Ordered Face Structures and Many-to-one Computads

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Many-to-one computads are ω-categories that are free 'levelwise' i.e. we adjoin $n+1$ dimensional generators (= *indets*) and we fix their *domains* and *codomains* only after generation on all n dimensional cell. Moreover we insists that the codomain of the indets are indets again (= many-to-one).

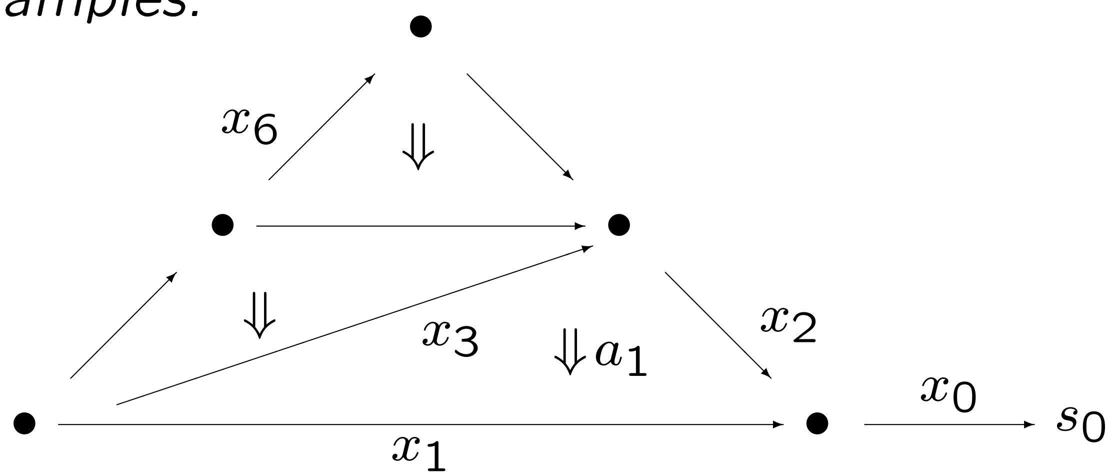
Why many-to-one computads? They seem to be in the center of two approaches to weak ω-categories: multitopic and opetopic.

The category $\mathbf{Comp}^{m/1}$ of many-to-one computads and computads map is equivalent to the category of multitopic sets $\mathbf{MltSets}$...

...and probably to the category of opetopic sets, as well.

Ordered face structures are combinatorial structures describing the 'types' of (all) cells in many-to-one computads.

Examples.



Domains: $\delta(a_1) = \{x_2, x_3\}$

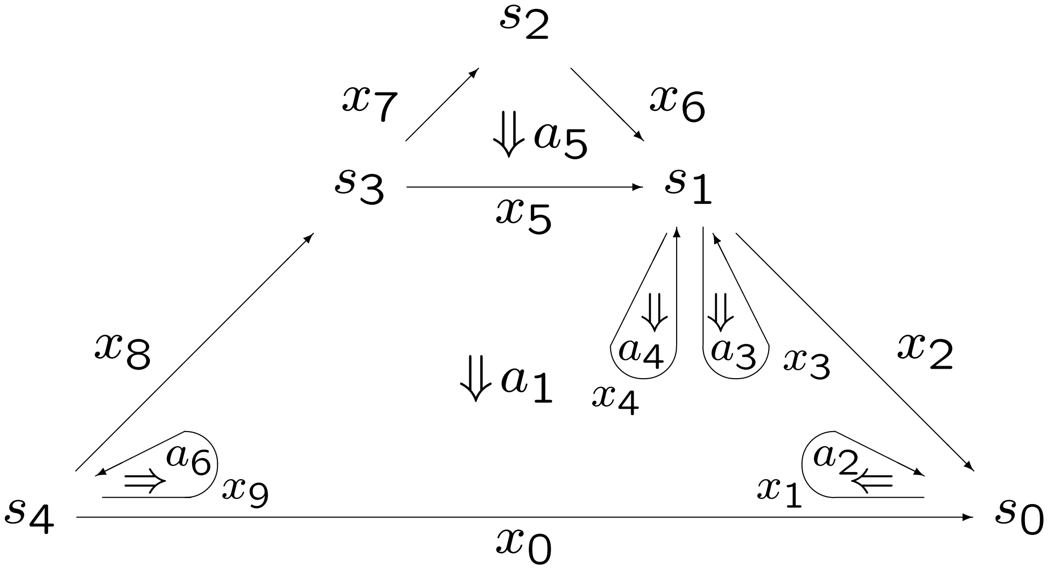
Codomains: $\gamma(a_1) = x_1$

Two orders.

