Aspects of Cartesian Differential Categories

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Preliminaries (→ cartesian differential categories)

Left (semi) additive category : each hom set is a commutative monoid

$f(g+h) = fg + fh$ ... diagrammatic order of composition

$f0 = 0$

A map h is (semi) additive if also

$(f+g)h = fh + gh$

$0h = 0$

Prop The additive maps of a left additive category $\mathbb{X}$ form an additive subcategory $\mathbb{X}_+$.
The inclusion $\mathbb{X}_+ \hookrightarrow \mathbb{X}$ reflects isos.

Eg Commutative monoids with "set" maps (no preservation properties) form a left additive, not additive, category

Left additive: because operations are def^{d} pointwise
Not additive: because maps need not preserve monoid str
Those that do are in $\mathbb{X}_+$.

Motivating example: Vector spaces with differentiable & additive maps

Cartesian left additive category: has products s.t.

- π_0, π_1, Δ are additive
- f, g additive $\Rightarrow f \times g$ is additive

Equivalent:
- π_1, π_2 additive
- f, g additive $\Rightarrow \langle f, g \rangle$ is additive

Equivalent: $\mathbb{X}_+$ has biproducts ; $\mathbb{X}_+ \hookrightarrow \mathbb{X}$ creates products

Equivalent: each obj A has a chosen commutative monoid structure $(+, 0)$ "compatible with $\times$" ... $\left\{ \begin{array}{l} +_{A \times B} = \langle (\pi_0 \times \pi_0) +_A , \langle \pi_1 \times \pi_1 \rangle +_B \rangle \\ 0_{A \times B} = \langle 0_A , 0_B \rangle \end{array} \right.$

Cartesian left additive functor : pres $+, 0$, products

[Fact: S a comonad on Cart. left add^{v} cat $\mathbb{X} \Rightarrow \mathbb{X}_S$ is Cart left add^{v}
& the canonical right adjoint $G_S : \mathbb{X} \to \mathbb{X}_S$ is cart left add^{v}]

"Localisation" : Simple slice category $\mathbb{X}[A]$

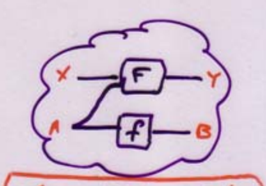
= the coKleisli cat of $A \times -$

Prelude: A "bundle" category $B(\mathbb{X})$
(where $\mathbb{X}$ is a left additive category.)

Semi

Objects : (X, A) of $\mathbb{X}$

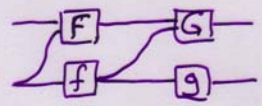
Morphisms : $\begin{array}{c} (X,A) \\ \downarrow (F,f) \\ (Y,B) \end{array}$ $\qquad$ $\begin{array}{ccc} X \times A & & A \\ \downarrow F & & \downarrow f \\ Y & & B \end{array}$



$\boxed{\text{where } F \text{ is additive in the } 1^{st} \text{ arg}^t.}$

Composition : $(F, f)(G, g) = (\langle F, \pi_1 f \rangle G, fg)$

Identity : $(\pi_0, 1)$



semi

$B(\mathbb{X})$ has additive structure $\quad 0 = (0, 0)$

$(F, f) + (G, g) = (F + G, f + g)$ $\quad$ [This is additive in the first arg^t]

If the cat $\mathbb{X}$ is Cartesian left additive, $B(\mathbb{X})$ has products

$1 = (1, 1) \qquad (X, A) \xleftarrow[(\pi_0 \pi_0, \pi_1)]{} (X \times Y, A \times B) \xrightarrow[\langle \pi_0 \pi_1, \pi_1 \rangle]{} (Y, B)$

[ditto]

In fact: If $\mathbb{X}$ is Cartesian left additive, so is $B(\mathbb{X})$

More: the 'projection' $B(\mathbb{X}) \xrightarrow{P} \mathbb{X}$ is a Cartesian left additive functor, and is a fibration, whose fibres are additive cats

Semi

in fibres
$(H, 1) + (K, 1) = (H + K, 1)$

SO ...

