Topology for $\mathcal{V}$-categories and $(\mathbb{T}, \mathcal{V})$-categories

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What is “Categorical Topology”?

- Today’s answer: Topology driven by Category Theory, *not* Category Theory driven by Topology
- Inspiration:
  - Eilenberg and Moore 1965
  - Manes 1969, Barr 1970
  - Lawvere 1973

$\mathcal{V}$-Cat

$\mathcal{V} = (\mathcal{V}, \otimes, k)$ commutative unital quantale, $k > \bot$

$$u \otimes \bigvee v_i = \bigvee u \otimes v_i$$

$\mathcal{V}$-Cat: $X = (X, a)$ — $k \leq a(x,x)$

$a(x,y) \otimes a(y,z) \leq a(x,z)$

$f : X \longrightarrow Y = (Y, b)$ — $a(x,y) \leq b(f(x), f(y))$

$2 = (2, \wedge, \top)$ — 2-Cat = Ord

$\mathbb{P}_+ = ([0,\infty]^{\mathrm{op}}, +, 0)$ — $\mathbb{P}_+$-Cat = Met

$\mathcal{V}$-Rel

$\mathcal{V}$-Rel: $r : X \longrightarrow\!\!\!\!\!\!\!+\;\; Y,\ s : Y \longrightarrow\!\!\!\!\!\!\!+\;\; Z$

$$(s \cdot r)(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z)$$

Set $\longrightarrow$ $\mathcal{V}$-Rel: $f : X \longrightarrow Y$

$$f(x, y) = \begin{cases} k & \text{if } f(x) = y, \\ \bot & \text{else.} \end{cases}$$

$\mathcal{V}$-Cat:

(X, a)		$1_X \leq a$	$k \leq a(x, x)$
		$a \cdot a \leq a$	$a(x, y) \otimes a(y, z) \leq a(x, z)$
$f : X \longrightarrow Y$		$f \cdot a \leq b \cdot f$	
		$a \leq f^\circ \cdot b \cdot f$	$a(x, x') \leq b(f(x), f(x'))$

Some properties

$\mathcal{V}$-Rel	is sup-enriched (a quantaloid), symmetric monoidal
$\mathcal{V}$-Cat	is symmetric monoidal closed, involutive

$X \otimes Y$	$a \otimes b\big((x,y),(x',y')\big) = a(x,x') \otimes b(y,y')$
E	1_E
$X \multimap Y$	$a \multimap b(f,g) = \bigwedge b\big(f(x), g(x)\big)$
X^{op}	a°
$\mathcal{V}$	$\multimap \quad (z \leq u \multimap v \iff z \otimes u \leq v)$

$\mathcal{V}$-Mod

$$\varphi : X \multimap Y$$
$$\varphi \cdot a \leq \varphi, \quad b \cdot \varphi \leq \varphi$$
$$a(x, x') \otimes \varphi(x', y') \otimes b(y', y) \leq \varphi(x, y)$$

$\mathcal{V}$-Mod is a quantaloid

$$(f : X \longrightarrow Y) \longmapsto \quad \begin{array}{ll} (f_* : X \multimap Y) & f_* = b \cdot f \\ (f^* : Y \multimap X) & f^* = f^\circ \cdot b \\ f_* \dashv f^* & \end{array}$$

The 2-category $\mathcal{V}$-Cat

$\mathcal{V}$-Cat is Ord-enriched:

$$f \leq g \iff f^* \leq g^* \iff g_* \leq f_*$$
$$\iff \forall x \in X : k \leq b\big(f(x), g(x)\big)$$

$$(-)_* : (\mathcal{V}\text{-Cat})^{\mathrm{co}} \longrightarrow \mathcal{V}\text{-Mod}, \quad (-)^* : (\mathcal{V}\text{-Cat})^{\mathrm{op}} \longrightarrow \mathcal{V}\text{-Mod}$$

$$f \dashv g \iff g^* \dashv f^* \iff g_* \dashv f_* \iff f_* = g^*$$
$$\iff \forall x, y : a\big(x, g(y)\big) = b\big(f(x), y\big)$$

$$\varphi : X \,{-\!\!\circ\!\!\rightarrow}\, Y \iff \varphi : X^{\mathrm{op}} \otimes Y \longrightarrow \mathcal{V}$$

$\mathcal{V}$-Cat is topological

$\mathcal{V}$-Cat $\longrightarrow$ Set is a fibration and opfibration with complete fibres

$$\begin{array}{ccc} (X,a) & \xrightarrow{f} & (Y,b) \\ & \downarrow & \\ X & \xrightarrow{f} & Y \end{array} \qquad \begin{array}{l} a = f^\circ \cdot b \cdot f \\ a(x,x') = b\big(f(x), f(x')\big) \end{array}$$

f fully faithful: $\quad 1_X^* = f^* \cdot f_* (= f^\circ \cdot b \cdot b \cdot f)$

Where is the topology?

