Weak Complicial Sets and Internal Quasi-Categories

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Motivation

Simplicial sets are lovely objects about which algebraic topologists know a lot. If something is described as a simplicial set, it is ready to be absorbed into topology. Or, in other words, no matter which definition of weak ω-category eventually becomes dominant, it will be valuable to know its simplicial nerve.

Ross Street, 2003

Nerves of (Strict) ω-Categories

We start by assuming that everyone is familiar with the categories Δ and Δ_+ of finite ordinals, $\underline{\text{Simp}} = [\Delta^{\text{op}}, \underline{\text{Set}}]$ of simplicial sets and ω-$\underline{\text{Cat}}$ of (strict) ω-categories.

To construct nerves of ω-categories, Ross Street started by constructing a functor $\mathcal{O}\colon \Delta \longrightarrow \omega\text{-}\underline{\text{Cat}}$ which maps the ordinal $[n]$ to the "free ω-category on the n-simplex", which he called the n^{th} *oriental*.

To this he applied *Kan's construction* to obtain an adjoint pair:

$$\omega\text{-}\underline{\text{Cat}} \underset{N}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \underline{\text{Simp}}$$

Here N is called the *ω-categorical nerve* and F is the corresponding *ω-categorical realisation*.

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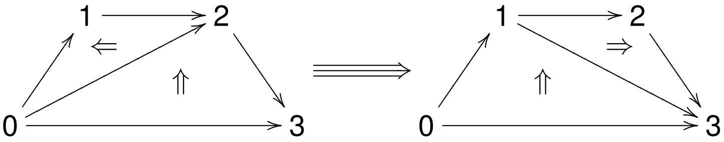
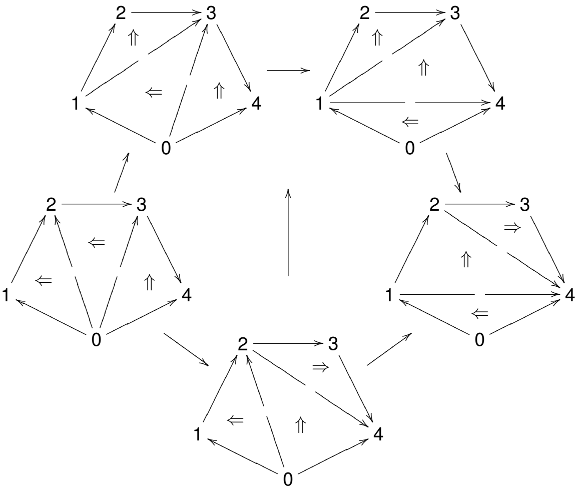
ω-Categorical Nerves in Pictures

For example, a 3-simplex in the nerve of a ω-category $\mathbb{C}$ may be drawn as:



Wherein single arrows are 1-cells, double arrows are 2-cells and so forth.

A 4-Simplex in $N(\mathbb{C})$



Some Simplicial Notation

- $\Delta[n]$ the *standard n-simplex* is simply the representable in $\underline{\text{Simp}}$ on $[n] \in \Delta$.
- $\partial\Delta[n]$ the *boundary* of $\Delta[n]$ is its simplicial subset generated by its $(n-1)$-dimensional faces $\delta_i \colon [n-1] \longrightarrow [n]$ $(i = 0, 1, 2, ..., n)$.
- $\Lambda^k[n]$ the *standard* $(n-1)$*-dimensional k-horn* is the simplicial subset of $\Delta[n]$ generated by the faces δ_i for $i \in [n] \setminus \{k\}$.
- We say that an n-simplex $x\colon \mathcal{O}[n] \longrightarrow \mathbb{C}$ of $N(\mathbb{C})$ is *thin* if it maps the unique non-trivial n-cell of $\mathcal{O}[n]$ to an identity.