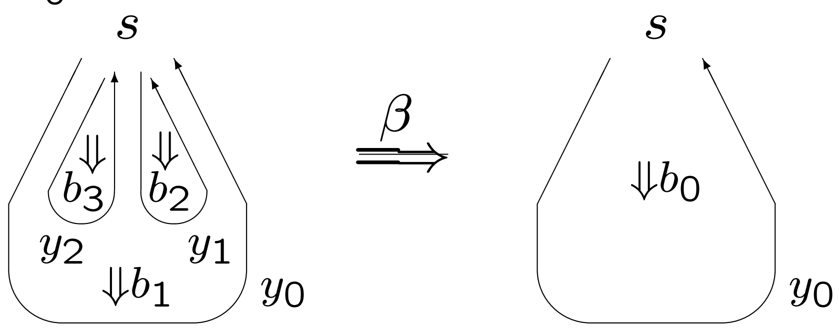
Lower: $x_1 <^- x_0$ as $\gamma(x_1) \in \delta(x_0)$,
... and by transitivity: $x_6 <^- x_0$

Upper: $x_3 <^+ x_1$ as $x_3 \in \delta(a_3)$ and $\gamma(a_3) = x_1$,
... and by transitivity: $x_6 <^- x_1$

However we may have *empty-domain faces*... and hence *empty faces* and *loops*



$\delta(a_2) = 1_{s_0}$, $\gamma(a_2) = x_1$ and more loops



What about lower order of x_3 and x_4?... or y_2 and y_3? We need this order as an additional part of data $<^{\sim}$ contained in $<^{-}$!
We have $x_4 <^{\sim} x_3$ but not $x_3 <^{\sim} x_4$!
Similarly $y_2 <^{\sim} y_1$.

Data for ordered face structures

faces: $\{S_n\}_{n\in\omega}$; S_n is a finite set of faces of dimension n; almost all S_n's are empty;

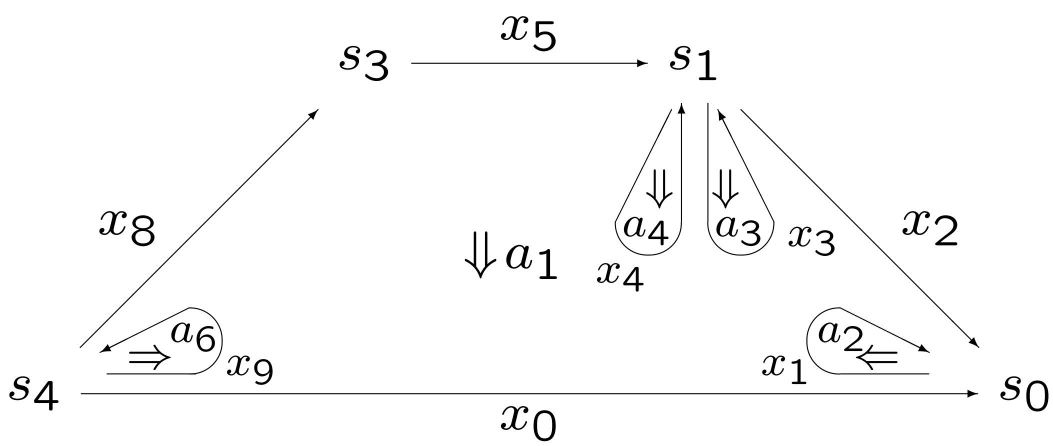
domain relation δ: $\delta(\alpha)$ is either a finite non-empty set of faces or an empty faces;