$f \quad L$-dense: $\quad 1_Y^* = f_* \cdot f^* \;\; (= b \cdot f \cdot f^\circ \cdot b)$

$$b(y, y') = \bigvee_{x \in X} b\big(y, f(x)\big) \otimes b\big(f(x), y'\big)$$

$\overline{M} = \bigcup\{N \subseteq X \mid M \hookrightarrow N \, L\text{-dense}\}$:

- an idempotent closure operator
- even finitely additive if $\big(k \le u \vee v \implies k \le u \text{ or } k \le v\big)$
- (L-dense, L-closed embedding)-factorizations

X is L-separated

$$\iff \delta_X : X \longrightarrow X \times X \;\; L\text{-closed}$$

$$\iff \forall f, g : Z \longrightarrow X \;\; \big(f \simeq g \implies f = g\big)$$

$$\iff \forall x, y : E \longrightarrow X \big(x \simeq y \implies x = y\big)$$

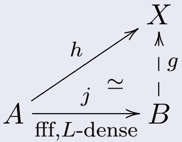
$$\iff \forall x, y \in X \big(k \leq a(x,y) \wedge a(y,x) \implies x = y\big)$$

$$\iff \forall\; D \xrightarrow[L\text{-dense}]{j} W \overset{f}{\underset{g}{\rightrightarrows}} X \quad \big(f \cdot j = g \cdot j \implies f = g\big)$$

X L-complete

$$\iff \forall \varphi : Z \rightarrow\!\!\!\circ\!\!\!\rightarrow X \ (\varphi \text{ map} \implies \exists f : \varphi = f_*)$$

$$\iff X \ \ L\text{-injective}$$



$$\iff \forall \varphi : E \rightarrow\!\!\!\circ\!\!\!\rightarrow X \ \ (\varphi \text{ map} \implies \exists x : \varphi = x_*)$$

Choice!

Preservation properties

$\mathcal{V}$ is L-separated and L-complete

Y L-separated $\Longrightarrow$ $X \multimap Y$ L-separated
Y L-complete $\Longrightarrow$ $X \multimap Y$ L-complete

$X \otimes Y$ L-separated if $k = \top$
$X \otimes Y$ L-complete if $k = \top$ and $\left(k \leq \bigvee u_i \implies k \leq \bigvee u_i \otimes u_i\right)$
(for X, Y L-separated/L-complete)

Yoneda

$$\frac{\dfrac{a = 1_X^* : X \multimap\!\!\to X}{a : X^{\mathrm{op}} \otimes X \longrightarrow \mathcal{V}}}{y : X \longrightarrow (X^{\mathrm{op}} \multimap \mathcal{V}) = \hat{X}}$$

- $\hat{a}(\varphi, \psi) = \bigwedge_{x \in X} \varphi(x) \multimap \psi(x)$
- Yoneda Lemma: y is fully faithful
- $\hat{X}$ is L-complete and L-separated

Reflectivity

$$X \longrightarrow \underline{y(X)} \text{ is the } (\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep}})\text{-reflection}$$
$$X \longrightarrow \overline{y(X)} =: \tilde{X} \text{ is the } (\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{cpl}})\text{-pseudo-reflection}$$

Equivalent:

(i) $\psi \in \tilde{X} \;\; (\psi : X^{\mathrm{op}} \longrightarrow \mathcal{V})$

(ii) ψ right adjoint $(\psi : X \multimap E)$

(iii) $k \leq \bigvee_{y \in X} \left(\bigwedge_{x \in X} a(x,y) \multimap \psi(x) \right) \otimes \psi(y)$

$\mathcal{V}$ cogenerates $\mathcal{V}$-$\mathsf{Cat}_{\mathrm{sep}}$ and $\mathcal{V}$-$\mathsf{Cat}_{\mathrm{cpl}}$.