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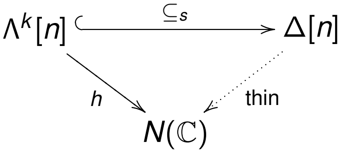
Horn Filling in ω-Categorical Nerves

We might think of a simplicial map $h\colon \Lambda^k[n] \longrightarrow N(\mathbb{C})$, known as a *horn in* $N(\mathbb{C})$, as a "co-cycle problem" in $\mathbb{C}$ and ask the question *can we "solve" this problem by filling the horn with a thin simplex?*



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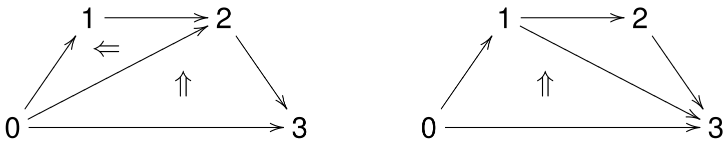
For instance here is a 1-dimensional 0-horn in $\mathbb{C}$



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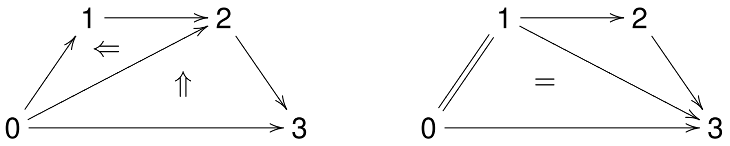
To solve it we must insist that two of its cells are identities.



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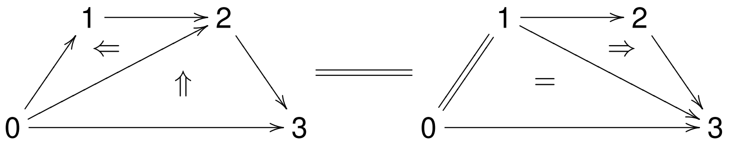
Filling this horn pastes together the diagram on the LHS and puts the resulting 2-cell in the free space on the RHS.



Complicial Horns

Ross also provides us with a *complicial decomposition* of each oriental $\mathcal{O}[n]$.

This allows us to prove that the horn $h\colon \Lambda^k[n] \longrightarrow N(\mathbb{C})$ has a (unique) solution if the simplex $h(\alpha) \in N(\mathbb{C})$ is thin for each simplex α of $\Lambda^k[n]$ whose vertices include those in $[n] \cap \{k-1, k, k+1\}$.

A horn which satisfies this condition is said to be *complicial*.

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Keeping Track of Thin Simplices

A *stratified simplicial set* is a simplicial set X equipped with a specified subset $tX \subseteq X$ of *thin* simplices satisfying the conditions:

- no 0-simplex is thin, and
- all degenerate simplices are thin.

A simplicial map $f\colon X \longrightarrow Y$ between stratified sets is said to be *stratified* if it preserves thinness - that is if $f(tX) \subseteq tY$. We get a category (quasi-topos) of stratified sets and stratified maps called Strat.

We implicitly promote every simplicial set to a stratified set by giving it the *minimal* stratification, in which only the degenerate simplices are thin.

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The nerve $N(\mathbb{C})$ has a canonical stratification under which whose elements are the thin simplices identified before. This stratification augments the nerve functor to a functor $N\colon \omega\text{-}\underline{\text{Cat}} \longrightarrow \underline{\text{Strat}}$.

So if we let $\Delta^k[n]$ denote the *standard k-complicial n-simplex* whose stratification makes thin all of those simplices identified as “in need of inversion” by Ross’ decomposition theorem ...

... and we also let $\Lambda^k[n]$ denote the *standard* $(n-1)$*-diml k-horn*, whose underlying simplicial set it the usual k-horn and which inherits its stratification from $\Delta^k[n]$ ...

... then we can rephrase the unique horn filler condition given earlier, to say that any stratified simplicial map $\Lambda^k[n] \longrightarrow N(\mathbb{C})$ has a *unique* extension along the inclusion $\Lambda^k[n] \hookrightarrow N(\mathbb{C})$.

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With a little effort, we may show that the nerve functor $N\colon \omega\text{-}\underline{\text{Cat}} \longrightarrow \underline{\text{Strat}}$ is full and faithful.

A deeper analysis also reveals a simple characterisation of the stratified simplicial sets in the replete image of this functor.

They are precisely the *complicial sets*, which were introduced by John Roberts' in his work on local cohomology theories.

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So we know that strict ω-categories live quite happily in Strat - how about *weak* ω-categories?

We already have a candidate model for the theory of those weak ω-categories in which all cells are equivalences above dimension 1 - Joyal's *quasi-categories*.

