Cartesian Bicategories as Symmetric Monoidal Bicategories

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The Axioms

(All bicategories are *normal*) A bicategory $\mathbf{B}$ is precartesian if

- $\mathbf{M} = \mathrm{Map}\mathbf{B}$ has $\times$ and 1, products (as a bicategory)
- Each $\mathbf{B}(X, A)$ has $\wedge$ and $\top$, products

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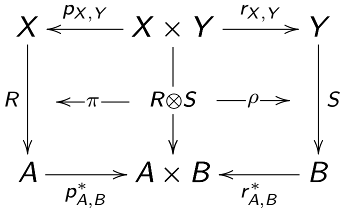
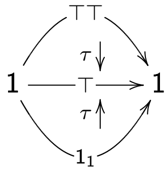
A precartesian bicategory **B** admits *lax* functors

$$\mathbf{B} \times \mathbf{B} \xrightarrow{\otimes} \mathbf{B} \xleftarrow{I} \mathbf{1}$$

$$R \otimes S := p^* R p \wedge r^* S r$$



I is the monad



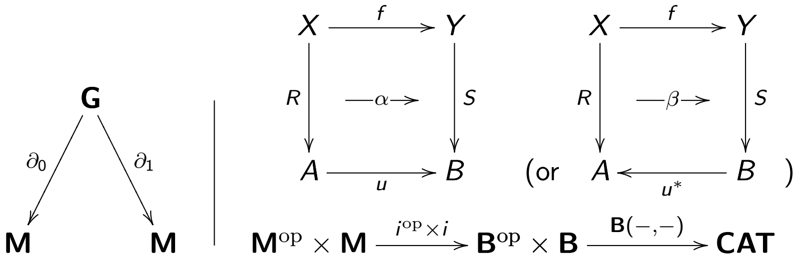
The Grothendieck Bicategory

$$\begin{array}{ccc} & \mathbf{G} & \\ {\scriptstyle \partial_0}\swarrow & & \searrow{\scriptstyle \partial_1} \\ \mathbf{M} & & \mathbf{M} \end{array}$$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle R}\downarrow & \xrightarrow{\alpha} & \downarrow{\scriptstyle S} \\ A & \xrightarrow[u]{} & B \end{array} \qquad \left(\text{or} \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle R}\downarrow & \xrightarrow{\beta} & \downarrow{\scriptstyle S} \\ A & \xleftarrow[u^*]{} & B \end{array} \right)$$

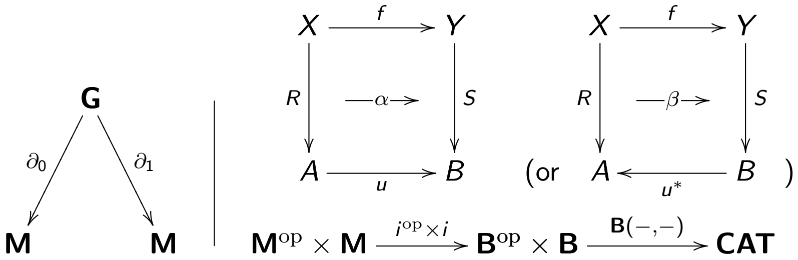
$$\mathbf{M}^{\mathrm{op}} \times \mathbf{M} \xrightarrow{i^{\mathrm{op}} \times i} \mathbf{B}^{\mathrm{op}} \times \mathbf{B} \xrightarrow{\mathbf{B}(-,-)} \mathbf{CAT}$$

The Grothendieck Bicategory



- **G** has $\times$ and 1, products preserved by ∂_0 and ∂_1
- On objects of **G**, $\times$ is $\otimes$

The Grothendieck Bicategory



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- On objects of **G**, $\times$ is $\otimes$

A precartesian bicategory **B** is cartesian if

- $\otimes$ and I are pseudofunctors (iff $\tilde{\otimes}{:}(T \otimes U)(R \otimes S) \to TR \otimes US$
  $\otimes^\circ{:}1_{X \otimes Y} \to 1_X \otimes 1_Y$ and $I^\circ{:}1_1 \to \top$ are invertible)

Note that, in general, **B** does not have $\times$ and 1, products

Why Cartesian Bicategories?

