Free Constructions on Double Categories

Dorette Pronk, Dalhousie University

with

Tom Fiore, University of Chicago

Simona Paoli, Macquarie University

CT2007, Portugal, June 19, 2007

Overview

- Definitions
  - Double Categories
  - Double Derivation Schemes

- Free Constructions
  - Free double category on a DDS
  - Horizontal Categorification

- Motivation: Model Structures on **DblCat**

- Some Pushouts of Double Categories

Double Categories

Definition A **double category** $\mathbb{D}$ is an internal category in **Cat**:

$$\mathbb{D} = \mathbf{D}_1 \rightrightarrows \mathbf{D}_0 = \begin{array}{ccc} \mathbf{D}_{1,1} & \rightrightarrows & \mathbf{D}_{0,1} \\ \downdownarrows & & \downdownarrows \\ \mathbf{D}_{1,0} & \rightrightarrows & \mathbf{D}_{0,0} \end{array}$$

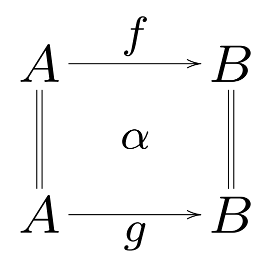
We will write

$$\mathsf{Obj}\,\mathbb{D} = \mathbf{D}_{0,0} \quad D_0, D_1, \ldots \qquad \mathsf{Hor}\,\mathbb{D} = \mathbf{D}_{1,0} \quad D_0 \xrightarrow{h} D_1$$

$$\mathsf{Ver}\,\mathbb{D} = \mathbf{D}_{0,1} \quad \begin{array}{c} D_0 \\ v \bullet\!\!\downarrow \\ D_1 \end{array} \qquad \mathsf{Sq}\,\mathbb{D} = \mathbf{D}_{1,1} \quad \begin{array}{ccc} D_0 & \xrightarrow{h_0} & D_0' \\ v \bullet\!\!\downarrow & \alpha & \downarrow\!\!\bullet v' \\ D_1 & \xrightarrow[h_1]{} & D_1' \end{array}$$

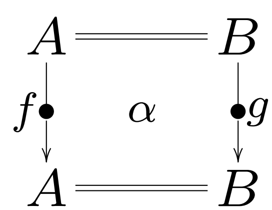
Examples

1. The horizontal embedding $I_H(\mathbf{C})$ of a 2-category $\mathbf{C}$, with squares



for $\alpha\colon f \Rightarrow g$ in $\mathbf{C}$.

2. The vertical embedding $I_V(\mathbf{C})$ of a 2-category $\mathbf{C}$, with squares



for $\alpha\colon f \Rightarrow g$ in $\mathbf{C}$.

3. The **external product** $\mathbf{C} \boxtimes \mathbf{D}$ of two 2-categories $\mathbf{C}$ and $\mathbf{D}$:

objects (C, D)

vertical morphisms $(f, D) : (C, D) \to\!\!\bullet\!\!\to (C', D)$

horizontal morphisms $(C, g) : (C, D) \longrightarrow (C, D')$

squares

$$\begin{array}{ccc} (C, D) & \xrightarrow{(C, g_1)} & (C, D') \\ {\scriptstyle (f_1, D)} \bullet\!\downarrow & (\gamma, \delta) & \downarrow\!\bullet {\scriptstyle (f_2, D')} \\ (C', D) & \xrightarrow[(C, g_2)]{} & (C', D') \end{array} \qquad \begin{array}{c} f_1 \overset{\gamma}{\Longrightarrow} f_2 \\ g_1 \overset{\delta}{\Longrightarrow} g_2 \end{array}$$

Lemma The external product is a functor

$$\boxtimes : \textbf{2-Cat} \times \textbf{2-Cat} \longrightarrow \textbf{DblCat}.$$

Note: If $\mathbf{C}$ and $\mathbf{D}$ are categories, the squares of $\mathbf{C} \boxtimes \mathbf{D}$ are

$$\begin{array}{ccc} (C,D) & \xrightarrow{(C,g)} & (C,D') \\ {\scriptstyle (f,D)}\downarrow & & \downarrow{\scriptstyle (f,D')} \\ (C',D) & \xrightarrow[(C',g)]{} & (C',D'). \end{array}$$

If $\mathbf{D} = [n] = (\mathbf{0} \overset{s_{1,0}}{\to} \mathbf{1} \overset{s_{2,1}}{\to} \cdots \overset{s_{n,n-1}}{\to} \mathbf{n})$, every square has a canonical horizontal factorization

$$\begin{array}{ccc} (C,\mathbf{k}) & \xrightarrow{(C,s_{m,k})} & (C,\mathbf{m}) \\ {\scriptstyle (f,\mathbf{k})}\downarrow & & {\scriptstyle (f,\mathbf{m})}\downarrow \\ (C',\mathbf{k}) & \xrightarrow[(C',s_{m,k})]{} & (C',\mathbf{m}) \end{array} =$$

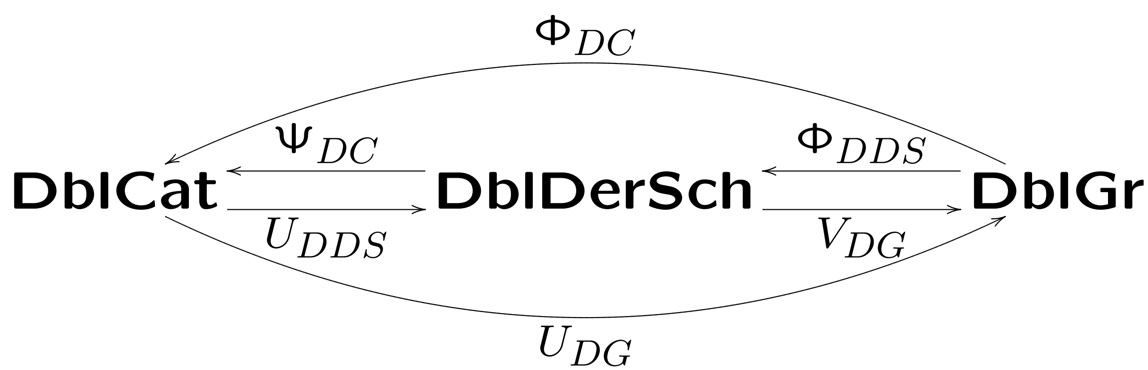
$$\begin{array}{ccccccc} (C,\mathbf{k}) & \xrightarrow{(C,s_{k+1,k})} & (C,\mathbf{k+1}) & \xrightarrow{(C,s_{k+2,k+1})} & & \xrightarrow{(C,s_{m,m-1})} & (C,\mathbf{m}) \\ {\scriptstyle (f,\mathbf{k})}\downarrow & & {\scriptstyle (f,\mathbf{k+1})}\downarrow & & \cdots & & {\scriptstyle (f,\mathbf{m})}\downarrow \\ (C',\mathbf{k}) & \xrightarrow[(C',s_{k+1,k})]{} & (C',\mathbf{k+1}) & \xrightarrow[(C',s_{k+2,k+1})]{} & & \xrightarrow[(C',s_{m,m-1})]{} & (C',\mathbf{m}). \end{array}$$

Double Graphs and Double Derivation Schemes

Definition 1. A **double graph** is an internal reflexive graph in **RGraph**.