Below we have: $\delta(a_2) = 1_{s_0}$ and

$$\delta(a_1) = \{x_1, x_2, x_3, x_4, x_5, x_8, x_9\}$$

codomain function γ: e.g. $\gamma(a_1) = x_0$

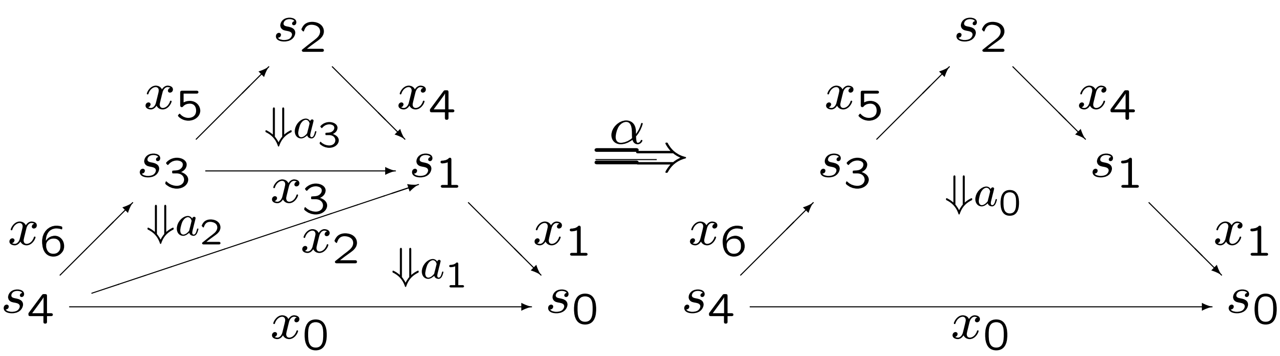
lower order $<^{\sim}$: e.g. $x_4 <^{\sim} x_3$



The upper order $<^{+}$ is definable from γ and δ.

Axioms for ordered face structures

In the ordered faces structure



we have

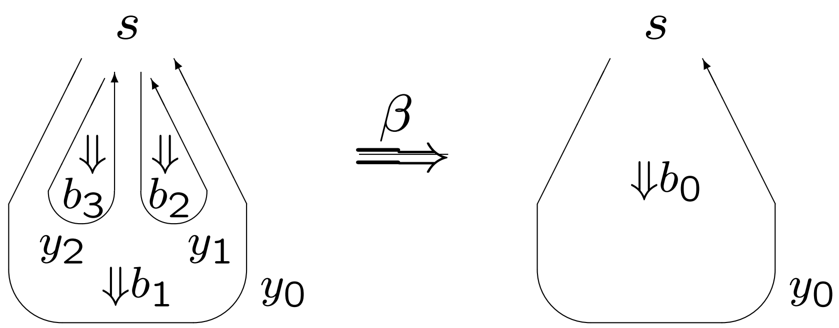
$$\gamma\gamma(\alpha) = x_0, \quad \delta\delta(\alpha) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$$

$$\delta\gamma(\alpha) = \{x_1, x_4, x_5, x_6\}, \quad \gamma\delta(\alpha) = \{x_0, x_2, x_3\}$$

Globularity axiom (positive case)

$$\gamma\gamma(\alpha) = \gamma\delta(\alpha) - \delta\delta(\alpha), \quad \delta\gamma(\alpha) = \delta\delta(\alpha) - \gamma\delta(\alpha)$$

But when we have loops in the domain as in b_1 or empty-domain loop as in $b2$



we have $\gamma\gamma(b_i) = \delta\delta(b_i) = \delta\gamma(b_i) = \gamma\delta(b_i) = s$, and the above formulas does not work. We have to drop both loops and empty faces.

Globularity axiom

$$\gamma\gamma(\alpha) = \gamma\delta(\alpha) - \delta\dot{\delta}^{-\lambda}(\alpha)$$

$$\delta\gamma(\alpha) \equiv_1 \delta\delta(\alpha) - \gamma\dot{\delta}^{-\lambda}(\alpha)$$

$\equiv_1$ is 'equality' that ignores empty faces, i.e. the empty faces that might occur on the right side of the sign $\equiv_1$ must be empty on ether domain or codomain of a face that belongs to the left side.