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warning! we've dropped uniqueness

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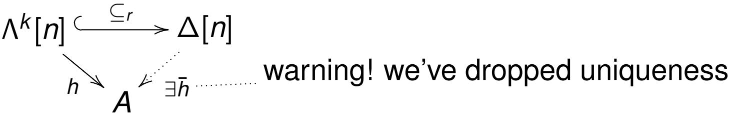

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- dropping the first of the complicial set axioms and
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In this approach the thin simplices of a stratified simplicial set are no longer to be regarded as equalities but as a (chosen class of) *equivalences*.

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Weak Complicial Sets

We need to be a little careful with Street's idea, since the original complicial set notion says nothing about fillers for *outer* horns. In that context

Degeneracy Condition + Inner Horn Fillers $\Rightarrow$ Outer Horn Fillers

To compensate for the loss of the degeneracy condition we ask for (non-unique) fillers for outer horns as well.

We thus arrive at the definition of structures called *weak complicial sets*, in which:

- Fillers for complicial horns provide weakened, non-unique composition / boundary mutation operations on simplices.
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- A quasi-category may be given a canonical stratification under which it becomes a weak complicial set. We do this by making thin all simplices above dimension 1 and making thin the *simplicial equivalences* at dimension 1.
- Complicial sets (and thus ω-categories) are all weak complicial sets, amongst which they may be characterised by a simple axiom relating thinness and degeneracy.
- The *homotopy coherent nerve* of a category enriched in weak complicial sets (under the cartesian product) is also a weak complicial set.

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- The *homotopy coherent nerve* of a category enriched in weak complicial sets (under the cartesian product) is also a weak complicial set.

Slogan

Somewhat facetiously, at this stage, our slogan might be:

Quasi-categories + Strict ω-Categories $\sim$ Weak Complicial Sets

Even more speculatively:

Weak Complicial Sets $\sim$ Weak ω-Categories

Indeed, to even get the second of these slogans off the ground we will need to add one last ingredient.

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The Equivalence Property of Simplicial Weak ω-Categories

Let E denote the *generic simplicial equivalence*, that is to say the nerve of the generic isomorphism category $\mathbb{I}$.

For $X \in \underline{\text{Strat}}$ let $\text{th}_n(X) \in \underline{\text{Strat}}$ denote the stratified simplicial set obtained from X by making thin all of its simplices above dimension n.

Let $\oplus$ denote the *simplicial join* bifunctor extended in an appropriate way to $\underline{\text{Strat}}$ - this extends the ordinal sum on Δ via Day's convolution formula.

Then we say that a weak complicial set is a simplicial weak ω-category iff it is injective with respect to each inclusion

$$\Delta[n] \oplus \text{th}_2(E) \oplus \Delta[m] \hookrightarrow \Delta[n] \oplus \text{th}_1(E) \oplus \Delta[m]$$

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The Lax Gray-Tensor Product

The category Strat supports a tensor product which is an analogue of the *lax Gray-tensor product*, in the precise sense that if

- $\otimes$ denotes both the lax Gray-tensor product on ω-Cat and this *simplicial lax Gray-tensor* in Strat, and
- $F\colon \text{Strat} \longrightarrow \omega\text{-Cat}$ denotes the realisation functor left adjoint the nerve construction,

then F is a strong monoidal functor from $(\text{Strat}, \otimes, 1)$ to $(\omega\text{-Cat}, \otimes, 1)$.

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Constructing the Simplicial Lax Gray-Tensor

The simplicial lax Gray-tensor of stratified simplicial sets $X, Y \in \underline{\text{Strat}}$ is constructed by taking the cartesian product of the underlying simplicial sets $|X|$ and $|Y|$ in $\underline{\text{Simp}}$ and giving it a stratification which is a suitable sub-stratification of that of the stratified product $X \times Y$.

The category $\underline{\text{Strat}}$ hosts a canonical *complicial Quillen model structure*, whose fibrant objects are precisely the simplicial weak ω-categories. Most importantly, this model structure is monoidal with respect to $\otimes$.

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Closures for $\otimes$

Unfortunately, the bifunctor $\otimes$ is not co-continuous in each variable, so it does not make Strat into a biclosed category.

However, we can show that if $A \in$ Strat satisfies the third of the complicality axioms then we can form left and right closures $\text{lax}_l(X, A)$ and $\text{lax}_r(X, A)$ for any stratified set X.

