots$$

$\mathbf{E}^{\mathbb{R}+}_{fc}$ rayed spaces first countable at infinity. There is an induced functor

$$\varepsilon\colon \mathbf{E}^{\mathbb{R}_+}_{fc} \to \mathrm{tow}^+\mathbf{Top}^*$$

We also can consider the Telescopic construction Tel: $\mathrm{tow}^{+}\mathbf{Top}^{*} \to \mathbf{E}_{\mathrm{fc}}^{\mathbb{R}_{+}}$
Using all these functors one can prove

Theorem 6.1 *The functors* $\mathrm{tow}^{+}\rho_{2}\varepsilon$ *and Tel* $\mathrm{tow}^{+}\mathrm{B}$ *induce an equivalence of categories*

$$\mathbf{E}_{fc}^{\mathbb{R}_{+}}[\Sigma_{\mathbb{N}}]^{-1} \to \mathrm{tow}^{+}\mathbf{CG}[\Sigma]^{-1}$$

where Σ *is the class of maps in* $\mathrm{tow}^{+}\mathbf{CG}$ *given by the closure of the level weak equivalences.*

7. Exterior $\mathbb{R}_+$-2-types and the $\mathbb{R}_+$-fundamental exterior categorical group

Consider the functor $p\colon \mathbf{Top}^* \longrightarrow \mathbf{E}^{\mathbb{R}_+}$ given by

$$p(X) = \mathbb{R}_+ \bar{\times} X$$

The functor p induces a covariant functor

$$p(S \wedge \Delta^+)\colon E(E(\bar{4}) \times EC(\Delta/2)) \longrightarrow \mathbf{E}^{\mathbb{R}_+}$$

and the corresponding singular an realization functors

$$\mathsf{Sing}^{\mathbb{R}_+}\colon \mathbf{E}^{\mathbb{R}_+} \longrightarrow \mathbf{Set}_{pp}^{E(E(\bar{4})\times EC(\Delta/2)))^{op}}$$

$$|\cdot|^{\mathbb{R}_+}\colon \mathbf{Set}_{pp}^{E(E(\bar{4})\times EC(\Delta/2)))^{op}} \longrightarrow \mathbf{E}^{\mathbb{R}_+}$$

On the other hand, we also have the adjunction

$$G\colon \mathbf{Set}_{pp}^{E(E(\bar{4})\times EC(\Delta/2)))^{op}} \longrightarrow \mathbf{CG}$$

$$U\colon \mathbf{CG} \longrightarrow \mathbf{Set}_{pp}^{E(E(\bar{4})\times EC(\Delta/2)))^{op}}$$

Taking the composites $G\mathsf{Sing}^{\mathbb{R}_+} \cong \rho_2^{\mathbb{R}_+}$ and $B^{\mathbb{R}_+} = |\cdot|^{\mathbb{R}_+}U$, one has that

Theorem 7.1 *The functors $\rho_2^{\mathbb{R}_+}$ and $B^{\mathbb{R}_+}$ induce an equivalence of categories*

$$\mathbf{E}^{\mathbb{R}_+}[\Sigma_{\mathbb{R}_+}]^{-1} \longrightarrow \mathbf{CG}[\Sigma]^{-1}$$

where Σ is the class weak equivalences (equivalences) in $\mathbf{CG}$.

8. Exterior $\mathbb{N}$-2-types and the $\mathbb{N}$-fundamental exterior categorical group

Consider the functor $c\colon \mathbf{Top}^* \to \mathbf{E}^{\mathbb{R}_+}$ given by

$$c(X) = (\mathbb{R}_+ \sqcup (\sqcup_n X))/n \sim *_n$$

where $n \geq 0$ is a natural number and $*_n$ denotes the base point of the corresponding copy of X .
The functor c induces the covariant functor

$$c(S \wedge \Delta^+)\colon E(E(\bar{4}) \times EC(\Delta/2)) \to \mathbf{E}^{\mathbb{R}_+}$$

and the corresponding singular an realization functors

$$\mathsf{S}^{\mathbb{N}}\colon \mathbf{E}^{\mathbb{R}_+} \to \mathbf{Set}_{pp}^{E(E(\bar{4})\times EC(\Delta/2)))^{op}}$$

$$R^{\mathbb{N}}\colon \mathbf{Set}_{pp}^{E(E(\bar{4})\times EC(\Delta/2)))^{op}} \to \mathbf{E}^{\mathbb{R}_+}$$

but the composites $G\,\mathsf{S}^{\mathbb{N}} \cong \rho_2^{\mathbb{N}}$ and $R^{\mathbb{N}}\,U$ does not induce an equivalence of exterior $\mathbb{N}$-2-types and categorical groups up to equivalence.

Take an exterior rayed space X (for example, $X = \mathbb{R}_+ \bar{\times} S^1$) such that

$\text{lim} \mathrm{tow}\, \pi_1 \varepsilon(\mathrm{X}) \neq 1$

We can prove that the space $R^{\mathbb{N}} U \rho_2^{\mathbb{N}}(X)$ satisfies that

$\text{lim} \mathrm{tow}\, \pi_1 \varepsilon(\mathrm{X}) = 1$

This implies that X and $R^{\mathbb{N}} U \rho_2^{\mathbb{N}}(X)$ have different $\mathbb{N}$-1-type and then different $\mathbb{N}$-2-type.

Open question: Is it possible to modify the notion of categorical group to obtain an new algebraic model for $\mathbb{N}$-2-types?

A partial answer is obtained by taking a monoid $\mathbb{M}$ of endomorphisms of the exterior space $\mathbb{R}_+ \sqcup (\sqcup_n S^1))/n \sim *_n$, and a new extension of the category $\bar{4}$ obtained by adding an arrow for each element of the monoid. This gives a new type of presheaf that will induce a categorical group enriched with an action of the monoid $\mathbb{M}$.

We think that the new enriched categorical group and realization functors will give an equivalence of a large class of exterior $\mathbb{N}$-2-types and the corresponding $\mathbb{M}$-categorical groups. This class of exterior $\mathbb{N}$-2-types contains the subcategory of proper $\mathbb{N}$-2-types. Consequently, we will obtain a category of algebraic models for proper $\mathbb{N}$-2-types.

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