Closed Multicategory of A_∞-Categories

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Motivation

Sources of interest in A_∞-categories:

- Kontsevich's Homological Mirror Symmetry Conjecture;
- recent advances in homological algebra (Bondal–Kapranov, Drinfeld, Keller, Kontsevich-Soibelman, ...).

Question: What do A_∞-categories form?

Our answer: A closed symmetric multicategory.

A short review of A_∞-categories

Throughout, $\Bbbk$ is a commutative ground ring.

Definition. A *graded quiver* $\mathcal{A}$ consists of a set $\operatorname{Ob}\mathcal{A}$ of objects and a graded $\Bbbk$-module $\mathcal{A}(X,Y)$, for each $X,Y \in \operatorname{Ob}\mathcal{A}$. A *morphism of graded quivers* $f:\mathcal{A}\to\mathcal{B}$ consists of a function $\operatorname{Ob} f:\operatorname{Ob}\mathcal{A}\to\operatorname{Ob}\mathcal{B}$, $X\mapsto Xf$ and a $\Bbbk$-linear map $f=f_{X,Y}:\mathcal{A}(X,Y)\to\mathcal{B}(Xf,Yf)$ of degree 0, for each $X,Y\in\operatorname{Ob}\mathcal{A}$.

Let $\mathscr{Q}$ denote the category of graded quivers. It is symmetric monoidal. The *tensor product* of graded quivers $\mathcal{A}$ and $\mathcal{B}$ is the graded quiver $\mathcal{A}\boxtimes\mathcal{B}$ given by

$$\operatorname{Ob}(\mathcal{A}\boxtimes\mathcal{B}) = \operatorname{Ob}\mathcal{A}\times\operatorname{Ob}\mathcal{B}$$

$$(\mathcal{A}\boxtimes\mathcal{B})((X,U),(Y,V)) = \mathcal{A}(X,Y)\otimes\mathcal{B}(U,V).$$

The *unit object* is the graded quiver $\mathbb{1}$ with $\operatorname{Ob}\mathbb{1}=\{*\}$ and $\mathbb{1}(*,*)=\Bbbk$.

Definition. For a set S, denote by $\mathscr{Q}/S$ the subcategory of $\mathscr{Q}$ whose objects are graded quivers $\mathcal{A}$ such that $\operatorname{Ob}\mathcal{A} = S$ and whose morphisms are morphisms of graded quivers $f : \mathcal{A} \to \mathcal{B}$ such that $\operatorname{Ob} f = \mathrm{id}_S$.

The category $\mathscr{Q}/S$ is (non-symmetric) monoidal. The *tensor product* of graded quivers $\mathcal{A}$, $\mathcal{B}$ is the graded quiver $\mathcal{A} \otimes \mathcal{B}$ given by

$$(\mathcal{A} \otimes \mathcal{B})(X, Z) = \bigoplus_{Y \in S} \mathcal{A}(X, Y) \otimes \mathcal{B}(Y, Z), \qquad X, Z \in S.$$

The *unit object* is the *discrete quiver* $\Bbbk S$ given by $\operatorname{Ob} \Bbbk S = S$ and

$$(\Bbbk S)(X, Y) = \begin{cases} \Bbbk & \text{if } X = Y, \\ 0 & \text{if } X \neq Y, \end{cases} \qquad X, Y \in S.$$

Definition. An *augmented graded cocategory* is a graded quiver $\mathcal{C}$ equipped with the structure of an augmented counital coassociative coalgebra in the monoidal category $\mathscr{Q}/\operatorname{Ob}\mathcal{C}$. Therefore, $\mathcal{C}$ comes with

- a *comultiplication* $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$,
- a *counit* $\varepsilon : \mathcal{C} \to \Bbbk \operatorname{Ob}\mathcal{C}$, and
- an *augmentation* $\eta : \Bbbk \operatorname{Ob}\mathcal{C} \to \mathcal{C}$,

which are morphisms in $\mathscr{Q}/\operatorname{Ob}\mathcal{C}$ satisfying the usual axioms.