The left closure $\text{lax}_l(X, A)$ can be thought of as a simplicial generalisation of the bicategory of homomorphisms, lax natural transformations and modifications between any two bicategories.

The fact that the complicial Quillen model structure on Strat is monoidal immediately implies that $\text{lax}_l(X, A)$ (resp. $\text{lax}_r(X, A)$) is a simplicial weak ω-category whenever A is.

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The Internal Quasi-Category of Prisms

Given a weak complicial set (or a simplicial weak ω-category) A, we can form the following simplicial object in Strat

$$\text{lax}_l(\text{th}_1(\Delta[0]), A) \rightleftarrows \text{lax}_l(\text{th}_1(\Delta[1]), A) \rightleftarrows \text{lax}_l(\text{th}_1(\Delta[2]), A) \cdots\cdots$$

which we denote by $\mathbb{P}_l(A)$.

We might think of this as the (internal) nerve of the complicial set, whose n-arrows are simply the *(left) prisms of A* with cross section $\Delta[n]$.

From the monoidality of the complicial Quillen model structure we already know that each stratified set $\mathbb{P}_l(A)([i])$ is a weak complicial set (or simplicial weak ω-category), but we can derive much more...

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$\mathbb{P}(A)$ as an Internal Quasi-Category

We may show that each inclusion $\mathrm{th}_1(\Lambda^k[n]) \overset{j}{\hookrightarrow} \mathrm{th}_1(\Delta[n])$ is a trivial cofibration in the complicial model structure.

So applying monoidality we see that the *Leibniz tensor* of that map with any cofibration (inclusion) $U \overset{i}{\hookrightarrow} V$ in Strat

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is also a trivial cofibration. It follows therefore that this map has the left lifting property with respect to the weak complicial set A.

Taking duals it follows that each simplicial map $- \circ i\colon \underline{\mathrm{Strat}}(V, \mathbb{P}_l(A)(-)) \longrightarrow \underline{\mathrm{Strat}}(U, \mathbb{P}_l(A)(-))$ has the right lifting property with respect to $\Lambda^k[n] \hookrightarrow \Delta[n]$.

In other words, this simplicial map is an *inner or mid anodyne extension*.

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We might think of the property developed in the last slide as the homotopically correct, representable internalisation (to $\underline{\text{Strat}}$) of the quasi-category notion.

With a few minor modifications, we may apply this *internal quasi-categoricity* definition to simplicial objects in any left proper Quillen model category.

Beyond internal quasi-categoricity, we can use similar arguments to show that $\mathbb{P}(A)$ is also:

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Internal Quasi-Categories in a Model Category

To do quasi-category theory in a model category $\mathcal{M}$ we will need to assume extra technical conditions on that category.

For example, it should at least be *combinatorial* so that we can easily build Quillen model structures on categories constructed from $\mathcal{M}$.

We also require a bunch of *left "properness"* properties.

For our purposes here we'll assume that all objects in $\mathcal{M}$ are cofibrant. Then we'll assume that the map $: B \vee_A C \longrightarrow D$ induced by applying the pushout property to a commutative square of cofibrations is itself a cofibration. Finally, we'll assume that all split monomorphism are cofibrations.

These last two properties ensure that all pointwise cofibrations in $\mathcal{M}$ are actually Reedy cofibrations.

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Internal Quasi-Category Theory

As an example of how we can do genuine (quasi-)category theory with internal quasi-categories, let $\mathcal{M}$ be a model category as in the last slide and suppose that $\mathbb{A}$ is an internal quasi-category in $\mathcal{M}$.

Now $\underline{\text{Simp}}(\mathcal{M})$ is canonically enriched in simplicial sets, and if $X \in \underline{\text{Simp}}$ and $\mathbb{A} \in \underline{\text{Simp}}(\mathcal{M})$ then their cotensor $X \pitchfork \mathbb{A}$ is given by the weighted limit formula $(X \pitchfork \mathbb{A})(n) = \lim(X \times \Delta[n], \mathbb{A})$.