Other axioms of ordered face structures
talks about the upper $<^{+}$ and lower $<^{\sim}$ orders.

They are strict, disjoint and $<^{\sim}$ is maximal such contained in $<^{-}$. The upper order on 0-cells is linear.

No two faces in a domain of a face might be comparable in the upper order $<^{+}$.

Incident faces must be comparable in one of these orders.

Every loop must be filled in, i.e. must be a codomain of a cell which is not a loop.

There are two basic kinds of morphisms of ordered face structures.

A *local morphism of ordered face structures* $f : S \to T$ is a family of functions $f_k : S_k \to T_k$, for $k \in \omega$, such that the diagrams

$$\begin{array}{ccc} S_{k+1} & \xrightarrow{f_{k+1}} & T_{k+1} \\ \gamma \downarrow & & \downarrow \gamma \\ S_k & \xrightarrow[f_k]{} & T_k \end{array} \qquad \begin{array}{ccc} S_{k+1} & \xrightarrow{f_{k+1}} & T_{k+1} \\ \delta \downarrow & & \downarrow \delta \\ S_k \uplus \mathbf{1}_{S_{k-1}} & \xrightarrow[f_k + \mathbf{1}_{f_{k-1}}]{} & T_k \uplus \mathbf{1}_{T_{k-1}} \end{array}$$

commute. For the right square it means more then commutation of relations, we demand that for any $a \in S_{\geq 1}$,

$$f_a : (\dot{\delta}(a), <^{\sim}) \longrightarrow (\dot{\delta}(f(a)), <^{\sim})$$

be an order isomorphism, where f_a is the restriction of f to $\dot{\delta}(a)$ (if $\delta(a) = 1_u$ we mean by that $\delta(f(a)) = 1_{f(u)}$).

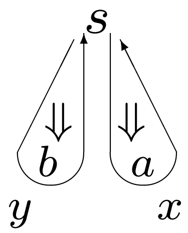
A *global (monotone) morphism of ordered face structures* $f : S \to T$ is a local morphism that preserves lower order $<^{\sim}$ (globally).

Examples. $f_1 : T_1 \to S$ is monotone:

T_1 :

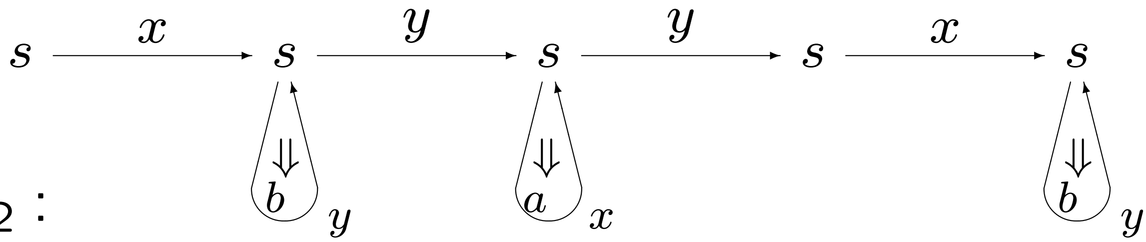
$$s_2 \xrightarrow{y} s_1 \xrightarrow{x} s_0$$

S :

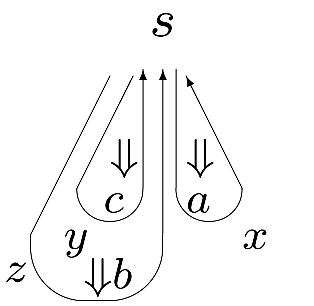


$f_2 : T_2 \to S$ is not monotone but it is local:

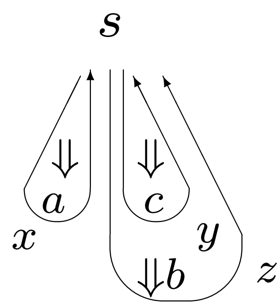
T_2 :



