Model categories were introduced as the foundation of homotopy theory by Quillen in 1967.

Definition. A *model category* is a complete and cocomplete category $\mathcal{K}$ together with three classes of morphisms $\mathcal{C}$, $\mathcal{F}$ and $\mathcal{W}$ such that

(1) $\mathcal{W}$ has the 2-out-of-3 property and is closed under retracts,

(2) $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are weak factorization systems.

Morphisms from $\mathcal{C}$, $\mathcal{F}$, $\mathcal{W}$, $\mathcal{C} \cap \mathcal{W}$ and $\mathcal{F} \cap \mathcal{W}$ are called cofibrations, fibrations, weak equivalences, trivial cofibrations and trivial fibrations.

The model structure is determined by cofibrations and trivial cofibrations alone.

$\mathcal{K}[\mathcal{W}^{-1}]$ is the *homotopy category* of $\mathcal{K}$.

Approximately 10 years ago, J. H. Smith introduced *combinatorial model categories* as model categories where $\mathcal{K}$ is locally presentable and both $\mathcal{C}$ and $\mathcal{C} \cap \mathcal{W}$ are generated by a set of morphisms. One can say that they are model categories determined by a set of data.

While **SSet** is a combinatorial model category, **Top** is not. The reason is that it is not locally presentable. J. H. Smith has suggested to replace **Top** by Δ-generated topological spaces. A topological space X is Δ-*generated* if its subset $S \subseteq X$ is open iff $f^{-1}(S)$ is open for each continuous map $f : \Delta_n \to X$, $n = 0, 1, \dots$.

Δ-generated spaces form the closure of simplices Δ_n under colimits in **Top**. They are analogous to compactly generated spaces where one takes the colimit closure of all compact Hausdorff spaces (they are usually called k-spaces). They were used to cure one disadvantage of **Top**: topological spaces are not cartesian closed. The point of J. H. Smith is that the category $\mathbf{Top}_\Delta$ of Δ-generated spaces is also locally presentable. Then, together with the usual model category structure of **Top**, $\mathbf{Top}_\Delta$ is a combinatorial model category. He has never given any proof of his claim and, up to now, nobody was able to supply it. It is instructive to look at the non-published Notes on Delta-generated spaces of D. Dugger.

Given any small full subcategory $\mathcal{S}$ of **Top** one can form $\mathcal{S}$-generated spaces. The category $\mathbf{Top}_{\mathcal{S}}$ of these spaces is cocomplete and $\mathcal{S}$ is dense in it. Assuming Vopěnka's principle, any such category is locally presentable.

Theorem 1. *Let $\mathcal{K}$ be a fibre-small topological category and $\mathcal{S}$ a small full subcategory of $\mathcal{K}$. Then the category $\mathcal{K}_{\mathcal{S}}$ of $\mathcal{S}$-generated spaces is locally presentable.*

The proof is based on my result from 1981 saying that each fibre-small topological category is the category of models of a suitable large (= given by a class of relation symbols) infinitary theory T. We express T as a union of an increasing chain of small subtheories T_i, which yields the chain

$$\mathbf{Mod}(T_0) \to \mathbf{Mod}(T_1) \to \dots \mathbf{Mod}(T_i) \to \dots$$

of left adjoints to reducts

$$\mathbf{Mod}(T_j) \to \mathbf{Mod}(T_i).$$

The point is that these left adjoints are full embeddings and $\mathbf{Mod}(T)$ is its union. Then $\mathcal{K}_{\mathcal{S}}$ is calculated in some $\mathbf{Mod}(T_i)$ which is locally presentable. Thus $\mathcal{K}_{\mathcal{S}}$ is locally presentable.

I do not know whether this argument works in Hausdorff or in compact Hausdorff spaces. The only missing thing is that the left adjoints above are full embeddings. Observe that Theorem 1 implies that $\mathcal{K}$ has the filtration consisting of coreflective full subcategories which are locally presentable. It suffices to express $\mathcal{K}$ as a union of an increasing chain of small full subcategories $\mathcal{S}_i$ and pass to $\mathcal{K}_{\mathcal{S}_i}$.