A *morphism of augmented graded cocategories* $f : \mathcal{C} \to \mathcal{D}$ is a morphism of graded quivers that preserves the comultiplication, counit, and augmentation.

The category of augmented graded cocategories is a symmetric monoidal category with the tensor product inherited from $\mathscr{Q}$.

Example. Let $\mathcal{A}$ be a graded quiver. The quiver

$$T\mathcal{A} = \bigoplus_{n=0}^{\infty} T^n\mathcal{A},$$

where $T^n\mathcal{A}$ is the n-fold tensor product of $\mathcal{A}$ in $\mathscr{Q}/\operatorname{Ob}\mathcal{A}$, equipped with the 'cut' comultiplication

$$\Delta_0 : f_1 \otimes \cdots \otimes f_n \mapsto \sum_{k=0}^{n} f_1 \otimes \cdots \otimes f_k \bigotimes f_{k+1} \otimes \cdots \otimes f_n,$$

the counit

$$\varepsilon = \mathrm{pr}_0 : T\mathcal{A} \to T^0\mathcal{A} = \Bbbk \operatorname{Ob}\mathcal{A},$$

and the augmentation

$$\eta = \mathrm{in}_0 : \Bbbk \operatorname{Ob}\mathcal{A} = T^0\mathcal{A} \hookrightarrow T\mathcal{A}$$

is an augmented graded cocategory.

For a graded quiver $\mathcal{A}$, denote by $s\mathcal{A}$ its *suspension*:

$$\operatorname{Ob} s\mathcal{A} = \operatorname{Ob} \mathcal{A}, \qquad (s\mathcal{A}(X,Y))^d = \mathcal{A}(X,Y)^{d+1}, \quad X, Y \in \operatorname{Ob} \mathcal{A}.$$

Let $s : \mathcal{A} \to s\mathcal{A}$ denote the 'identity' map of degree -1.

Definition. An A_∞*-category* consist of a graded quiver $\mathcal{A}$ and a differential $b : Ts\mathcal{A} \to Ts\mathcal{A}$ of degree 1 such that $(Ts\mathcal{A}, \Delta_0, \mathrm{pr}_0, \mathrm{in}_0, b)$ is an *augmented differential graded cocategory*, i.e.,

$$b^2 = 0, \qquad b\Delta_0 = \Delta_0(1 \otimes b + b \otimes 1), \qquad b\,\mathrm{pr}_0 = 0, \qquad \mathrm{in}_0\, b = 0.$$

For A_∞-categories $\mathcal{A}$ and $\mathcal{B}$, an A_∞-functor $f : \mathcal{A} \to \mathcal{B}$ is a morphism of augmented differential graded cocategories $f : (Ts\mathcal{A}, b) \to (Ts\mathcal{B}, b)$.

Even better: we can define A_∞-functors of many arguments!

A short review of multicategories

A *multicategory* is just like a category, the only difference being the shape of arrows. An arrow in a multicategory looks like

$$X_1, X_2, \ldots, X_n \longrightarrow Y$$

with a finite family of objects as the source and one object as the target, and composition turns a tree of arrows into a single arrow.

Example. An arbitrary (symmetric) monoidal category $\mathcal{C}$ gives rise to a (symmetric) multicategory $\widehat{\mathcal{C}}$ with the same set of objects. A morphism

$$X_1, \ldots, X_n \longrightarrow Y$$

in $\widehat{\mathcal{C}}$ is a morphism

$$X_1 \otimes \cdots \otimes X_n \longrightarrow Y$$

in $\mathcal{C}$. Composition in $\widehat{\mathcal{C}}$ is derived from composition and tensor in $\mathcal{C}$.