2. A **double derivation scheme** is a double graph whose vertical and horizontal graphs are categories.

There are adjunctions:

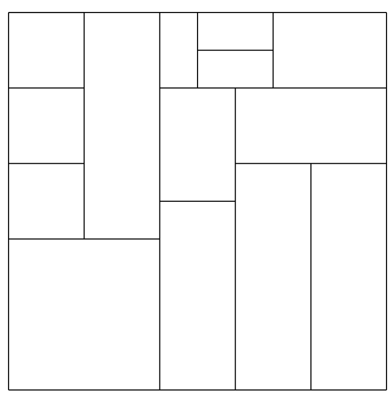


The free double category on a double derivation scheme

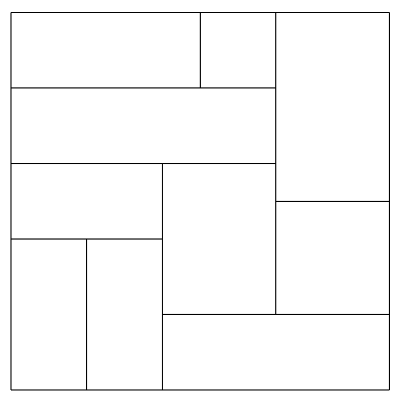
Let $\mathbb{S}$ be a double derivation scheme. The squares of the free double category $\Psi_{DC}(\mathbb{S})$ are equivalence classes of allowable pasting diagrams of squares in $\mathbb{S}$.

1. The squares of $\mathbb{S}$ are allowable.
2. A pasting diagram is allowable when it admits a vertical or horizontal cut which creates two allowable pasting diagrams.

Allowable:



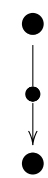
Not allowable:



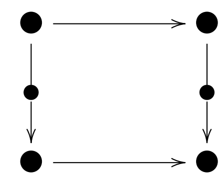
The Horizontal Nerve

Definition The horizontal nerve of a double category is the simplicial category:

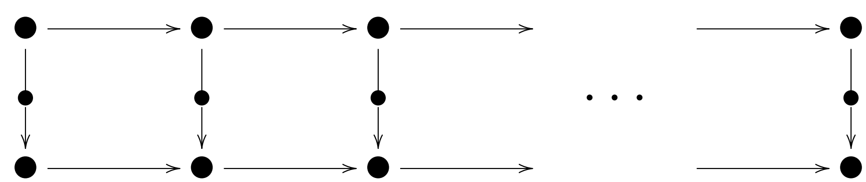
$$(N_h\mathbb{D})_0 = (\mathsf{Obj}\,\mathbb{D}, \mathsf{Ver}\,\mathbb{D})$$



$$(N_h\mathbb{D})_1 = (\mathsf{Hor}\,\mathbb{D}, \mathsf{Sq}\,\mathbb{D})$$



$$(N_h\mathbb{D})_n = (\text{paths of } n \text{ horizontal arrows}, \text{rows of } n \text{ squares})$$



Composition in each category is vertical composition.

Example $N_h(\mathbf{A} \boxtimes \mathbf{B}) = (\underline{\underline{\mathbf{A}}})_* \times dN\mathbf{B}_*$, where

$$(\underline{\underline{\mathbf{A}}})_n = \mathbf{A} \text{ for all } n$$

and

$$dN\mathbf{B}_n = (N\mathbf{B}_n)_{\mathsf{disc}}.$$

Categorification

Recall: the ordinary nerve functor

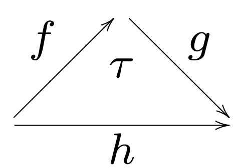
$$N\colon \mathbf{Cat} \to [\Delta^{\mathsf{op}}, \mathbf{Set}]$$

has a left adjoint

$$c\colon [\Delta^{\mathsf{op}}, \mathbf{Set}] \to \mathbf{Cat},$$

the **fundamental category** or **categorification** functor.

For a simplicial set X_*, $c(X_*)$ is the free category on the reflexive graph (X_0, X_1), modulo the smallest congruence such that for every $\tau \in X_2$, with boundaries



we have that $g \circ f \sim h$.

Horizontal Categorification

We define the functor $c_h \colon [\Delta^{\mathsf{op}}, \mathbf{Cat}] \to \mathbf{DblCat}$ in two steps.

Step 1:

$$[\Delta^{\mathsf{op}}, \mathbf{Cat}] \xrightarrow{\quad s_h \quad} \mathbf{DblDerSch}$$

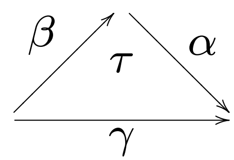
$$\mathbf{X}_* \mapsto s_h(\mathbf{X}_*), \text{ defined by:}$$

Vertical category:	$\mathbf{X}_0$
Horizontal category:	$c(\mathsf{Obj}\,(\mathbf{X}_*))$
Squares:	$\mathsf{Mor}\,\mathbf{X}_1$

Step 2: $c_h(\mathbf{X}_*)$ is a quotient of $\Psi_{DC}(c_h(\mathbf{X}_*))$ by the smallest congruence relation on the squares such that

a. $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \sim [\beta \circ \alpha]$, where the composition is taken from $\mathbf{X}_1$;

b. For each $\tau \in \operatorname{Mor} \mathbf{X}_2$ with simplicial boundaries



we have $[\alpha\, \beta] \sim [\gamma]$.

Proposition Horizontal categorification c_h is left adjoint to the horizontal nerve N_h.

Examples

1. For a simplicial set X_*, let dX_* be the simplicial category with discrete categories,

$$dX_n = (X_n)_{\mathsf{disc}}.$$

Then $c_h(dX_*) = I_H(cX_*)$.