A distinguished advantage of a locally presentable category are the following two facts.

Theorem 2. *Let $\mathcal{K}$ be a locally presentable category and $\mathcal{I}$ a set of morphisms. Then* $(\mathrm{cof}(\mathcal{I}), \mathcal{I}^{\square})$ *is a weak factorization system.*

Theorem 3. *Let $\mathcal{K}$ be a locally presentable category and $\mathcal{I}$ a set of morphisms. Then* $(\mathrm{colim}(\mathcal{I}), \mathcal{I}^{\perp})$ *is a factorization system.*

Here, $\mathrm{cof}(\mathcal{I})$ is the closure of $\mathcal{I}$ under pushout, transfinite composition and retract while $\mathrm{colim}(\mathcal{I})$ is the colimit closure of $\mathcal{I}$ in $\mathcal{K}^{\to}$. Further, $\mathcal{I}^{\square}$ consists of morphisms g having the right lifting property w.r.t. each $f \in \mathcal{I}$ while $\mathcal{I}^{\perp}$ consists of morphisms g having the unique right lifting property w.r.t. each $f \in \mathcal{I}$.

Here, $f \Box g$ ($f \perp g$) means that that in each commutative square

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow[v]{} & D \end{array}$$

there is a (unique) diagonal $d : B \to C$ with $df = u$ and $gd = v$.

$(\mathrm{cof}(\mathcal{I}), \mathcal{I}^{\Box})$ is called a weak factorization system because each morphism h of $\mathcal{K}$ has a factorization $h = gf$ with $f \in \mathrm{cof}(\mathcal{I})$ and $g \in \mathcal{I}^{\Box}$. Moreover,

$$\mathrm{cof}(\mathcal{I}) = {}^{\Box}(\mathcal{I}^{\Box}).$$

We say that this factorization system is *cofibrantly generated* by $\mathcal{I}$. This completes the definition of a combinatorial model category.

$\mathcal{I}^\square$ is an accessible full subcategory of $\mathcal{K}^\to$ and the resulting weak factorization is functorial and accessible. It means that there is an accessible functor

$$F : \mathcal{K}^\to \to \mathcal{K}$$

and natural transformations

$$\alpha : \mathrm{dom} \to F \qquad \beta : F \to \mathrm{cod}$$

such that

$$h = \alpha_h \beta_h$$

is the weak factorization of h.

The accessibility of $\mathcal{I}^\square$ in $\mathcal{K}^\to$ follows from the fact that $f \square g$ iff g is injective to $(f, \mathrm{id}_B) : f \to \mathrm{id}_B$ in $\mathcal{K}^\to$ where $f : A \to B$.

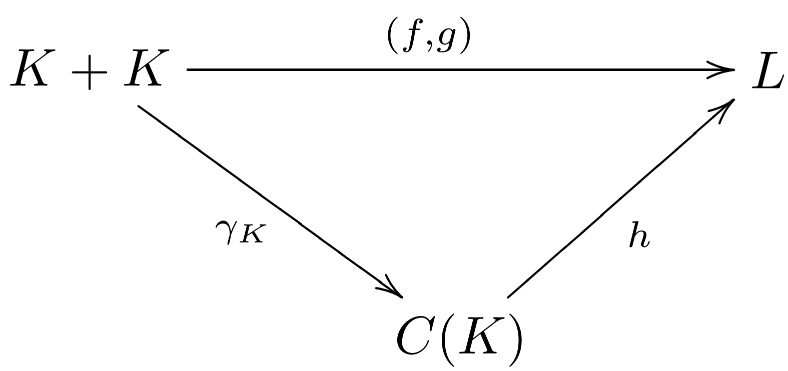
A weak factorization of the codiagonal

$$\nabla : K + K \xrightarrow{\gamma_K} C(K) \xrightarrow{\sigma_K} K$$

yields the *cylinder functor* $C : \mathcal{K} \to \mathcal{K}$ which is accessible. We denote by

$$\gamma_{1K}, \gamma_{2K} : K \to C(K)$$

the compositions of γ_K with the coproduct injections. Now, given two morphisms $f, g : K \to L$, we say that f and g are *homotopic* and write $f \sim g$ if there is a morphism $h : C(K) \to L$ such that the following diagram commutes