A **bundle fibration** $\begin{array}{c}\mathbb{Y}\\ \downarrow p\\ \mathbb{X}\end{array}$ is a fibration satisfying:

- $\mathbb{X}$ is left additive
- fibres $p^{-1}A$ are additive ($\forall$ obj A) $\mathbb{Y}_A := p^{-1}A$
- $f^*: \mathbb{Y}_B \to \mathbb{Y}_A$ is additive ($\forall$ map $f: A \to B$)
- For any $f: A \to B$ of $\mathbb{X}$, any obj X of $\mathbb{Y}_B$ ("X over B"), the domain of the cartesian lifting of f to X depends only on X and A:

$$\begin{tikzcd} f^*X := \widehat{(X,A)} \arrow[r, "\hat{f}"] \arrow[d, dotted, no head] & X \arrow[d, dotted, no head] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

THEN: if $\mathbb{Y} \xrightarrow{p} \mathbb{X}$ is a bundle fibration

(i) $\mathbb{Y}$ is left additive;

(ii) If $\mathbb{X}$ is Cartesian left additive, each $\mathbb{Y}_A$ cartesian additive, then $\mathbb{Y}$ is Cartesian left additive. ... products? Pull back to appropriate fibre & do it there

The addition in $\mathbb{Y}$:

$$\begin{tikzcd} Y \arrow[d, shift right, "f'"'] \arrow[d, shift left, "g'"] \arrow[drr, shift left, "g"] \arrow[drr, shift right, "f"'] & & \\ \widehat{(X,A)} \arrow[rr, "\widehat{pf+pg}"'] \arrow[d, dotted, no head] & & X \arrow[d, dotted, no head] \\ A \arrow[rr, shift left, "pf"] \arrow[rr, shift right, "pg"'] & & B \end{tikzcd}$$

$$f+g = (f'+g')(\widehat{pf+pg})$$

$$0 = 0\,\hat{0}$$

ie factor maps into "vertical" & "cartesian", + add each factor

So: If $\mathbb{X}$ is left additive, $B(\mathbb{X}) \xrightarrow{p} \mathbb{X}$ is a bundle fib^{n}

If $\mathbb{X}$ is Cartesian left add^{ve}, so p a Cartesian left add^{v} functor

suppose we have a (Cart) left additive **section** of p:

$$B(\mathbb{X}) \underset{p}{\overset{d}{\rightleftarrows}} \mathbb{X} \qquad \begin{array}{l} d(A) = (d_0A, A) \\ d(f) = (D[f], A) \end{array}$$

This has some interesting consequences:

Eg. $d(f)\,d(g) = d(fg)$ becomes (consider 1st component)

$$\langle D[f], \pi_1 f \rangle D[g] = D[fg]$$

chain rule

Eg. $d(f+g) = df + dg \Rightarrow D[f+g] = D[f] + D[g]$

Eg $d(\langle f,g \rangle) = \langle df, dg \rangle \Rightarrow D[\langle f,g \rangle] = \langle D[f], D[g] \rangle$

Eg $d(1) = 1 = (\pi_0, 1) \Rightarrow D[1] = \pi_0$

$d(\pi_0) = \pi_0 = (\pi_0\pi_0, \pi_0) \Rightarrow D[\pi_0] = \pi_0\pi_0$

(& similarly $D[\pi_1] = \pi_0\pi_1$)

If you recall my CT06 talk these are just the "derivative rules" I listed then...

There's even more structure lying around here:

Call a map f in $\mathbb{X}$ <u>linear</u> if there is a map f' so that $D[f] = \pi_0 f'$

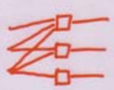
<u>Then</u>:
- such a map f' is uniquely determined by f (since π_0 is epi, having section $\langle 1,0\rangle$); call it $d_0(f)$ $(:= f')$
- d_0 is a functor; and linear maps form an additive sub category
- if f is additive, so is $d_0(f)$
- projections are linear (see last slide); if f, g linear, so is $\langle f, g\rangle$

If we add the condition that d_0 is the identity, we can push this further; in particular "$\langle 1,0\rangle D[f]$ is linear" (for any f) is (a variant of) [CD.6]

This can be pushed a <u>lot</u> further, via the Faà di Bruno categories

$$\mathbb{X} \underset{p}{\overset{d}{\rightleftarrows}} B(\mathbb{X}) \underset{p}{\overset{d}{\rightleftarrows}} B^{(2)}(\mathbb{X}) \underset{p}{\overset{d}{\rightleftarrows}} B^{(3)}(\mathbb{X}) \leftarrow \cdots$$

where composition is 'given' by the higher order chain rules.