The following two ordered face structures are not isomorphic (globally) but they are isomorphic locally:



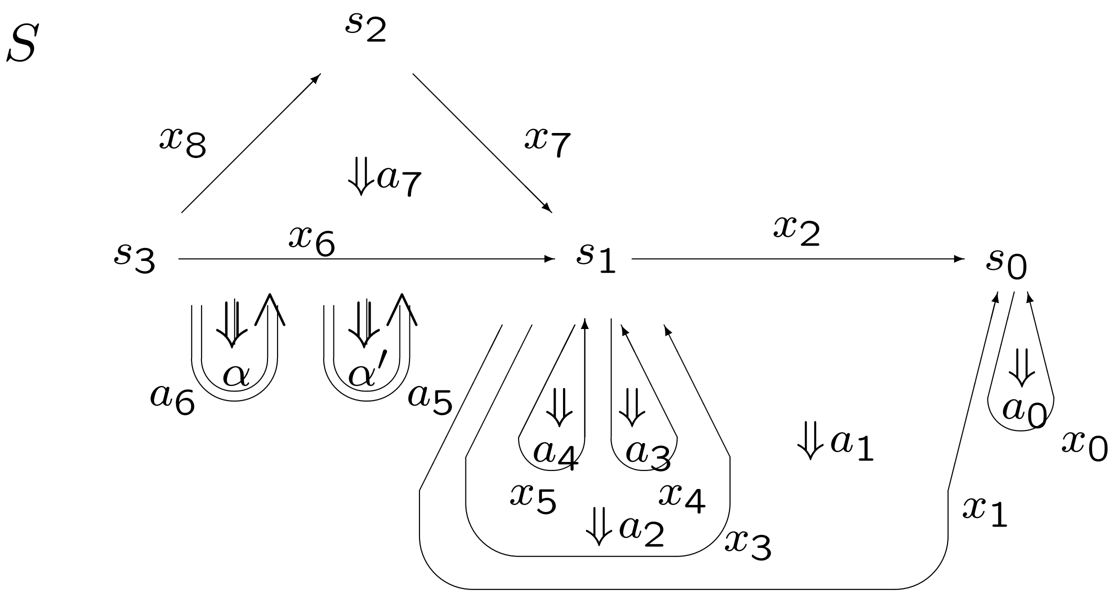
$z <^{\sim} x$



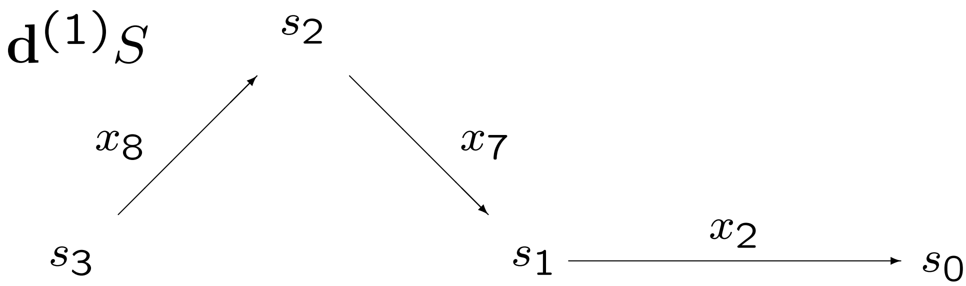
$x <^{\sim} z$

$\mathbf{oFs}$ ($\mathbf{oFs}_{loc}$) - is the category of ordered face structures and monotone (local) maps

In $\mathbf{oFs}$ we have operations of the *k-th domain* $\mathbf{d}^{(k)}$ and *k-th codomain* $\mathbf{c}^{(k)}$. For an order face structure S as follows