The homotopy relation $\sim$ is clearly reflexive, symmetric and compatible with the composition. But, it is not transitive in general and we denote its transitive hull by $\approx$. We say that $h : K \to L$ is a *homotopy equivalence* if there is $g : L \to K$ such that $gh \approx \mathrm{id}_K$ and $hg \approx \mathrm{id}_L$. Thus, we have a homotopy theory for each weak factorization system.

In a combinatorial model category, $\mathcal{F}$ and $\mathcal{F} \cap \mathcal{W}$ are accessible subcategories of $\mathcal{K}^{\to}$. Both the *fibrant replacement functor* R_f and the *cofibrant replacement functor* R_c are accessible. Recall that

$$K \to R_f(K) \to 1$$

is a (trivial cofibration, fibration)-factorization and

$$0 \to R_c(K) \to K$$

is a (cofibration, trivial fibration) one. Hence the *replacement functor* $R = R_f R_c$ is accessible.

The following result is due to J. H. Smith, but he has not presented a proof.

Theorem 4. *In a combinatorial model category $\mathcal{K}$, $\mathcal{W}$ is an accessible subcategory of $\mathcal{K}^{\to}$.*

Let $\mathcal{H}$ be the category of homotopy equivalences f together with a homotopy inverse g and the witnessing homotopies $h_1, \dots, h_n$. Let

$$\begin{array}{ccc} \mathcal{K}^{\to} & \xrightarrow{\quad R \quad} & \mathcal{K}^{\to} \\ \uparrow & & \uparrow \\ \mathcal{V} & \longrightarrow & \mathcal{H} \end{array}$$

be a pseudopullback. Thus $\mathcal{V}$ is accessible. Since $\mathcal{W}$ is the full image of the projection $\mathcal{V} \to \mathcal{K}^{\to}$, it satisfies the smallness condition in the definition of an accessible category. Since it is known that $\mathcal{W}$ is always closed under λ-filtered colimits in $\mathcal{K}^{\to}$ for some λ, it is accessible.

Theorem 5. (J. H. Smith) *Let $\mathcal{I}$ be a set of morphisms in a locally presentable category $\mathcal{K}$. Then $\mathcal{C} = \mathrm{cof}(\mathcal{I})$ and $\mathcal{W}$ make $\mathcal{K}$ a combinatorial model category if and only if*

(1) $\mathcal{W}$ has the 2-out-of-3 property and is closed under retracts,

(2) $(\mathcal{I})^{\square} \subseteq \mathcal{W}$,

(3) $\mathrm{cof}(\mathcal{I}) \cap \mathcal{W}$ is closed under pushout and transfinite composition, and

(4) $\mathcal{W}$ satisfies the solution set-condition at $\mathcal{I}$.

The Smith's proof of sufficiency was presented by T. Beke in 2000. Necessity follows from Theorem 4.

The last two theorems imply that classes $\mathcal{W}$ are closed under small intersections.

Assuming Vopěnka's principle, (4) is automatic and thus classes $\mathcal{W}$ are closed under all intersections. In particular, the smallest class $\mathcal{W}_{\mathcal{I}}$ satisfying (1)-(3) yields a combinatorial model category structure. Together with W. Tholen, we introduced this construction in 2000 and called it *left-determined*. The same construction was independently considered by D.-C. Cisinski who proved, without any set theory, that one gets a combinatorial model category in the special case when $\mathcal{K}$ is a Grothendieck topos and $\mathrm{cof}(I) = \mathrm{Mono}$.