Eg $B^{(2)}(\mathbb{X})$ has triplets for objects, maps 

composition ("glue these circuits") is the 2^{nd} order chainrule at the "top":

$$G''(F'(x,z), F'(y,z), F(z)) + G'(F''(x,y,z), F(z))$$

(and $B(\mathbb{X})$ composition at lower level.)

<u>Introduction to Differential Categories</u>

- Motivating example of linear logic

$$A \Rightarrow B = !A \multimap B$$

- coKleisli category of cotriple (comonad) ! {stable domains & coherence spaces}

- Differential λ-calculus of Ehrhard & Regnier {Köthe spaces, Finiteness spaces}

<u>Our aim</u>:

Categorically "reconstruct" the E-R differential structure

Basic setting: (symmetric) monoidal category with comonad on it

<u>Intuition</u>: The "base category" maps are "linear"

CoKleisli maps are "smooth"

An illustration of how this works

A smooth map $f : \mathbb{R}^3 \to \mathbb{R}^2$ $\quad f\langle x,y,z\rangle = \langle x^2+xyz,\ z^3-xy\rangle$

Its Jacobian $\begin{pmatrix} 2x+yz & xz & xy \\ -y & -x & 3z^2 \end{pmatrix}$

For chosen $\langle x,y,z\rangle$ this is a linear map $\mathbb{R}^3 \to \mathbb{R}^2$

ie from
$$\frac{f : A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m}{D[f] : A \longrightarrow \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^m)}$$
we get

These are both smooth ie CoKleisli maps

So: in our setting we would have this:

$$\frac{f : !A \longrightarrow B}{D[f] : !A \longrightarrow (A \multimap B)}$$

all maps in base cat ✗

Linear Hom

To avoid the need for closed structure, we shall take $D[f] : A \otimes !A \longrightarrow B$

Outline

- Basic notions (Comm monoid enriched)
  - Differential Category:
    - coalgebra modality on a (semi)additive symmetric monoidal category
    - differential combinator (we'll show this in 2 presentations)
- Examples
  - sets & relations (with "bag" functor)
  - sup lattices (with dual of free algebra functor)
  - commutative polynomials + "ordinary" derivatives (on Vec^{op})
  - S_∞ construction (generalises previous eg)
- Extending the theory
  - storage (M Fiore)
  - $\longrightarrow$ Ehrhard & Regnier (a not-necessarily-closed version of their structure)

Basic context

- (semi) additive symmetric monoidal category $\mathbb{X}$

commutative monoid enriched
no assumption of biproducts - yet!

Eg Sets & relations is (semi)additive but not AbGrp-enriched

- coalgebra modality !
  - a cotriple (comonad)
  - $\top \xleftarrow{e} !X \xrightarrow{\Delta} !X \otimes !X$ natural coalgebra str.
  - $(!X, \Delta, e)$ is a comonoid

$$\begin{tikzcd} !X \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & !X \otimes !X \arrow[d, "\Delta \otimes 1"] \\ !X \otimes !X \arrow[r, "1 \otimes \Delta"'] & !X \otimes !X \otimes !X \end{tikzcd} \qquad \begin{tikzcd} & !X \arrow[dl] \arrow[d, "\Delta"] \arrow[dr] & \\ !X & !X \otimes !X \arrow[l, "1 e"] \arrow[r, "e \otimes 1"'] & !X \end{tikzcd}$$

commute

- $\delta : !X \to !!X$ is a comonoid morphism

$$\begin{tikzcd} !X \arrow[r, "\delta"] \arrow[dr, "e"'] & !!X \arrow[d, "e"] \\ & \top \end{tikzcd} \qquad \begin{tikzcd} !X \arrow[r, "\delta"] \arrow[d, "\Delta"'] & !!X \arrow[d, "\Delta"] \\ !X \otimes !X \arrow[r, "\delta \otimes \delta"'] & !!X \otimes !!X \end{tikzcd}$$

commute

[we don't assume that δ, or any of these transformations are monoidal - yet]

Intuition: $!A \to B$ is "a differentiable map $A \to B$"

(but we need more structure to realize this)