Good axiom base to characterize bicategories of:

- (Carboni & Walters 1987 (locally ordered case))
  - relations in a regular category
  - ordered objects and order ideals in an exact category
  - additive relations in an abelian category
  - relations in a Grothendieck topos
- (CKWW @ CT06) spans in a category with finite limits
- profunctors in an elementary topos

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Observe that a cartesian bicategory is a bicategory with *properties*

? What is the derived *structure* provided by the pseudofunctors

$$\mathbf{B} \times \mathbf{B} \xrightarrow{\otimes} \mathbf{B} \xleftarrow{I} \mathbf{1}$$

Cartesian Bicategories as Symmetric Monoidal Bicategories

For a cartesian $\mathbf{B}$, if

$$\mathbf{G} \times \mathbf{G} \xrightarrow{\times} \mathbf{G} \xleftarrow{1} \mathbf{1}$$

underlies symmetric monoidal structure
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? Is a bicategory with $\times$ and 1, products symmetric monoidal

A bicategory **A** with $\times$ and 1, products

- For each pair (X, Y) of objects there is an object $X \times Y$
- and arrows $p_{X,Y}: X \times Y \to X$ and $r_{X,Y}: X \times Y \to Y$
- such that, for each A,

$$\mathbf{A}(A, X \times Y) \xrightarrow{\langle \mathbf{A}(A,p), \mathbf{A}(A,r) \rangle} \mathbf{A}(A, X) \times \mathbf{A}(A, Y)$$

  is an equivalence of categories
- There is an object 1 such that, for each A,

$$\mathbf{A}(A, 1) \xrightarrow{\ !\ } \mathbf{1}$$

  is an equivalence of categories

A monoidal bicategory is a one-object tricategory

- $X \otimes Y$ pseudofunctorial and I
- $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ pseudonatural equivalence
- $l_X : I \otimes X \to X$ pseudonatural equivalence
- $r_X : X \otimes I \to X$ pseudonatural equivalence
- $\pi_{W,X,Y,Z} : Wa.a.aZ \to a.a$ invertible modification
- $\mu_{X,Y} : XI.a \to rY$ invertible modification

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A *symmetric* monoidal bicategory is a one-object, one-arrow, one-2-cell, one-3-cell, weak 6-category

- $s_{X,Y}:X \otimes Y \to Y \otimes X$ pseudonatural equivalence
- $\rho_{X,Y,Z}:a.s.a \to Ys.a.sZ$ invertible modification
- $\lambda_{X,Y,Z}:a^*.s.a^* \to sY.a^*.Xs$ invertible modification
- $\sigma_{X,Y}:1_\otimes \to s_{Y,X}.s_{X,Y}$ invertible modification

The modifications are to satisfy $3+4+2+1$ equations (!Rather a lot to dig out of a rather weak universal property)

A bicategory $\mathbf{A}$ has *finite products* if

- For each finite I and $X = (X_i)_{i \in I}$ in $|\mathbf{A}|$, there is P in $|\mathbf{A}|$
- and $p = (p_i : P \to X_i)_{i \in I}$
- such that, for each A,

$$\mathbf{A}(A, P) \to \prod_{i \in I} \mathbf{A}(A, X_i)$$

  is an equivalence. (Call such p a product cone over X)

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$$\Pi_n : \mathbf{A} \times \cdots \times \mathbf{A} = \mathbf{A}^n \to \mathbf{A}$$

Can write, say, $\Pi_3(X, Y, Z) = X \times Y \times Z$ *without parentheses*.

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For $X = (X_i)_{i \in I}$, write $\mathbf{A}(X) = \mathbf{A}((X_i)_{i \in I})$ for the bicategory

- whose objects are the product cones over X
- $(b_i : B \to X_i) \to (c_i : C \to X_i)$ is $R : B \to C$ with $\mu_i : c_i R \xrightarrow{\simeq} b_i$
- $(R, \mu_i) \to (S, \nu_i)$ is $\alpha : R \to S$ with $\nu_i . (c_i \alpha) = \mu_i$.