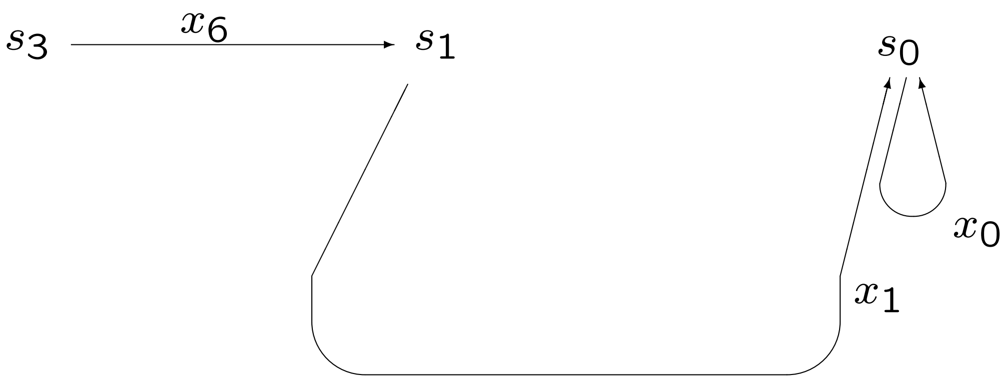


its 1-domain is



the convex subset of S defining 1-codomain is

$c^{(1)}S$



and finally the 1-codomain of S is the stretching of $c^{(1)}S$

$\mathbf{c}^{(1)}S$

$$s_3 \xrightarrow{x_6} s_1 \xrightarrow{x_1} (s_0,\emptyset,\{x_0\}) \xrightarrow{x_0} (s_0,\{x_0\},\emptyset)$$

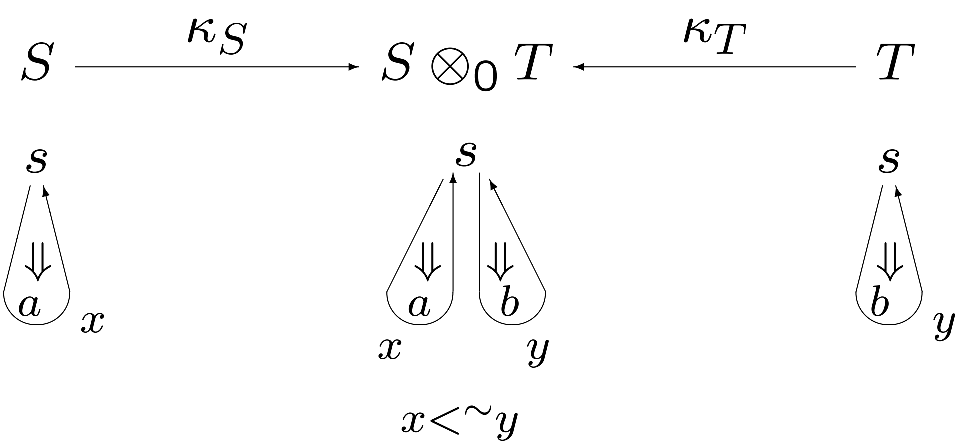
We have monotone maps embedding k-th domain and k-th codomain into an order face structure S:

$$\mathbf{d}^{(k)}S \xrightarrow{\mathbf{d}^{(k)}_S} S \xleftarrow{\mathbf{c}^{(k)}_S} \mathbf{c}^{(k)}S$$

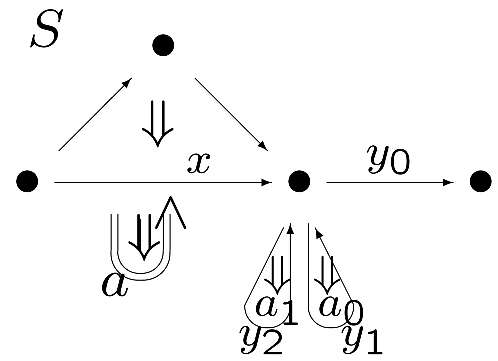
We also have an operation of k-*tensor* of two ordered face structures S and T such that $\mathbf{c}^{(k)}S = \mathbf{d}^{(k)}T$.