Storage

Given a s.m.cat with products and a comonad !
a comonoidal transformation $s : ! \to !$
from $(\mathbb{X}, \times, 1)$ to $(\mathbb{X}, \otimes, \top)$ amounts to

$$s_0 : !(1) \to \top \quad \text{and} \quad s_2 : !(X \times Y) \to !X \otimes !Y$$

st

$$\begin{tikzcd} !((X \times Y) \times Z) \arrow[r, "s_2"] \arrow[d, "!(a_\times)"'] & !(X \times Y) \otimes !Z \arrow[r, "s_2 \otimes 1"] & (!X \otimes !Y) \otimes !Z \arrow[d, "a_\otimes"] \\ !(X \times (Y \times Z)) \arrow[r, "s_2"'] & !X \otimes !(Y \times Z) \arrow[r, "1 \otimes s_2"'] & !X \otimes (!Y \otimes !Z) \end{tikzcd}$$

$$\begin{tikzcd} !(1 \times X) \arrow[r, "s_2"] \arrow[d, "!\pi_1"'] & !(1) \otimes !X \arrow[d, "s_0 \otimes 1"] \\ !X & \top \otimes !X \arrow[l, "u_\otimes"] \end{tikzcd} \qquad \begin{tikzcd} !(X \times 1) \arrow[r, "s_2"] \arrow[d, "!\pi_1"'] & !X \otimes !(1) \arrow[d, "1 \otimes s_0"] \\ !X & !X \otimes \top \arrow[l, "u_\otimes"] \end{tikzcd}$$

(+ diagram for symmetry if appropriate)

In our setting, requiring that ! be comonoidal (as a comonad) is too strong - we'd want δ to be so, but not ϵ (The Id functor is not comonoidal)

$$\begin{tikzcd} F(X \times Y) \arrow[r, "\alpha"] \arrow[d, "s^F"'] & G(X \times Y) \arrow[d, "s^G"] \\ FX \otimes FY \arrow[r, "\alpha \otimes \alpha"'] & GX \otimes GY \end{tikzcd} \qquad \begin{tikzcd} F(1) \arrow[r, "\alpha"] \arrow[dr, "s^F"'] & G(1) \arrow[d, "s^G"] \\ & \top \end{tikzcd}$$

no such s^G for $G = Id$

A (s)m cat $\mathbb{X}$ with comonad $!$, products has a <u>storage transformation</u> if there is a comonoidal transformation

$$s : ! \to ! : (\mathbb{X}, \times, 1) \longrightarrow (\mathbb{X}, \otimes, \top)$$

so that δ is comonoidal

(using the canonical comonoidal trans $(\mathbb{X}, \times, 1)$
ie $!(X \times Y) \to !X \times !Y$
$!(1) \to 1$)

Key Fact:

For a (symm) monoidal cat with products: to have a comonad with (symm) storage trans. is equiv. to having a (cocommutative) coalgebra modality.

$(\Downarrow)$ Define $\Delta : !X \xrightarrow{!(\Delta_X)} !(X \times X) \xrightarrow{s_2} !X \otimes !X$

$e : !X \xrightarrow{!(\pi)} !(1) \xrightarrow{s_0} \top$

$(\Uparrow)$ Define $s_2 : !(X \times Y) \xrightarrow{\Delta} !(X \times Y) \otimes !(X \times Y) \xrightarrow{!\pi_0 \otimes !\pi_1} !X \otimes !Y$

$s_0 : !(1) \xrightarrow{e} \top$

This works!

Examples

- id on any cat with finite products
- $!$ in linear logic
- Dual of "algebra modality"
  - The free algebra $T(X) = \bigoplus_{n=0}^{\infty} X^{\otimes n}$
  - The free symmetric algebra $Sym(X) = \bigoplus_0^{\infty} X^{\otimes n}/S_n$
  - The "exterior algebra" $\Lambda(X) = \bigoplus_0^{\infty} X^{\otimes n}/A$

    so $xy = -yx$

Differential Combinators

$$D_{AB} : \mathbb{X}(!A, B) \longrightarrow \mathbb{X}(A \otimes !A, B)$$

$$\frac{!A \xrightarrow{f} B}{A \otimes !A \xrightarrow[D[f]]{} B}$$

Think $!A \to (A \multimap B)$

This must satisfy:

- naturality (for combinators), additivity
- "constants have deriv $= 0$" $D[e] = 0$
- product rule $(1 \otimes \Delta)(D[f] \otimes g) + (1 \otimes \Delta)(c \otimes 1)(f \otimes D[g]) = D[\Delta(f \otimes g)]$
- "Linear maps have constant deriv" $D[\epsilon f] = (1 \otimes e) f$
- chain rule $D[\delta !f g] = (1 \otimes \Delta)(D[f] \otimes \delta !f) D[g]$

There is a "circuit calculus" for all this ...