$\mathcal{W}_\mathcal{I}$ is constructed, starting from $\mathcal{I}^\square$ by a transfinite iteration of taking the closure of $\mathcal{X}$ under (1) and the cofibrant closure of $\operatorname{cof}(\mathcal{I}) \cap \mathcal{X}$. The starting full subcategory $\mathcal{I}^\square \subseteq \mathcal{K}^\to$ is accessible but one cannot expect that the two closures preserve this property. But they preserve the property of being a full image of an accessible functor into $\mathcal{K}^\to$. This is quite tricky for the cofibrant closure because one has to use the fact that transfinite compositions form an accessible subcategory of $\mathcal{K}^\to$. It follows from a remarkable result of Makkai and Paré that the Grothendieck construction of an accessible pseudo-functor is accessible. Since full images of accessible functors are closed under small unions, each $\mathcal{W}_i$ on the way to $\mathcal{W}_\mathcal{I}$ is such a full images. Whenever $\mathcal{W}_\mathcal{I}$ satisfies (4) it is accessible and thus the construction stops at some ordinal i. On the other hand, if the construction stops at some i, $\mathcal{W}_\mathcal{I}$ is a full image of an accessible functor and thus it satisfies (4).

We have just seen that a class $\mathcal{C}$ of cofibrations in a combinatorial model category $\mathcal{K}$ is an image of an accessible functor into $\mathcal{K}^{\to}$. However, $\mathcal{C}$ does not need to be accessible. It suffices to find a cofibrantly generated weak factorization system $(\mathcal{C}, \mathcal{C}^{\square})$ such that $\mathcal{C}$ is not accessible. One can complete it to a combinatorial model category by putting $\mathcal{W} = \mathcal{K}^{\to}$ (the greatest one).

In the category **Pos** of posets, take $\mathcal{C}$ consisting of split monomorphisms. It yields a weak factorization system cofibrantly generated by split monomorphisms between finite posets. The closure of split monomorphisms under λ-filtered colimits in $\mathcal{K}^{\to}$ precisely consists of λ-pure monomorphisms. It is easy to see that, for each regular cardinal λ, there is a λ-pure monomorphism which does not split.

Consequently, one cannot expect that the full subcategory $\mathcal{K}_c$ consisting of cofibrant objects is always accessible, which is true for the full subcategory $\mathcal{K}_f$ of fibrant objects. An interesting example is the category **Ab** of abelian groups and $\mathcal{C}$ cofibrantly generated by

$$0 \to \mathbf{Z}.$$

Cofibrant objects are then free abelian groups whose accessibility is set-theoretical. It follows from the existence of a compact cardinal and contradicts the axiom of constructibility.

Thus the full subcategory

$$\mathcal{K}_{cf} = \mathcal{K}_c \cap \mathcal{K}_f$$

does not need to be accessible. Remind that the homotopy category $\mathcal{K}[\mathcal{W}^{-1}]$ is equivalent to the quotient $\mathcal{K}/\approx$.

We have just discussed accessibility of cofibrantly generated classes of morphisms. There is a converse question asking whether a cofibrantly closed accessible class of morphisms is cofibrantly generated. It cannot be true in general because pure monomorphisms in a locally finitely presentable category are always accessible and cofibrantly closed. If they are cofibrantly generated there is enough pure injectives, which is not always true. On the other hand, the class of all morphisms in a locally presentable category is always cofibrantly generated and M. Hébert has recently shown that it cane be generalized to each accessible category with pushouts.

In this connection, I want to mention the problem asked by G. Maltsiniotis. Let $\mathcal{C}$ be the largest cofibrantly closed class of morphisms in **Cat** whose pushouts preserve weak equivalences. Is $\mathcal{C}$ cofibrantly generated? In the positive case, these cofibrations together with weak equivalences form a left proper combinatorial model category structure on **Cat** where all objects are cofibrant. The Thomason model category structure does not have the last property.