2. Let $\mathbf{A}$ be a category and Y_* a simplicial set. Then

$$c_h((\underline{\mathbf{A}})_* \times dY_*) = \mathbf{A} \boxtimes cY.$$

Motivation - Model Structures on Double Categories

One way to obtain these is to transfer model structures along the adjunction:

$$\mathbf{DblCat} \underset{N_h}{\overset{c_h}{\rightleftarrows}} [\Delta^{\mathrm{op}}, \mathbf{Cat}]$$

Specifically, we want to transfer the levelwise Thomason structure and the levelwise categorical structure on $[\Delta^{\mathrm{op}}, \mathbf{Cat}]$.

Main Technical Lemma

For the pushouts j_1 and j_2

$$\begin{array}{ccc} (cSd^2\Lambda^k[m]) \boxtimes [n] & \longrightarrow & \mathbb{D} \\ {\scriptstyle i\boxtimes 1_{[n]}}\downarrow & & \downarrow{\scriptstyle j_1} \\ (cSd^2\Delta[m]) \boxtimes [n] & \longrightarrow & \mathbb{P}_1 \end{array} \qquad \begin{array}{ccc} * \boxtimes [n] & \longrightarrow & \mathbb{D} \\ {\scriptstyle i\boxtimes 1_{[n]}}\downarrow & & \downarrow{\scriptstyle j_2} \\ I \boxtimes [n] & \longrightarrow & \mathbb{P}_2 \end{array}$$

in **DblCat**, the morphisms

$$N_h(j_1) \text{ and } N_h(j_2)$$

are weak equivalences in the Thomason and categorical (resp.) model structures on $[\Delta^{\mathrm{op}}, \mathbf{Cat}]$.

Theorem Let $\mathbf{A} \subseteq \mathbf{B}$ be a full subcategory satisfying the (FRC), $\mathbb{D}$ a double category, and $\mathbb{P}$ the pushout

$$\begin{array}{ccc} \mathbf{A} \boxtimes [n] & \longrightarrow & \mathbb{D} \\ {\scriptstyle i \boxtimes 1_{[n]}} \downarrow & & \downarrow \\ \mathbf{B} \boxtimes [n] & \longrightarrow & \mathbb{P} \end{array}$$

in **DblCat**. Then the induced map on horizontal nerves

$$N_h(\mathbf{B} \boxtimes [n]) \coprod_{N_h(\mathbf{A} \boxtimes [n])} N_h(\mathbb{D}) \longrightarrow N_h(\mathbb{P})$$

is an isomorphism of simplicial objects in **Cat**.

Pushouts of inclusions - in Set

Lemma If $A \subseteq B$ and D are sets, then the pushout in **Set**

$$\begin{array}{ccc} A & \longrightarrow & D \\ \big\downarrow & & \big\downarrow \\ B & \longrightarrow & P \end{array}$$

is $P = D \coprod (B \backslash A)$.

Pushouts of inclusions - in **Cat**

Lemma If $A \subseteq B$ are sets and $\mathbf{C}$ and $\mathbf{D}$ are categories, then the pushout in **Cat**

$$\begin{array}{ccc} A_{\mathsf{disc}} \times \mathbf{C} & \longrightarrow & \mathbf{D} \\ {\scriptstyle i \times 1_{\mathbf{C}}} \big\downarrow & & \big\downarrow \\ B_{\mathsf{disc}} \times \mathbf{C} & \longrightarrow & \mathbf{P} \end{array}$$

is

$$\mathbf{P} = \mathbf{D} \coprod ((B \backslash A)_{\mathsf{disc}} \times \mathbf{C}).$$

Lemma Let $\mathbf{A} \subseteq \mathbf{B}$ be a full subcategory. Then the pushout

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{F} & \mathbf{D} \\ \big\downarrow & & \big\downarrow \\ \mathbf{B} & \longrightarrow & \mathbf{P} \end{array}$$

in **Cat** can be desribed as:

$$\text{Obj}\,\mathbf{P} = \text{Obj}\,\mathbf{D} \coprod (\text{Obj}\,\mathbf{B} \backslash \text{Obj}\,\mathbf{A})$$

and the morphisms of $\mathbf{P}$ have two forms:

1. Arrows $B_0 \xrightarrow{f} B_1$ with $f \in \mathbf{B} \backslash \mathbf{A}$.
2. Paths

$$X_1 \xrightarrow{f_1} D_1 \xrightarrow{d} D_2 \xrightarrow{f_2} X_2$$

where $d \in \mathbf{D}$, and $f_1, f_2 \in \mathbf{B} \backslash \mathbf{A}$ or identities.

If f_1 is nontrivial, then $D_1 \in \mathbf{A}$.
If f_2 is nontrivial, then $D_2 \in \mathbf{A}$.

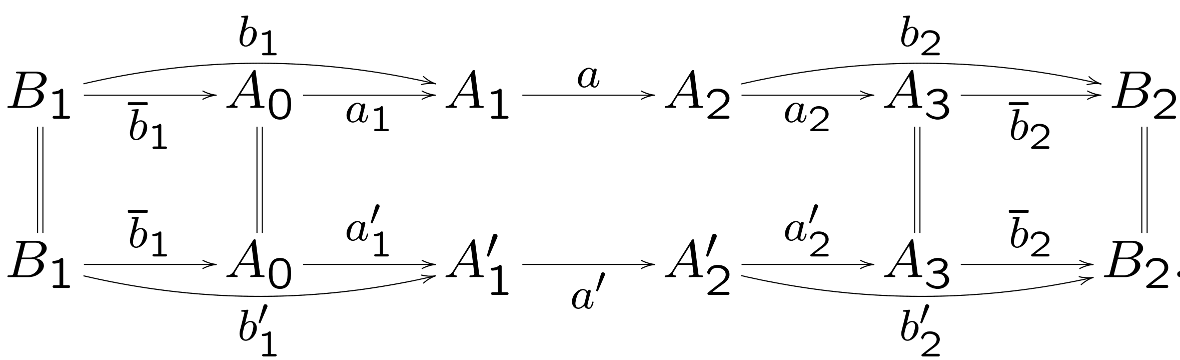
Definition A subcategory $\mathbf{A} \subseteq \mathbf{B}$ satisfies the **Factorization Refinement Condition** (FRC) if for any two paths of morphisms

$$B_1 \xrightarrow{b_1} A_1 \xrightarrow{a} A_2 \xrightarrow{b_2} B_2 \qquad \text{and} \qquad B_1 \xrightarrow{b_1'} A_1' \xrightarrow{a'} A_2' \xrightarrow{b_2'} B_2$$

with $a, a' \in \mathbf{A}$ and $b_2 a b_1 = b_2' a' b_1'$, there are

$$a_1, a_2, a_1', a_2' \in \mathbf{A} \text{ and } \bar{b}_1, \bar{b}_2 \in \mathbf{B}$$

such that



Propositon Let $\mathbf{A} \subseteq \mathbf{B}$ be a full subcategory satisfying the (FRC), and $\mathbb{D}$ a double category. Then the pushout

$$\begin{array}{ccc} \mathbf{A} \boxtimes [n] & \xrightarrow{F} & \mathbb{D} \\ {\scriptstyle i \boxtimes 1_{[n]}} \downarrow & & \downarrow \\ \mathbf{B} \boxtimes [n] & \longrightarrow & \mathbb{P} \end{array}$$

in **DblCat** has the following explicit description.