Cartesian Differential categories

- Left additive
- products

} Cartesian left additive

- a cartesian differential operator

$$\frac{X \xrightarrow{f} Y}{X \times X \xrightarrow[D_\times[f]]{} Y}$$

Think: 1^{st} arg^t is "linear"; 2^{nd} is "smooth"

sat several axioms:

[CD1] $D_\times[f+g] = D_\times[f] + D_\times[g]$; $D_\times[0] = 0$

[CD2] $\langle h+k, v\rangle D_\times[f] = \langle h,v\rangle D_\times[f] + \langle k,v\rangle D_\times[f]$
$\langle 0,v\rangle D_\times[f] = 0$

[CD3] $D_\times[1] = \pi_0$; $D_\times[\pi_i] = \pi_0 \pi_i$ $(i=0,1)$

[CD4] $D_\times[\langle f,g\rangle] = \langle D_\times[f], D_\times[g]\rangle$

[CD5] $D_\times[fg] = \langle D_\times[f], \pi_1 f\rangle D_\times[g]$

[CD6] $(\langle 1,0\rangle \times 1) D_\times[D_\times[f]] = (1 \times \pi_1) D_\times[f]$

Example: Fin dim vector spaces over $\mathbb{R}$ (or $\mathbb{C}$ ...) with ∞'ly differentiable maps

- $D_\times$ given by Jacobian

"just like" the diff cat eg

Repeat the earlier example:

$$f : \langle x,y,z\rangle \longmapsto \langle x^2 + xyz,\ z^3 - xy\rangle$$

$$D_\times[f] : \langle u,v,w\rangle, \langle r,s,t\rangle \longmapsto$$

linear in $\langle u,v,w\rangle$

$$\left.\begin{pmatrix} 2x+yz & xz & xy \\ -y & -x & 3z^2 \end{pmatrix}\right|_{\substack{x=r\\ y=s\\ z=t}} \bullet \langle u,v,w\rangle$$

But not in $\langle r,s,t\rangle$

$$= \langle (2r+st)u + rtv + rsw,\ -su - rv + 3t^2 w\rangle$$

"Linear maps"

In a cart diff cat, f is linear if

$$D_\times[f] = \pi_0 f$$

Prop: The linear maps form an additive subcat $\mathbb{X}_{lin}$ of a cart diff cat $\mathbb{X}$; $\mathbb{X}_{lin}$ has (bi)products; $\mathbb{X}_{lin} \hookrightarrow \mathbb{X}$ reflects isos & creates products

Prop: The coKleisli category of a differential cat with biproducts is a cartesian differential cat

define $D_\times[f]$, for $X \xrightarrow{f} Y$, to be:

$$\begin{tikzcd}
S(X\times X) \arrow[r, "\Delta"] \arrow[rr, bend right, "S_2"'] & S(X\times X) \otimes S(X\times X) \arrow[r, "S\pi_0 \otimes S\pi_1"] & S(X) \otimes S(X) \arrow[d, "\epsilon \otimes 1"] \\
& & X \otimes S(X) \arrow[d, "D_\otimes[f]"] \\
& & S(X)
\end{tikzcd}$$

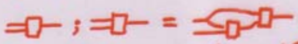
(The canonical storage trans$^{\underline{n}}$)

Notation: $D_\otimes$ is the diff combinator for the diff cat
($D_\times$ is the cart diff op)
S is the comonad
(called ! in linear logic)

Recall $\mathbb{X}[A]$ the "simple slice cat"

{$\mathbb{X}$ Cart. left add^{v}}

{ Obj = Obj of $\mathbb{X}$
Map: $A\times X \to Y : X \to Y$



Kleisli composition }

Note: each $\mathbb{X}[A]$ is Cartesian left additive. Also:

If $\mathbb{X}$ is a Cartesian differential cat, so is $\mathbb{X}[A]$ ($\forall A$:obj $\mathbb{X}$)

- its differential $D^A[f] = c'\, D_1[f]$, the "partial derivative"

$$\triangleq \langle \pi_1\pi_0, \langle \pi_0;\pi_1\pi_1\rangle\rangle (\langle 0,1\rangle \times 1)\, D[f]$$

To prove this, one could just work through the definition, but another approach suggests itself: develop a term calculus. With parameters, such a calculus naturally works "locally" — ie for slices.

We actually did the slice cat first, then we realised the logical approach would be "nicer"...

SO ...