Topological spaces

Manes

(X, c)

$$c : \beta X \longrightarrow X \quad \mathfrak{x} c y \iff \mathfrak{x} \longrightarrow y$$

$$\dot{x} \longrightarrow x$$

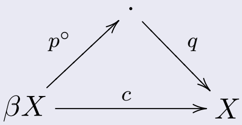
$$\mathfrak{X} \longrightarrow \mathfrak{y}, \mathfrak{y} \longrightarrow z \implies \sum \mathfrak{X} \longrightarrow z$$

$f : X \longrightarrow Y$

$$\mathfrak{x} \longrightarrow y \implies f(\mathfrak{x}) \longrightarrow f(y)$$

Barr

- $c : \beta X \longrightarrow\!\!\!\!\!\!+\; X$
- $1_X \leq c \cdot e_X$
- $c \cdot \bar{\beta} c \leq c \cdot m_X$



$$\bar{\beta} c = \beta q \cdot (\beta p)^\circ$$

The $(\mathbb{T},\mathcal{V})$-setting

$\mathcal{V} = (\mathcal{V}, \otimes, k)$ as before
$\mathbb{T} = (T, e, m)$ Set-monad with *lax extension* (G. Seal)
$\hat{T} : \mathcal{V}\text{-Rel} \longrightarrow \mathcal{V}\text{-Rel}$

(0) $\hat{T}X = TX$

(1) $1_{TX} \leq \hat{T}1_X$, $\hat{T}s \cdot \hat{T}r \leq \hat{T}(s \cdot r)$, $r \leq r' \implies \hat{T}r \leq \hat{T}r'$

(2) $Tf \leq \hat{T}f$, $(Tf)^\circ \leq \hat{T}(f^\circ)$

(3) $e_Y \cdot r \leq \hat{T}r \cdot e_X$, $m_Y \cdot \hat{T}\hat{T}r \leq \hat{T}r \cdot m_X$

$(\mathbb{T}, \mathcal{V})$-Cat

$$
\begin{array}{llll}
(X,c) & c: TX \longrightarrow\!\!\!\!\!\mapsto X & 1_X \leq c \cdot e_X & 1_X^{\#} \leq c \\
 & & c \cdot \hat{T}c \leq c \cdot m_X & c * c \leq c \\
f: X \longrightarrow Y = (Y,d) & & f \cdot c \leq d \cdot Tf & 1_X^{*} \leq f^{*} * f_{*}
\end{array}
$$

$1_X^{\#} = e_X^{\circ} \cdot \hat{T} 1_X$ (discrete structure on X)
$r: TX \longrightarrow\!\!\!\!\!\mapsto Y$, $s: TY \longrightarrow\!\!\!\!\!\mapsto Z$: $s * r = s \cdot \hat{T} r \cdot m_X^{\circ}$ (Kleisli)

$(\mathbb{T}, \mathcal{V})$-Cat is topological over Set
But: monoidal, monoidal closed, involutive, Yoneda ???

$(\mathbb{T}, \mathcal{V})$-Mod

$$\varphi : (X, c) \multimap (Y, d) : \qquad \varphi : TX \nrightarrow Y$$
$$\varphi * c \leq \varphi \ , \ d * \varphi \leq \varphi \qquad \varphi \cdot \hat{T}c \cdot m_X^\circ \leq \varphi, d \cdot \hat{T}\varphi \cdot m_X^\circ \leq \varphi$$

$f_* = d \cdot Tf : X \multimap Y, \;\; f^* = f^\circ \cdot d : Y \multimap X, \;\; f_* \dashv f^*.$

$(\mathbb{T}, \mathcal{V})$-Cat $\xrightarrow{(-)_*}$ $(\mathbb{T}, \mathcal{V})$-Mod $\qquad$ $\big((\mathbb{T}, \mathcal{V})\text{-Cat}\big)^{\mathrm{op}} \xrightarrow{(-)^*}$ $(\mathbb{T}, \mathcal{V})$-Mod

What $(\mathbb{T}, \mathcal{V})$-Mod actually is

$$\mathcal{M} = (\mathbb{T}, \mathcal{V})\text{-}\mathsf{Mod}, \;\; \mathcal{K} = (\mathbb{T}, \mathcal{V})\text{-}\mathsf{Cat}$$

$$\begin{array}{llll} \mathcal{M}\colon & \mathcal{K}^{\mathrm{op}} \longrightarrow & [\mathcal{K}^{\mathrm{op}}, \mathsf{Ord}] \\ & Y \longmapsto & \big(X \longmapsto \mathcal{M}(X,Y)\big) \\ & & \big(f \longmapsto \mathcal{M}(f,Y) : (\varphi \longmapsto \varphi * f_*)\big) \\ & g \longmapsto & \mathcal{M}(X,g) : (\varphi \longmapsto g^* * \varphi) \end{array}$$

"equipment with scalar category $\mathcal{K}$" (Carboni, Kelly, Wood)