- Objects:

- Horizontal Arrows:

- Vertical Arrows:

- Squares:

First create the pushout

$$\begin{array}{ccc} \mathbf{A} \boxtimes [n] & \xrightarrow{F} & \mathbb{D} \\ {\scriptstyle i \boxtimes 1_{[n]}} \downarrow & & \downarrow \\ \mathbf{B} \boxtimes [n] & \longrightarrow & \mathbb{S} \end{array}$$

in **DblDerSch** and use the previous lemmas to obtain:

- $\mathrm{Obj}\,\mathbb{P} = \mathrm{Obj}\,\mathbb{S}$: $(B, \mathbf{k})$ and D;

- $\mathrm{Hor}\,\mathbb{P} = \mathrm{Hor}\,\mathbb{S}$: $(B, s_{m,k})$ with $B \in \mathbf{B} \backslash \mathbf{A}$ or $d \in \mathrm{Hor}\,\mathbb{D}$

- $\mathrm{Ver}\,\mathbb{P} = \mathrm{Ver}\,\mathbb{S}$: 1. Arrows $(B_0, \mathbf{k}) \xrightarrow{(b,\mathbf{k})} (B_1, \mathbf{k})$.
  2. Paths $X_1 \xrightarrow{f_1} D_1 \xrightarrow{d} D_2 \xrightarrow{f_2} X_2$ where $d \in \mathrm{Ver}\,\mathbf{D}$, and each of $f_1, f_2 \in \mathbf{B} \backslash \mathbf{A} \times \{\mathbf{k}\}$ or identities.

The squares of $\mathbb{S}$ are:

$$\mathsf{Sq}\,\mathbb{S} = \mathsf{Sq}\,\mathbb{D} \coprod (\mathsf{Sq}\,(\mathbf{B} \boxtimes [n]) \backslash \mathsf{Sq}\,(\mathbf{A} \boxtimes [n])).$$

The squares of $\mathbb{P}$ are equivalence classes of allowable pasting diagrams of squares in $\mathbb{S}$. The equivalence relation is generated by:

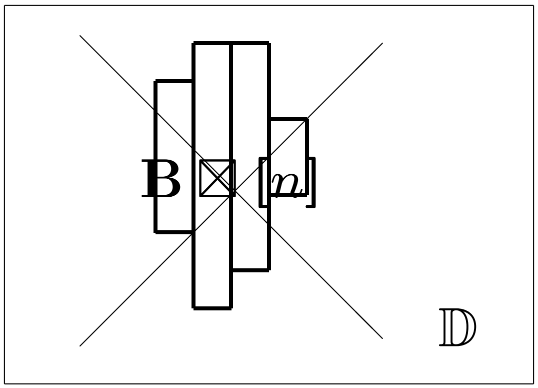
- any rectangular subdiagram containing only squares from one double category may be replaced by its composition in that double category;
- any square may be factored in the double category it came from;
- any $\mathbf{A} \boxtimes [n]$-square is considered as a $\mathbb{D}$-square.

Step 1: Layer the $\mathrm{B} \boxtimes [n]$-squares

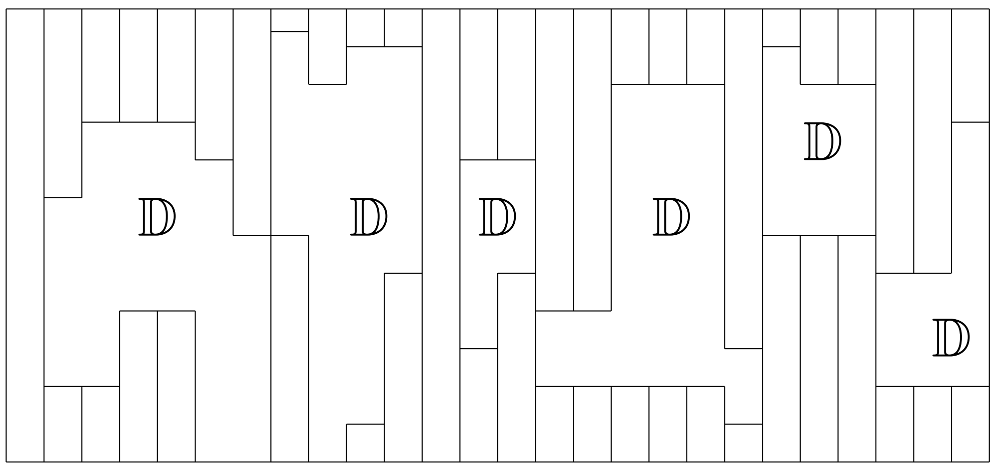
We can factor all $\mathbf{B} \boxtimes [n]$-squares into vertical layers, and then compose these vertically where possible.

Note any $\mathbf{B} \boxtimes [n]$-square which has $\mathbb{D}$-squares above and below it is an $\mathbf{A} \boxtimes [n]$-square and such squares will be viewed as $\mathbb{D}$-squares.

So there are no floating $\mathbf{B} \boxtimes [n]$-squares.