Closed multicategories

A multicategory C is *closed* if, for each $X_i, Z \in \operatorname{Ob}\mathsf{C}$, $i \in I$, there exist an *internal* Hom-*object* $\underline{\mathsf{C}}((X_i)_{i\in I}; Z)$ and an *evaluation* morphism

$$\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};Z} : (X_i)_{i\in I}, \underline{\mathsf{C}}((X_i)_{i\in I}; Z) \longrightarrow Z$$

satisfying the following universal property: an arbitrary morphism

$$(X_i)_{i\in I}, (Y_j)_{j\in J} \longrightarrow Z$$

can be written in a unique way as

$$\bigl[(X_i)_{i\in I}, (Y_j)_{j\in J} \xrightarrow{(1_{X_i})_{i\in I}, f} (X_i)_{i\in I}, \underline{\mathsf{C}}((X_i)_{i\in I}; Z) \xrightarrow{\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};Z}} Z\bigr].$$

Example. Let $\mathcal{C}$ be a monoidal category. It is closed if and only if so is the associated multicategory $\widehat{\mathcal{C}}$.

Main theorem

The symmetric multicategory A_∞ of A_∞-categories is defined as follows.

- Objects are A_∞-categories.
- A morphism

$$f : \mathcal{A}_1, \dots, \mathcal{A}_n \longrightarrow \mathcal{B},$$

  called an A_∞-*functor*, is a morphism of augmented differential graded cocategories

$$f : Ts\mathcal{A}_1 \boxtimes \cdots \boxtimes Ts\mathcal{A}_n \longrightarrow Ts\mathcal{B}.$$

Theorem. *The multicategory* A_∞ *is closed.*

Basic ideas of proof

Step 1. The category $\mathscr{Q}$ of graded quivers admits a different symmetric monoidal structure with tensor product given by

$$\mathcal{A} \boxtimes_u \mathcal{B} \overset{\text{def}}{=} (\mathcal{A} \boxtimes \mathcal{B}) \oplus (\Bbbk \operatorname{Ob} \mathcal{A} \boxtimes \mathcal{B}) \oplus (\mathcal{A} \boxtimes \Bbbk \operatorname{Ob} \mathcal{B}),$$

and the unit object being the graded quiver $\mathbb{1}_u$ with $\operatorname{Ob} \mathbb{1}_u = \{*\}$ and $\mathbb{1}_u(*, *) = 0$. Let $\mathscr{Q}_u$ denote the category $\mathscr{Q}$ with this symmetric monoidal structure.

Proposition. *The symmetric monoidal category $\mathscr{Q}_u$ is closed.*

Step 2. For a graded quiver $\mathcal{A}$, denote by

$$T^{\geq 1}\mathcal{A} = \bigoplus_{n=1}^{\infty} T^n \mathcal{A}$$

the reduced tensor quiver.

Proposition. *The functor $T^{\geq 1} : \mathscr{Q} \to \mathscr{Q}$ admits the structure of a lax symmetric monoidal comonad in the closed symmetric monoidal category $\mathscr{Q}_u$.*

In particular, $T^{\geq 1}$ gives rise to a symmetric multicomonad $T^{\geq 1}$ in the closed symmetric multicategory $\widehat{\mathscr{Q}_u}$.

Theorem. *Let T be a symmetric multicomonad in a closed symmetric multicategory C. Then the multicategory of free T-coalgebras is closed.*

Proposition. *There is an isomorphism of symmetric multicategories*

$$\left\{ \begin{array}{c} \text{free} \\ T^{\geq 1}\text{-coalgebras} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{augmented graded} \\ \text{cocategories of the form } T\mathcal{A} \end{array} \right\}.$$

In particular, the multicategory in the right hand side is closed.

Step 3. Add differentials.

Question. Is the symmetric monoidal category of augmented (differential) graded cocategories closed?

We do not know the answer in general…

Summary

- A_∞-categories naturally form a symmetric multicategory.
- This multicategory is closed.

Outlook

- Unital A_∞-categories (A_∞-categories with weak identities). We prove that unital A_∞-categories and unital A_∞-functors constitute a closed symmetric submulticategory of A_∞.
- Closed multicategories vs. closed categories in the sense of Eilenberg-Kelly. We prove that these are basically the same (suitably defined 2-categories of closed multicategories and closed categories are 2-equivalent).