There is a forgetful pseudofunctor $\mathbf{A}(X) \to \mathbf{A}$

KEY LEMMA: Each bicategory $\mathbf{A}(X)$ is biequivalent to $\mathbf{1}$. So ...

- the set of objects is not empty
- for any (B, b_i), (C, c_i) there is an $(R, \mu_i){:}(B, b_i) \to (C, c_i)$
- for any $(R, \mu_i), (S, \nu_i){:}B \to C$, $\exists! R \xrightarrow{\alpha} S[\nu_i.(c_i\alpha) = \mu_i]$

every $(R, \mu_i) \to (S, \nu_i)$ invertible, every $(R, \mu_i){:}(B, b_i) \to (C, c_i)$ an equivalence in $\mathbf{A}(X)$ and hence in $\mathbf{A}$ ...

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The $p_{X,Y}$ and $r_{X,Y}$ provide components for pseudonaturals

$$\text{Get} \quad p{:}P \leftarrow \Pi_2 \rightarrow R{:}r \quad \text{a product in } [\mathbf{A}^2, \mathbf{A}]$$

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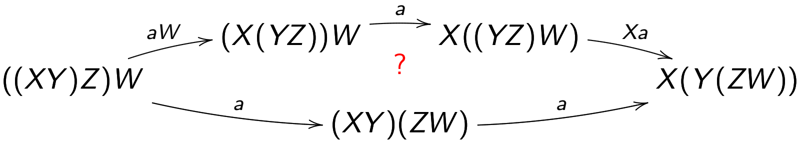
$$p_{X,Y} \cdot p_{(X \times Y),Z}, \quad r_{X,Y} \cdot p_{(X \times Y),Z}, \quad \text{and} \quad r_{(X \times Y),Z}$$

from $(X \times Y) \times Z$ to X, Y, and Z respectively, and

$$p_{X,(Y \times Z)}, \quad p_{Y,Z} \cdot r_{X,(Y \times Z)}, \quad \text{and} \quad r_{Y,Z} \cdot r_{X,(Y \times Z)}$$

from $X \times (Y \times Z)$ to X, Y, and Z respectively, give objects in $[\mathbf{A}^3, \mathbf{A}](P_1, P_2, P_3)$
Applying KL we get an $a{:}\Pi_2(\Pi_2 \times 1) \to \Pi_2(1 \times \Pi_2)$ and μ providing $a_{X,Y,Z}{:}(X \times Y) \times Z \to X \times (Y \times Z)$, pseudonatural equivalences

In $[\mathbf{A}^4, \mathbf{A}]$ (writing XY for $X \times Y$, X for P_1 etcetera)



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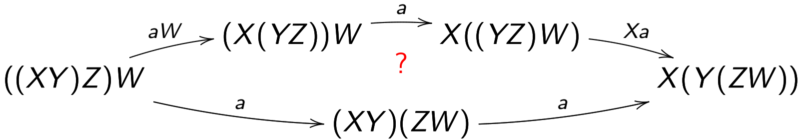


Actually in $[\mathbf{A}^4, \mathbf{A}](X, Y, Z, W)$, get unique $\pi{:}Xa.a.aW \to a.a$

In $[\mathbf{A}^4, \mathbf{A}]$ (writing XY for $X \times Y$, X for P_1 etcetera)

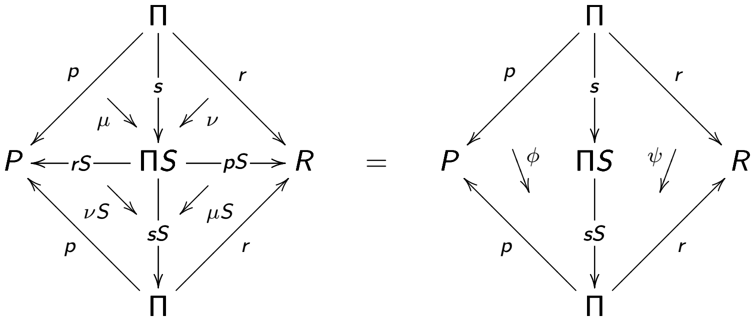
$$\begin{array}{ccccc} & (X(YZ))W & \xrightarrow{a} & X((YZ)W) & \\ {}^{aW}\nearrow & & {\color{red}?} & & \searrow^{Xa} \\ ((XY)Z)W & & & & X(Y(ZW)) \\ & \searrow_{a} & & \nearrow_{a} & \\ & & (XY)(ZW) & & \end{array}$$