The Term Calculus for Cartesian differential categories

Start with the 'usual' primitive types, function symbols (typed)
We can pair variables (think: our cat has products) to produce "patterns" (like (x,y)) (assume no repeated variables)

Usual term formation rules for products, sums, substitution
(as needed for Cartesian left additive cats)

The differential term formation rule is:

$$\frac{\Gamma, x : S \vdash t : T \qquad \Gamma \vdash s : S \qquad \Gamma \vdash u : S}{\Gamma \vdash \frac{\partial t}{\partial x}(s)\cdot u \;:\; T}$$

x is a pattern

Γ is a list of patterns

[Idea: $\frac{\partial t}{\partial x}$ is the Jacobian, (s) is the subst. of s for x (giving the linear transformation) $\cdot u$ is "application" at u (or "dot product")]

But these $\frac{\partial t}{\partial x}$ are "partial derivatives":
Γ may contain other "parameters"

Example: if $t = \langle ax^2 + bxyz,\ z^3 - xy \rangle$ then

$$\frac{\partial t}{\partial (xyz)} = \begin{pmatrix} 2ax + byz & bxz & bxy \\ -y & -x & 3z^2 \end{pmatrix}$$

Equations (usual substitution rules ... plus:)

(Dt.1) $\frac{\partial (t_1 + t_2)}{\partial x}(s)\cdot u = \frac{\partial t_1}{\partial x}(s)\cdot u + \frac{\partial t_2}{\partial x}(s)\cdot u$

$\frac{\partial 0}{\partial x}(s)\cdot u = 0$

(Dt.2) $\frac{\partial t}{\partial x}(s)\cdot(u_1 + u_2) = \frac{\partial t}{\partial x}(s)\cdot u_1 + \frac{\partial t}{\partial x}(s)\cdot u_2$

(Dt.3a) $\frac{\partial x}{\partial x}(s)\cdot u = u$

(Dt.3b) $\frac{\partial t}{\partial (x,x')}(s,s')\cdot(u,0) = \frac{\partial t[s'/x']}{\partial x}(s)\cdot u$

& dual: $\frac{\partial t}{\partial (x,x')}(s,s')\cdot(0,u') = \frac{\partial t[s/x]}{\partial x'}(s')\cdot u'$

(Dt.4) $\frac{\partial (t_1,t_2)}{\partial x}(s)\cdot u = \left(\frac{\partial t_1}{\partial x}(s)\cdot u,\ \frac{\partial t_2}{\partial x}(s)\cdot u\right)$

(Dt.5) $\frac{\partial t[t'/x']}{\partial x}(s)\cdot u = \frac{\partial t}{\partial x'}(t'[s/x])\cdot\left(\frac{\partial t'}{\partial x}(s)\cdot u\right)$

No variable in x may occur in t

(Dt.6) $\frac{\partial\left(\frac{\partial t}{\partial x}(s)\cdot x'\right)}{\partial x'}(r)\cdot u = \frac{\partial t}{\partial x}(s)\cdot u$

Facts: ("Basic lemma")

- $\frac{\partial t}{\partial ()}(())\cdot() = 0$
- $\frac{\partial t}{\partial x}(s)\cdot u = 0$ if no variable in x appears in t
- $\frac{\partial t}{\partial (x,x')}(s,s')\cdot(u,u') = \frac{\partial t}{\partial x}(s)\cdot u$ if no variable in x' appears in t (and dual)
- $\frac{\partial t}{\partial (x,x')}(s,s')\cdot(u,u') = \frac{\partial t[s'/x']}{\partial x}(s)\cdot u + \frac{\partial t[s/x]}{\partial x'}(s')\cdot u'$
- $\frac{\partial t}{\partial (x,x')}(s,s')\cdot(u,u') = \frac{\partial t}{\partial (x',x)}(s',s)\cdot(u',u)$

Soundness An interpretation M of a differential theory T [types ; typed function symbols , equations] consists of an assignment types $\mapsto$ objects , fcn symbols $\mapsto$ suitably typed maps