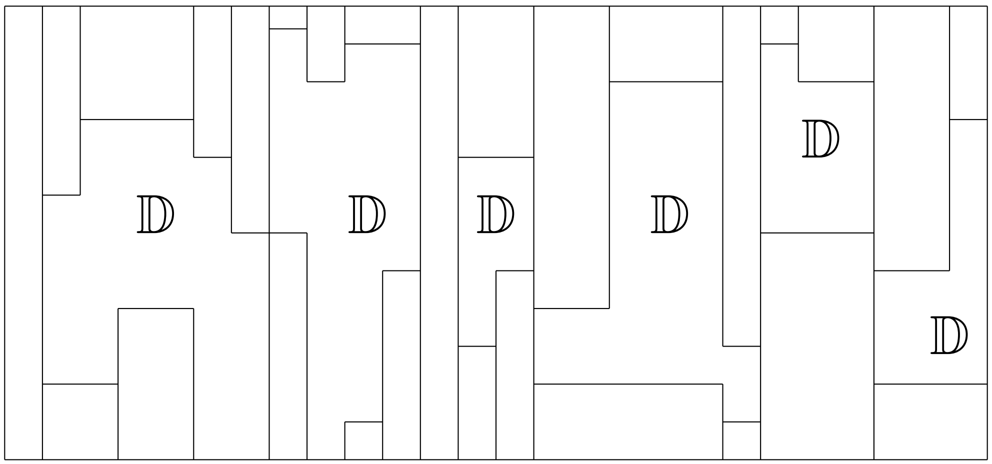


The pasting diagram has a shape as in

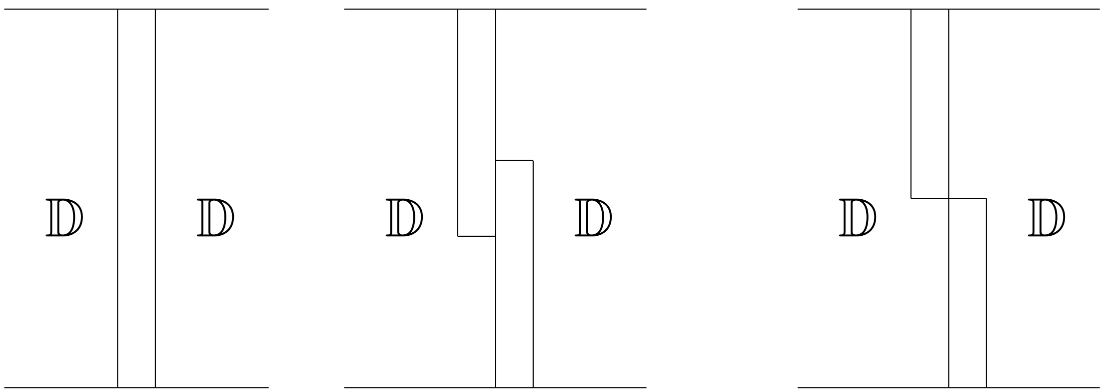


Step 2: Horizontally compose $\mathrm{B} \boxtimes [n]$-squares

Horizontally compose the $\mathrm{B} \boxtimes [n]$-squares where possible, to obtain a pasting diagram as in

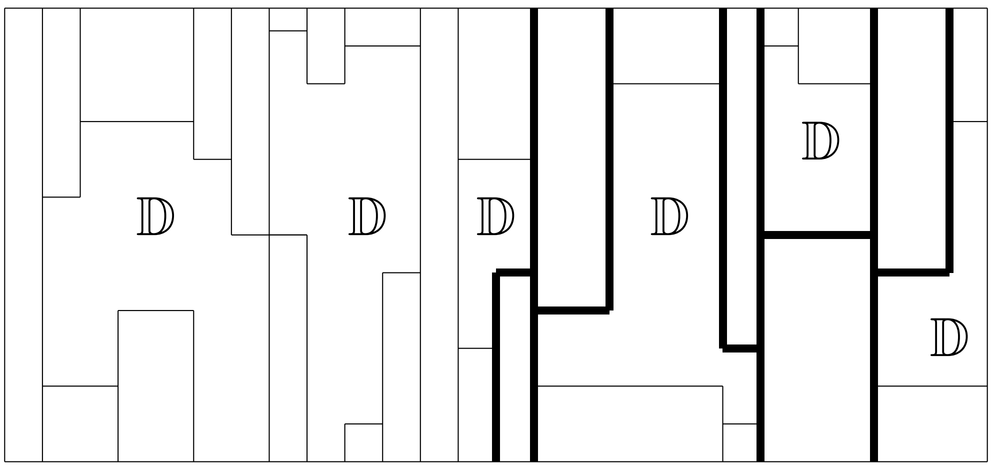


Note: The $\mathbb{D}$-regions are separated by configurations of the following three forms:



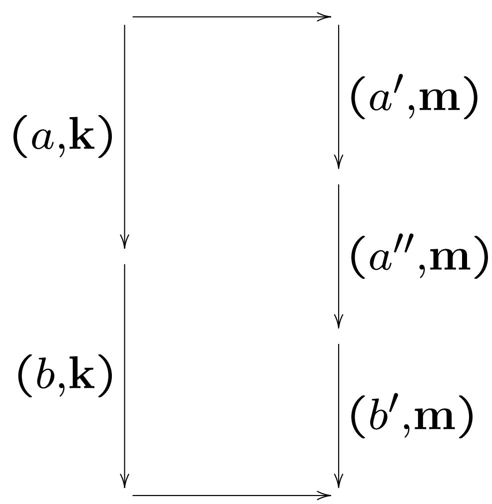
Step 3: Separation Removal

Separation Removal I: Clusters

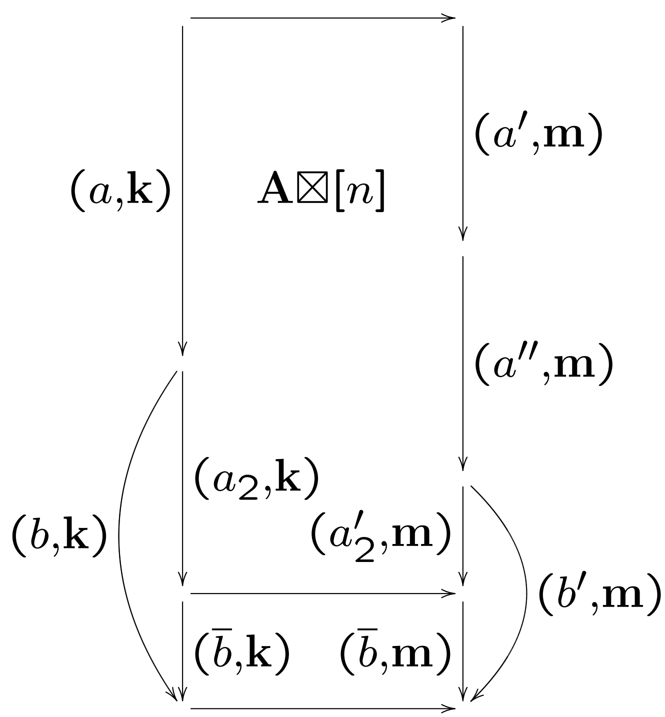


Note: any vertical arrow shared by two neighbouring columns in a cluster is of the form $(a, \mathbf{k})$.