Actually in $[\mathbf{A}^4, \mathbf{A}](X, Y, Z, W)$, get unique $\pi{:}Xa.a.aW \to a.a$
In $[\mathbf{A}^5, \mathbf{A}](X, Y, Z, U, V)$ pasting π's we get

$$X(Ya).Xa.a.(Xa)V.aV.(aU)V \overset{\hat{\pi}}{\underset{\check{\pi}}{\rightrightarrows}} a.a.a{:}((((XY)Z)U)V \rightrightarrows X(Y(Z(UV)))$$

and the equality $\hat{\pi} = \check{\pi}$ of pasting composites is the "non-abelian 4-cocycle condition" (TA1) of [GPS]. The proof of (TA2) and (TA3) is similar.

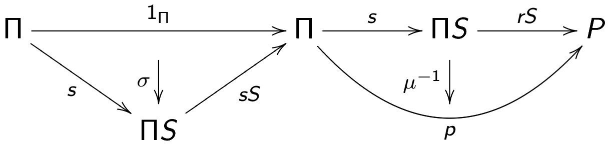
Have $S{:}\mathbf{A}^2 \to \mathbf{A}^2$ with $SS = 1_{\mathbf{A}^2}$, $PS = R$, and $RS = P$



There is a unique 2-cell $\sigma{:}1_\Pi \to (sS)s$ with $p\sigma = \phi$ and $r\sigma = \psi$ in $[\mathbf{A}^2, \mathbf{A}]$ invertible since μ and ν are

$$s_{X,Y}\sigma_{X,Y} = \sigma_{Y,X}s_{X,Y} \quad \text{symmetry equation} \quad s\sigma = (\sigma S)s$$

suffices to show $(rS)s\sigma = (rS)(\sigma S)s$ and $(pS)s\sigma = (pS)(\sigma S)s$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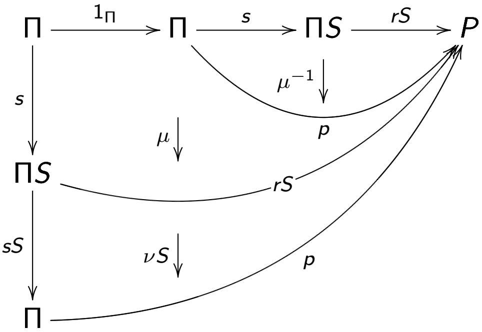

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= \quad
\begin{tikzcd}[column sep=huge]
\Pi \arrow[r, "s"] & \Pi S \arrow[rr, "rS"] \arrow[dr, "sS"'] & \arrow[d, Rightarrow, "\nu S"'] & P \\
& & \Pi \arrow[ur, "p"'] &
\end{tikzcd}
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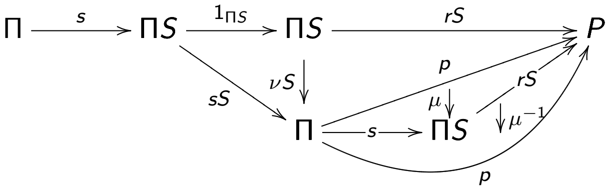
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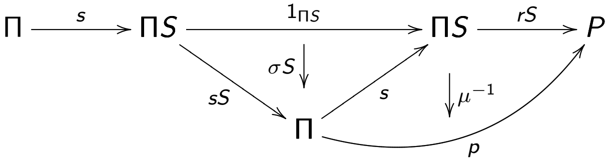

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