The assignment is extended as follows to all terms

$$[\![x:A \vdash x]\!] = 1_{M(A)} \qquad [\![x \vdash 0]\!] = 0$$

$$[\![x \vdash t_1 + t_2]\!] = [\![x \vdash t_1]\!] + [\![x \vdash t_2]\!]$$

(x could be a "pattern")

$$[\![x \vdash f(t_1 \cdots t_n)]\!] = [\![x \vdash (t_1 \cdots t_n)]\!] M(f)$$

$$[\![x \vdash (t_1 \cdots t_n)]\!] = \langle [\![x \vdash t_1]\!], \ldots, [\![x \vdash t_n]\!] \rangle$$

$$[\![(p,p') \vdash x]\!] = \begin{cases} \pi_0 [\![p \vdash x]\!] & x \in p \\ \pi_1 [\![p' \vdash x]\!] & x \in p' \end{cases}$$

$$[\![p \vdash \frac{\partial t}{\partial x}(s) \cdot u]\!] = \langle \langle [\![p \vdash u]\!], 0 \rangle, \langle [\![p \vdash s]\!], 1 \rangle \rangle \, D[[\![(p,x) \vdash t]\!]]$$

[$M \models T$ if all the equations are 'satisfied']

Prop

Such a translation gives a sound interpretation of terms into any cartesian differential category

An example: Here's the interpretation of this term

$$\frac{a:A, y:B, x:A \vdash f(x,y) : C \qquad a:A, y:B \vdash g(a,y) : A \qquad a:A, y:B \vdash a:A}{\underbrace{a:A, y:B}_{\Gamma} \vdash \frac{\partial f(x,y)}{\partial x}(g(a,y)) \cdot a \; : C}$$

(Supposing interpretations of A, B, C, & f, g already available)

Note: $[\![((a,y),x) \vdash f(x,y)]\!] = [\![((a,y),x) \vdash (x,y)]\!] M(f)$

$$= \langle [\![((a,y),x) \vdash x]\!], [\![((a,y),x) \vdash y]\!] \rangle M(f)$$

$$= \langle \pi_1, \pi_0 \pi_1 \rangle M(f)$$

So $[\![(a,y) \vdash \frac{\partial f}{\partial x}(g(a,y)) \cdot a]\!] =$

$$= \langle \langle [\![(a,y) \vdash a]\!], 0 \rangle, \langle [\![(a,y) \vdash g(a,y)]\!], 1 \rangle \rangle D[[\![((a,y),x) \vdash f(x,y)]\!]]$$

$$= \langle \langle \pi_0, 0 \rangle, \langle M(g), 1 \rangle \rangle D[\langle \pi_1, \pi_0 \pi_1 \rangle M(f)]$$

Completeness

Construct the classifying cartesian differential cat of a theory "as usual":

Objects: products of primitive types

Arrows: (equiv. classes of) sequents $x : T_1 \vdash t : T_2$

$x \mapsto t : T_1 \to T_2$

Composition:

$$(x \mapsto t)(x' \mapsto t') = (x \mapsto t'[t/x'])$$

Differential:

$$D[x \mapsto t : X \to Y] = (x', x) \vdash \frac{\partial t}{\partial x}(x) \cdot x'$$

For example, here's the verification of the chain rule (CD.5):

$$D[(x \mapsto t)(y \mapsto s)] = D[x \mapsto s[t/y]]$$

$$= (x', x) \mapsto \frac{\partial s[t/y]}{\partial x}(x) \cdot x'$$

No variable of s can occur in x

$$= (x', x) \mapsto \frac{\partial s}{\partial y}(t) \cdot \left(\frac{\partial t}{\partial x}(x) \cdot x'\right)$$

$$= \left\langle (x', x) \mapsto \frac{\partial t}{\partial x}(x) \cdot x',\ \pi_1(x \mapsto t) \right\rangle \left((y', y) \mapsto \frac{\partial s}{\partial y}(y) \cdot y' \right)$$

$$= \langle D[x \mapsto t], \pi_1(x \mapsto t) \rangle D[y \mapsto s]$$

(as required)

An example of how the term calculus "cleans up" combinator terms, consider the 2^{nd} order chain rule (seen earlier in defining composition in $B^{(2)}(\mathbb{X})$):

$$\frac{\partial^2 gf(x)}{\partial x}(x) \cdot x' \cdot x'' =$$

$$\frac{\partial^2 g}{\partial u^2}(f(x)) \cdot \left(\frac{\partial f}{\partial x}(x) \cdot x'\right) \cdot \left(\frac{\partial f}{\partial x}(x) \cdot x''\right)$$

$$+ \quad \frac{\partial g}{\partial u}(f(x)) \cdot \left(\frac{\partial^2 f}{\partial x^2}(x) \cdot x' \cdot x''\right)$$

Contrast: $G''(F'(x,z), F'(y,z), F(z)) + G'(F''(x,y,z), F(z))$

ie

$$\langle\langle (\langle 0,1\rangle \times 1) D^2[f], \underbrace{(\langle 1,0\rangle \times 1) D^2[f]}_{\overset{=}{[CD.6]} (1 \times \pi_1) D[f]} \rangle, \langle \pi_1 D[f], \pi_1 \pi_1 f \rangle\rangle D^2[g]$$

Appendix

3 slides from my FMCS 2007 talk
(Colgate University, June 2007)

showing how the comonad S may be drawn from the additive structure of a left additive category.