Now, we say that a simplicial map $p \colon \mathbb{E} \longrightarrow \mathbb{A}$ is a *left discrete fibration* iff the canonical map

$$\Delta[1] \pitchfork \mathbb{E} \longrightarrow (\Delta[1] \pitchfork \mathbb{A}) \;{}_{d_1}\!\times_p \mathbb{E}$$

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Some Properties of Left Discrete Fibrations

- The slice category $\underline{\mathrm{Simp}}(\mathcal{M})/\mathbb{A}$ has a canonical (strict) enrichment over $\underline{\mathrm{Simp}}(\mathcal{M})$. Under this enrichment, if $(\mathbb{E}, p)$ is a left discrete fibration then each $\underline{\mathrm{Simp}}(\mathcal{M})/\mathbb{A}((\mathbb{B}, f), (\mathbb{E}, p))$ is *quasi-discrete* - where we say that $\mathbb{C}$ is quasi-discrete if $d_1 : \Delta[1] \pitchfork \mathbb{C} \longrightarrow \mathbb{C}$ is a reedy weak equivalence).
- If $a \colon 1 \longrightarrow \mathbb{A}$ is an arrow in $\underline{\mathrm{Simp}}(\mathcal{M})$ then the *representable* $p \colon \mathbb{A} \downarrow a \longrightarrow \mathbb{A}$ formed via the pullback

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The Yoneda Lemma

- The unique map $!\colon \mathbb{A}\downarrow a \longrightarrow 1$ has a right inverse $t\colon 1 \longrightarrow \mathbb{A}\downarrow a$ which picks out the identity on a.
- We have a map $\epsilon\colon \Delta[1]\cdot(\mathbb{A}\downarrow a) \longrightarrow \mathbb{A}\downarrow a$ which evaluates to the identity on $\mathbb{A}\downarrow a$ at $0\in\Delta[1]$ and to the constant function mapping to a at $1\in\Delta[1]$.
- If $(\mathbb{E},p)$ is an arbitrary left discrete fibration, then $\underline{\mathrm{Simp}}(\mathcal{M})/\mathbb{A}(-,(\mathbb{E},p))$ maps the above data to a homotopy equivalence.
- Notice also that $\underline{\mathrm{Simp}}(\mathcal{M})/\mathbb{A}((1,a),(\mathbb{E},p))$ is isomorphic to the pullback $\mathbb{E}_a$ of $(\mathbb{E},p)$ along a.
- Finally, we find that the map $\underline{\mathrm{Simp}}(\mathcal{M})/\mathbb{A}((\mathbb{A}\downarrow a,p_0),(\mathbb{E},p)) \longrightarrow \mathbb{E}_a$, induced by pre-composition with the map t, is a Reedy trivial fibration.

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- The unique map $! \colon \mathbb{A} \downarrow a \longrightarrow 1$ has a right inverse $t \colon 1 \longrightarrow \mathbb{A} \downarrow a$ which picks out the identity on a.
- We have a map $\epsilon \colon \Delta[1] \cdot (\mathbb{A} \downarrow a) \longrightarrow \mathbb{A} \downarrow a$ which evaluates to the identity on $\mathbb{A} \downarrow a$ at $0 \in \Delta[1]$ and to the constant function mapping to a at $1 \in \Delta[1]$.
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A Coherence Result

Now enrich $\underline{\text{Simp}}(\mathcal{M})/\mathbb{A}$ over $\mathcal{M}$ rather than $\underline{\text{Simp}}(\mathcal{M})$, simply by taking the objects of 0-simplices of each of the homsets of this latter enrichment.

Then the Yoneda lemma, along with a few facts about Reedy trivial fibrations, tells us that the following internal quasi-categories are Reedy weakly equivalent:

- The full subcategory of the genuine enriched category $\underline{\text{Simp}}(\mathcal{M})/\mathbb{A}$ on the representables $p\colon \mathbb{A} \downarrow a \longrightarrow \mathbb{A}$, and
- The internal quasi-category obtained by "pulling back" $\mathbb{A}$ to the discrete set of points of its object of 0-simplices $\mathbb{A}([0])$.

Compare this with the Gordon, Power and Street proof of coherence for tricategories.

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A Conjecture

Based upon our experience with (strict) complicial sets, and the result of the last slide, we might pose the following conjecture:

Every simplicial weak ω-category A is homotopy equivalent to the homotopy coherent nerve of the $\underline{\text{Strat}}$-enriched category of representables over the internal quasi-category of prisms $\mathbb{P}(A)$.