Assume that the left-most column of the cluster is part of the bottom layer. Then it is of the form

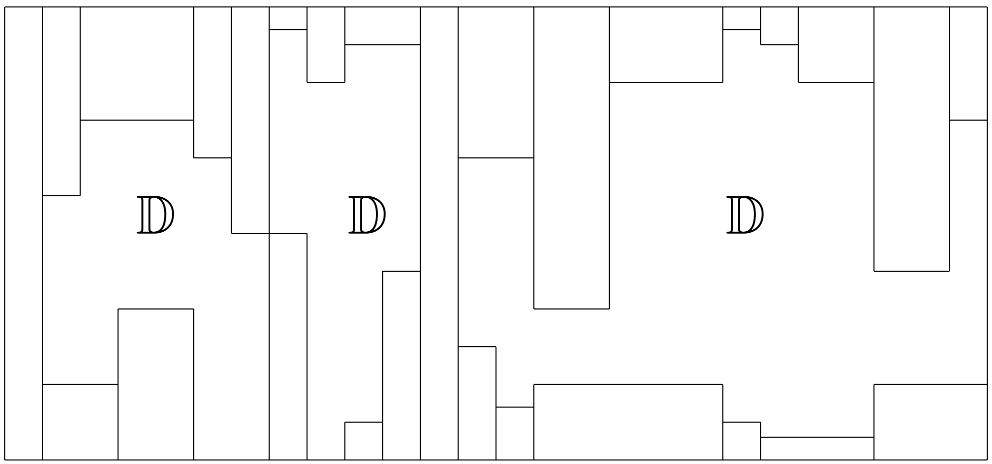


Use (FRC) to refactor this:



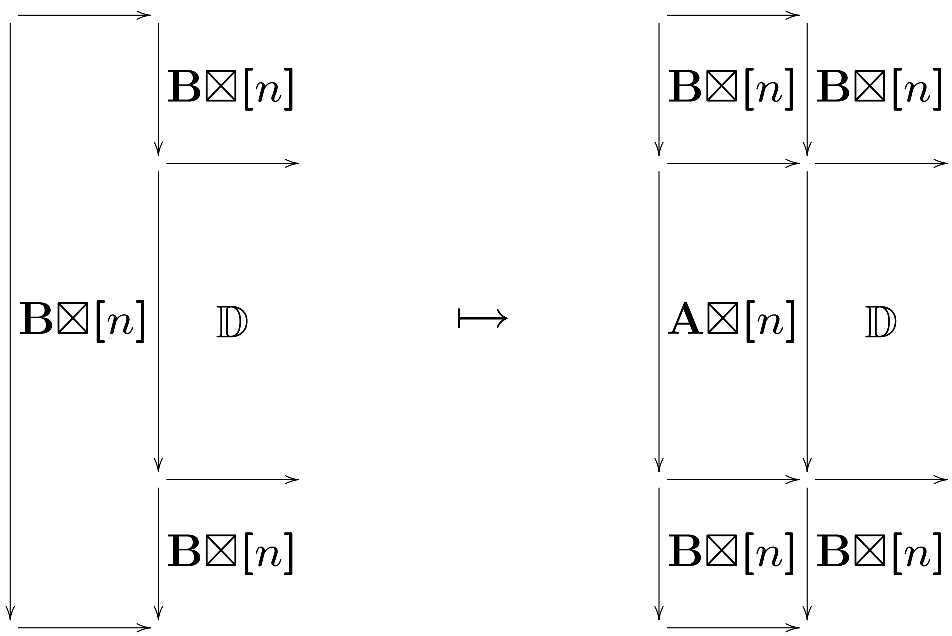
This will disconnect the neigbouring columns or turn them into touching columns.