$\mathbb{T}$ as a $\mathcal{V}$-Cat monad with $\mathcal{V}$-Mod extension

$$\begin{array}{ccc} \mathcal{V}\text{-}\mathsf{Cat} & \xrightarrow{T} & \mathcal{V}\text{-}\mathsf{Cat} \\ \uparrow & \geq & \uparrow \\ \mathsf{Set} & \xrightarrow{T} & \mathsf{Set} \end{array}$$

$$(X, a) \longmapsto (TX, \hat{T}a)$$

$$1_{TX} = T1_X \leq \hat{T}1_X$$

$$\begin{array}{ccc} \mathcal{V}\text{-}\mathsf{Mod} & \xrightarrow{\hat{T}} & \mathcal{V}\text{-}\mathsf{Mod} \\ (-)_* \uparrow & = & \uparrow (-)_* \\ \mathcal{V}\text{-}\mathsf{Cat} & \xrightarrow{T} & \mathcal{V}\text{-}\mathsf{Cat} \end{array} \qquad \begin{array}{ccc} \mathcal{V}\text{-}\mathsf{Mod} & \xrightarrow{\hat{T}} & \mathcal{V}\text{-}\mathsf{Mod} \\ (-)^* \uparrow & = & \uparrow (-)^* \\ (\mathcal{V}\text{-}\mathsf{Cat})^{\mathrm{op}} & \xrightarrow{T^{\mathrm{op}}} & (\mathcal{V}\text{-}\mathsf{Cat})^{\mathrm{op}} \end{array}$$

Strict vs. lax Eilenberg-Moore

$$\big(\mathcal{V}\text{-}\mathsf{Cat}\big)^{\mathbb{T}} \xrightarrow{K} (\mathbb{T},\mathcal{V})\text{-}\mathsf{Cat}$$
$$\big((X,a),\xi\big) \longmapsto (X, a\cdot\xi)$$

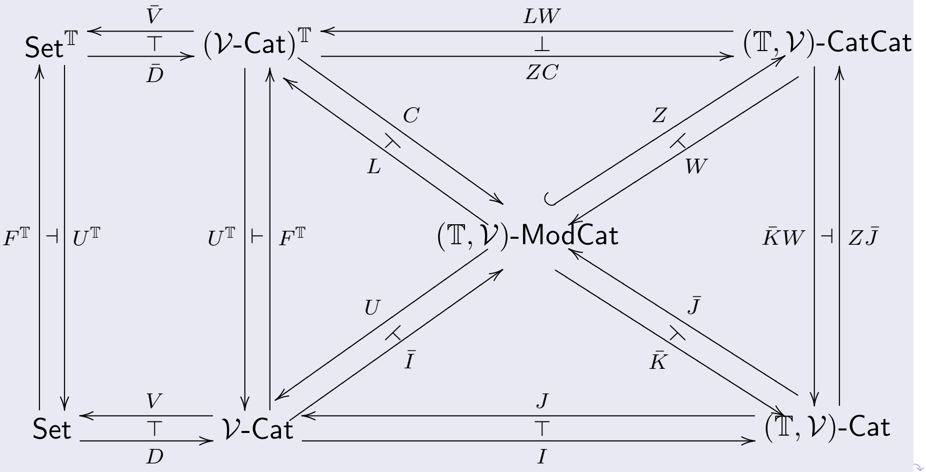
Properties?

$(\mathbb{T},\mathcal{V})$-ModCat: (X,a,c) $(X,a)\in\mathcal{V}$-Cat
$(X,c)\in(\mathbb{T},\mathcal{V})$-Cat
$c : T(X,a) \to\!\!\!\!\circ\; (X,a)$ in $\mathcal{V}$-Mod

$(\mathbb{T},\mathcal{V})$-ModCat is topological over $\mathcal{V}$-Cat: $(X,a,c)\longmapsto(X,a)$

$$\big(\mathcal{V}\text{-}\mathsf{Cat}\big)^{\mathbb{T}} \xrightarrow{C} (\mathbb{T},\mathcal{V})\text{-}\mathsf{ModCat}$$
$$\big((X,a),\xi\big) \longmapsto (X, a, \xi_*)$$

The whole picture



Dualizing a $(\mathbb{T}, \mathcal{V})$-category

$$
\begin{array}{ccccc}
\mathcal{V}\text{-}\mathsf{Cat} & \xrightarrow{\;I\;} & (\mathbb{T}, \mathcal{V})\text{-}\mathsf{Cat} & \xrightarrow{\;M\;} & \mathcal{V}\text{-}\mathsf{Cat} \\
{\scriptstyle (-)^{\mathrm{op}}}\downarrow & & {\scriptstyle (-)^{\mathrm{op}}}\downarrow & & \downarrow{\scriptstyle (-)^{\mathrm{op}}} \\
\mathcal{V}\text{-}\mathsf{Cat} & \xrightarrow[\;N\;]{} & (\mathbb{T}, \mathcal{V})\text{-}\mathsf{Cat} & \xrightarrow[\;J\;]{} & \mathcal{V}\text{-}\mathsf{Cat}
\end{array}
$$