Additional note:

During CT2007, we (B.C.S.) showed that in a differential storage category, the deriving transform does in fact have the universal property of the Kähler differential; so it is a "property", not "structure", answering one of Anders Kock's questions.

Classification & Representation

$\mathbb{X}$ left additive cat: additive maps are *weakly* classified if

$$\begin{tikzcd} \forall A \arrow[r, "\forall f"] \arrow[d, "\exists\varphi"'] & B \\ \exists\, S_+(A) \arrow[ur, dashed, "\exists!\, f_+ \text{ (additive)}"'] & \end{tikzcd}$$

[Equivalently $\mathbb{X}_+ \hookrightarrow \mathbb{X}$ has left adjoint S_+]

$S_+(f) = (f\varphi)_+$

[*strong* classification $= \mathbb{X}[A]$ classified $\forall A$]

Classification is *retentive* if for all f, additive in B

$$\begin{tikzcd} A\times B\times X \arrow[r, "f"] \arrow[d, "1\times 1\times\varphi"'] & Y \\ A\times B\times S_+(X) \arrow[ur, "f_+^{[AB]}"'] & \end{tikzcd}$$

$f_+^{[AB]}$ is also additive in B

$\mathbb{X}$ has *weak* binary $(\otimes)$ representation if

$$\begin{tikzcd} \forall\ X\times Y \arrow[r, "\forall\text{ ``biadditive''}", "g"'] \arrow[d, "\exists\varphi_\otimes"'] & Z \\ \exists\ X\otimes Y \arrow[ur, dashed, "\exists!\text{ additive } g_\otimes"'] & \end{tikzcd}$$

Also:
- nullary rep ($\equiv$ unit)
- strong rep ($\equiv$ rep for all slices)
- retentive rep.

The point of all this? :

If $\mathbb{X}$ is a Cartesian left additive category with retentive classification & representation, then there is a canonical comonoidal natural isomorphism

$s : S_+ \to S_+$

$\left\{\begin{array}{l} S_+(A\times B) \xrightarrow{\sim} S_+(A)\otimes S_+(B) \\ S_+(1) \xrightarrow{\sim} \top \end{array}\right.$

In fact S_+ is a monoidal comonad ($\equiv$ lax)

Thm If $\mathbb{X}$ left additive with retentive classification & representation then $\mathbb{X}_+$ is an S-category

(meaning: a symmetric monoidal cat with products & a comonad S with an iso storage transformation)

- - - - -

In the following slides, we'll examine an alternate route to such structure, supposing S from the start + deriving the rest of the structure ...

Let's go back a while to revisit our alternate idea: with suitable structure on $\mathbb{X}$, we induced S as the left adjoint S_+ to $\mathbb{X}_+ \hookrightarrow \mathbb{X}$ ["suitable" = retentive additive classification & tensor representation]

Definitions: • An S-cat is a symmetric monoidal category with products, a monoidal comonad S, & the s-isos

• A 'prestorage' cat is a cartesian cat with retentively classified strong 'system of maps' (think: a subfibration, closed under $\times$, of the simple fibration given by slices $\mathbb{X}[A]$)

'abstract linear'

• A 'storage' cat is a 'prestorage' cat with retentive tensor representation.

Thm A cartesian differential category with retentive strong tensor representation has an additive sym tensor on the subcat of linear maps (given by the induced $\otimes$).

If $\mathbb{X}$ also has retentive classification, $\mathbb{X}_{lin}$ is a differential S-cat

- - - - -

This is very close to what we want:

- In a 'storage' cat, the sub cat of "systemic" maps (those in the sub fib) is an S-cat
- The coKleisli cat of an S-cat is a 'pre storage' cat
- Exact S-cats $\leftrightarrow$ systemic maps in a 'storage' cat
- A cart. diff. cat which is a 'pre storage' category in which all linear idempotents split linearly IS a 'storage' cat