The resulting pasting diagram is of the form

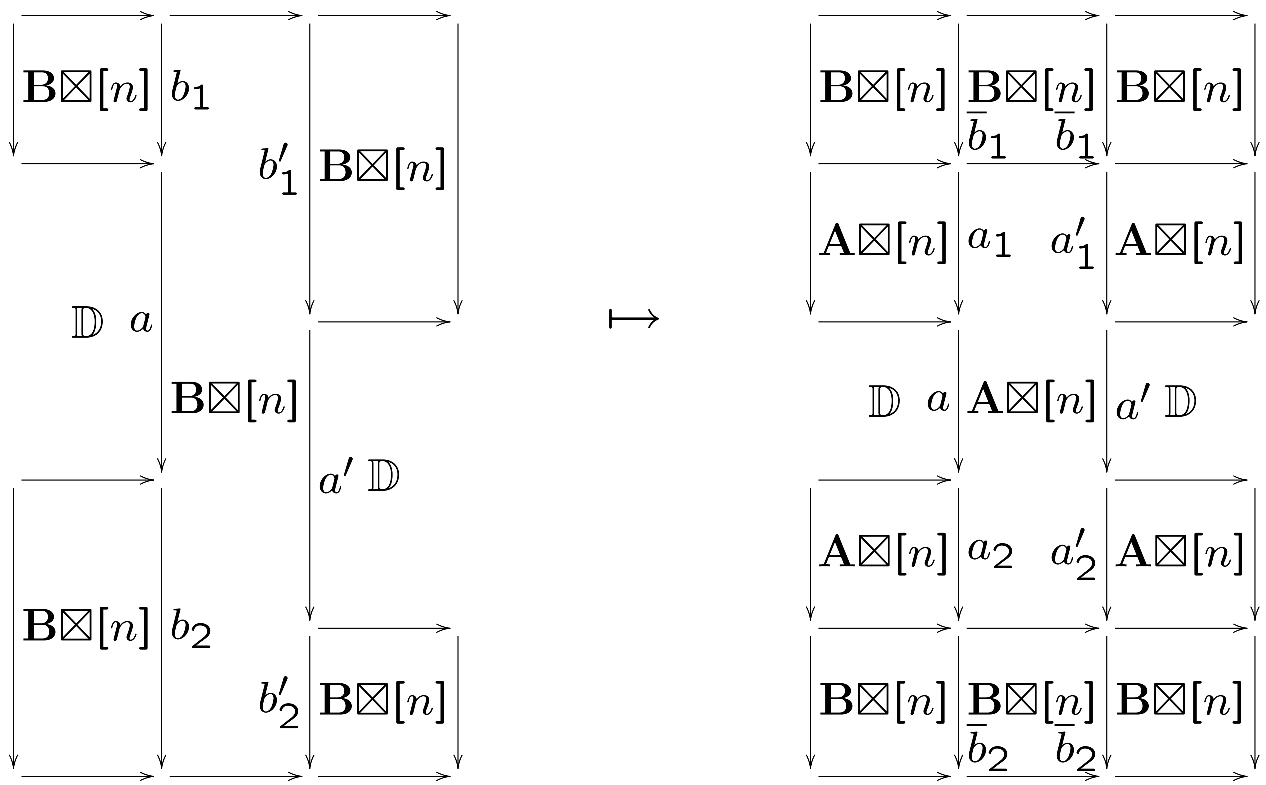


Separation Removal II: Full Height Columns

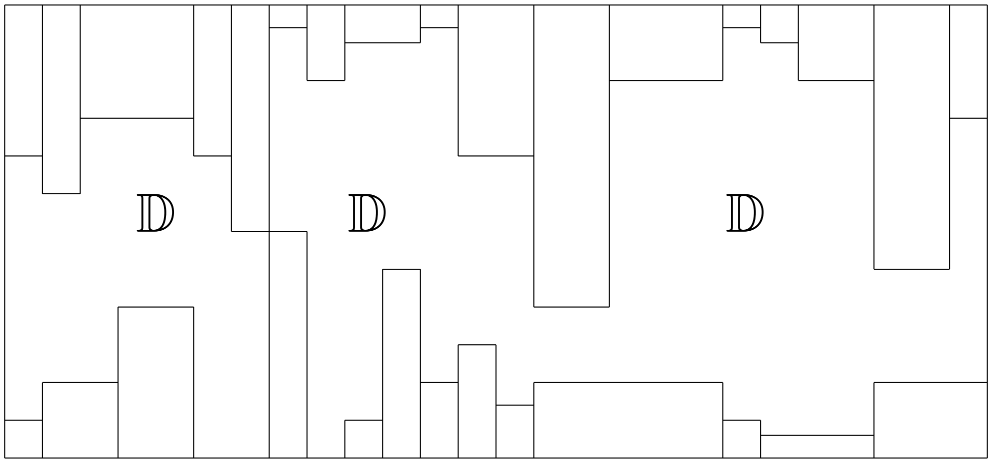
On the side of the diagram:



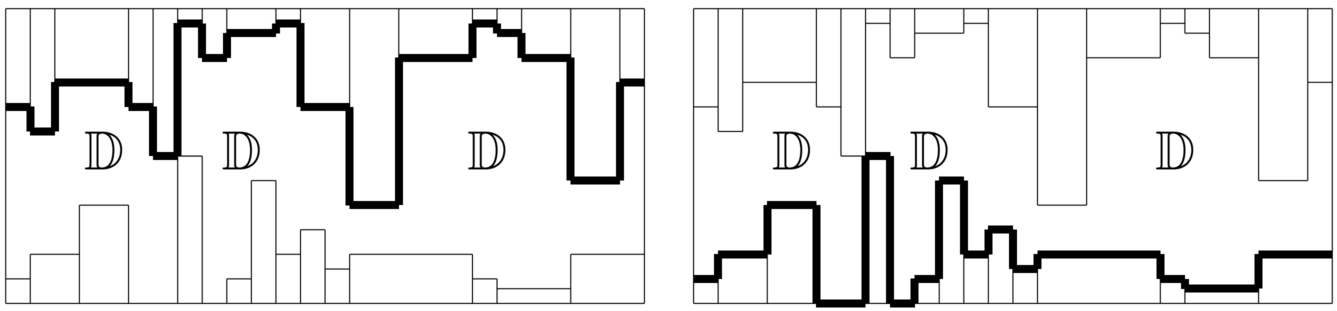
In the middle of the diagram:



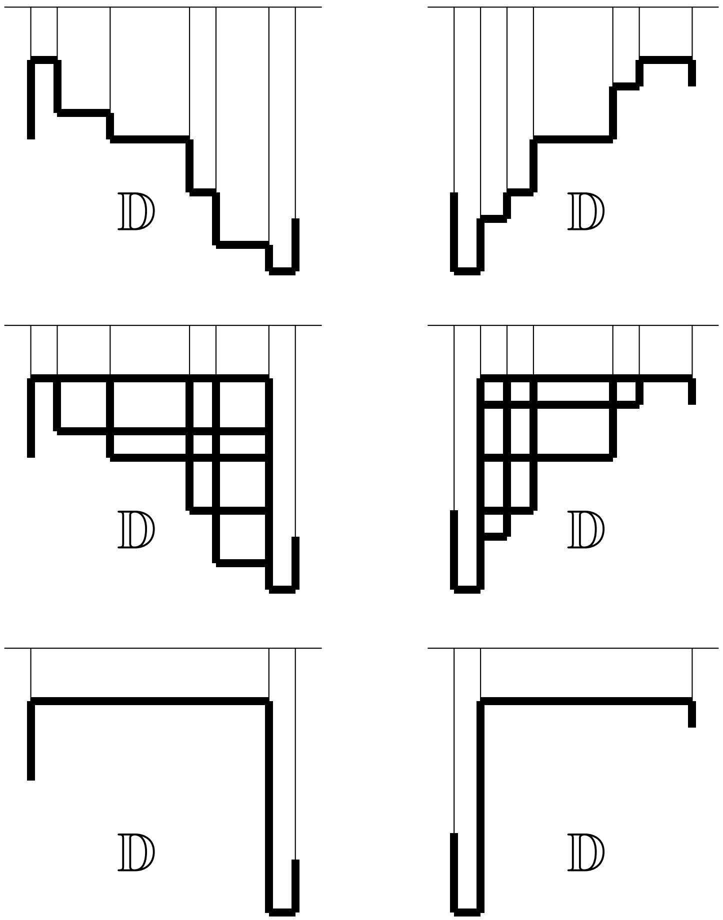
The resulting pasting diagram is of the form:



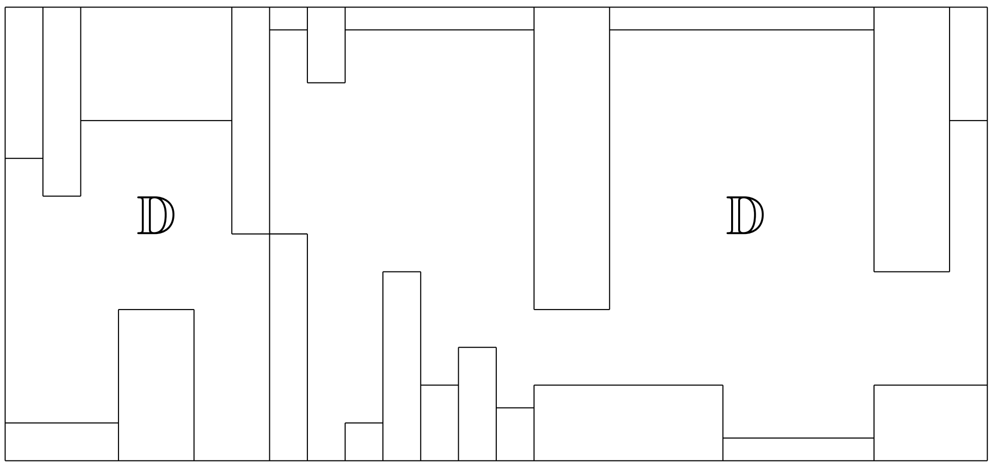
Note the boundaries of the $\mathbb{D}$-region:



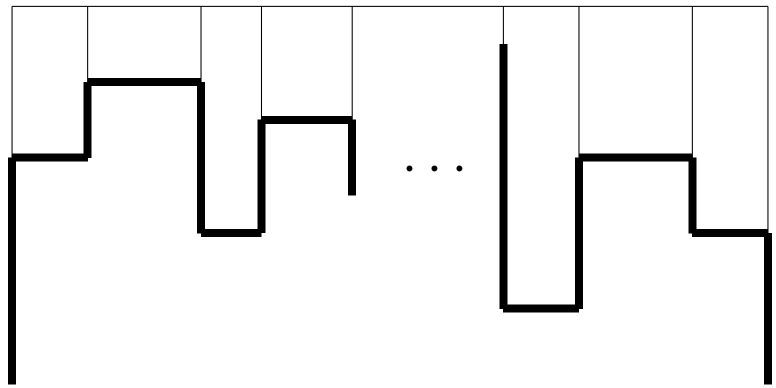
Straighten the boundaries I: Staircase removal



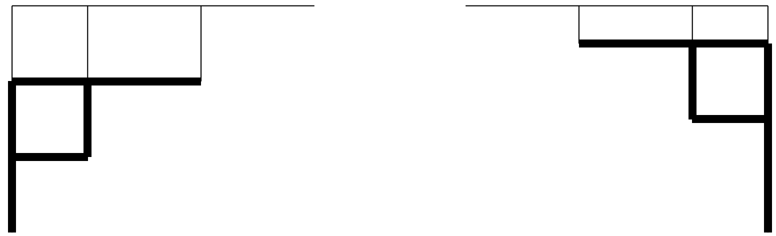
The resulting pasting diagram will be of the following form:



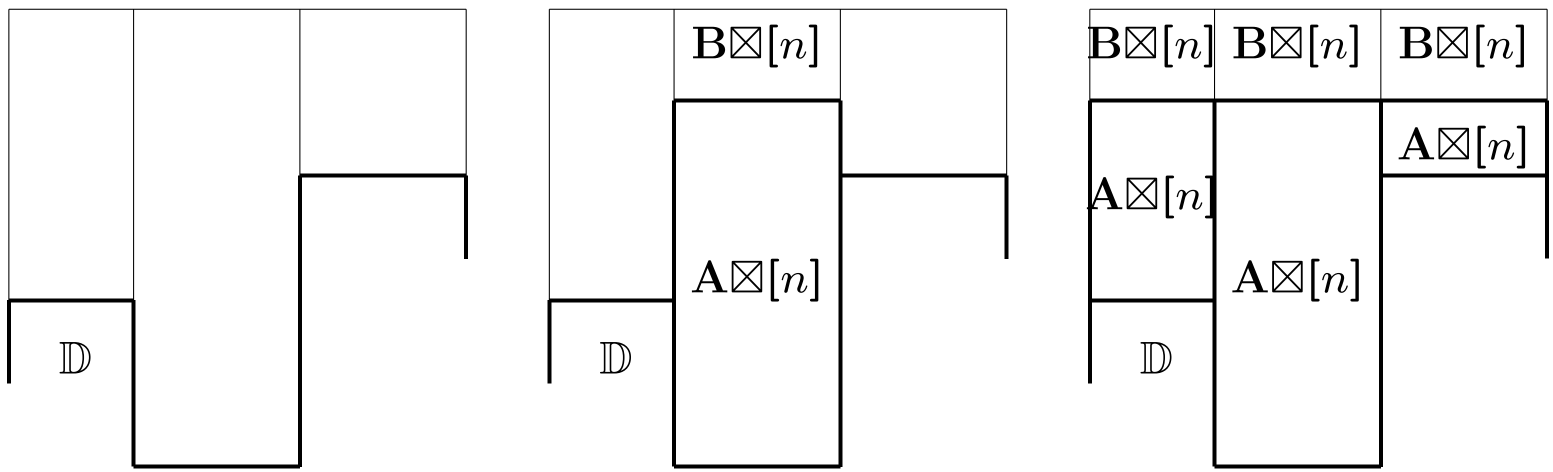
Straighten the boundaries II:
Minima removal



- Boundary minima:



- Other local minima:



(a) (b) (c)

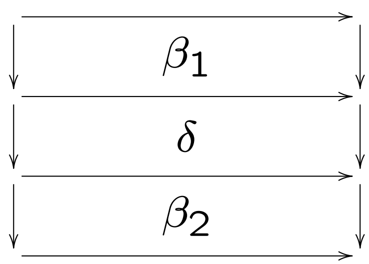
Conclusion: Squares of $\mathbb{P}$ have two forms:

1. A square

$$\begin{array}{ccc} (B,\mathbf{k}) & \xrightarrow{(B,s_{m,k})} & (B,\mathbf{m}) \\ {\scriptstyle (f,\mathbf{k})}\downarrow & & \downarrow{\scriptstyle (f,\mathbf{m})} \\ (B',\mathbf{k}) & \xrightarrow[(B',s_{m,k})]{} & (B',\mathbf{m}) \end{array}$$

in $\mathsf{Sq}\,(\mathbf{B} \boxtimes [n]) \backslash \mathsf{Sq}\,(\mathbf{A} \boxtimes [n])$.

2. A vertical path of squares



$$\delta \in \mathsf{Sq}\,\mathbb{D}$$
$$\beta_1, \beta_2 \in \mathsf{Sq}\,(\mathbf{B} \boxtimes [n]) \backslash \mathsf{Sq}\,(\mathbf{A} \boxtimes [n])$$
or vertical identity squares