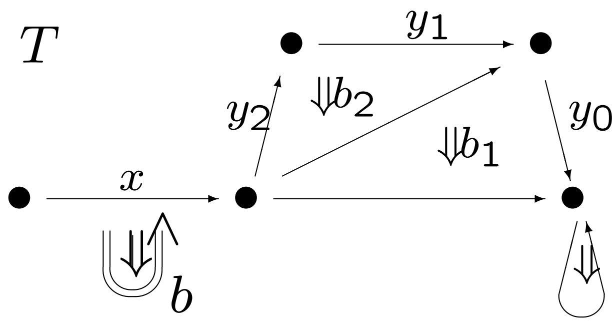
$$\begin{array}{ccc} S & \xrightarrow{\kappa_S} & S \otimes_k T \\ {\scriptstyle \mathbf{c}^{(k)}_S}\uparrow & & \uparrow{\scriptstyle \kappa_T} \\ \mathbf{c}^{(k)}S & \xrightarrow[\mathbf{d}^{(k)}_T]{} & T \end{array}$$

Examples

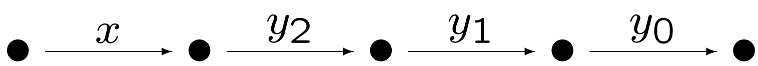


The tensor is a pushout locally. But $<^{\sim}$ is not uniquely determined by this. The additional rule is that in case of doubts faces from S comes before faces from T. This is why $x <^{\sim} y$, above.

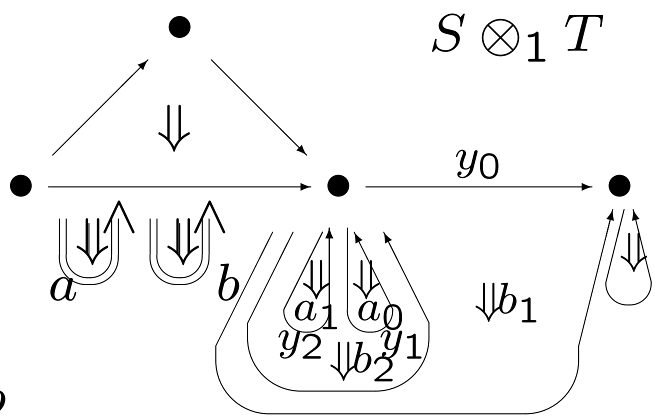




$\mathbf{c}^{(1)}S = \mathbf{d}^{(1)}T$



and the 1-tensor $S \otimes_1 T$ is



$a <^{\sim} b$

Theorem. *The category* $\mathbf{oFs}$ *is a monoidal globular category in the sense of Batanin, with* k*-tensor squares being pushout locally (i.e. in* $\mathbf{oFs}_{loc}$*).*

We have a full embedding functor

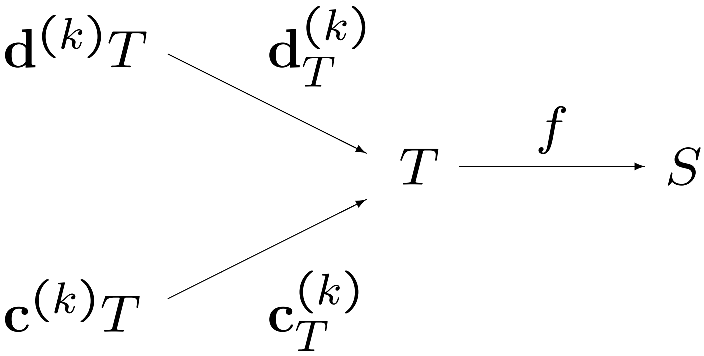
$$(-)^* : \mathbf{oFs}_{loc} \longrightarrow \mathbf{Comp}^{m/1}$$

Fix S in $\mathbf{oFs}_{loc}$.
k-cells in S^*: (monotone iso classes of) local maps $f : T \longrightarrow S$ with $dim(T) \leq k$;
f_0 is equivalent to f_1 iff there is a monotone iso h such that the triangle

$$\begin{array}{ccc} T_0 & \xrightarrow{\quad h \quad} & T_1 \\ & {\scriptstyle f_0} \searrow \quad \swarrow {\scriptstyle f_1} & \\ & S & \end{array}$$

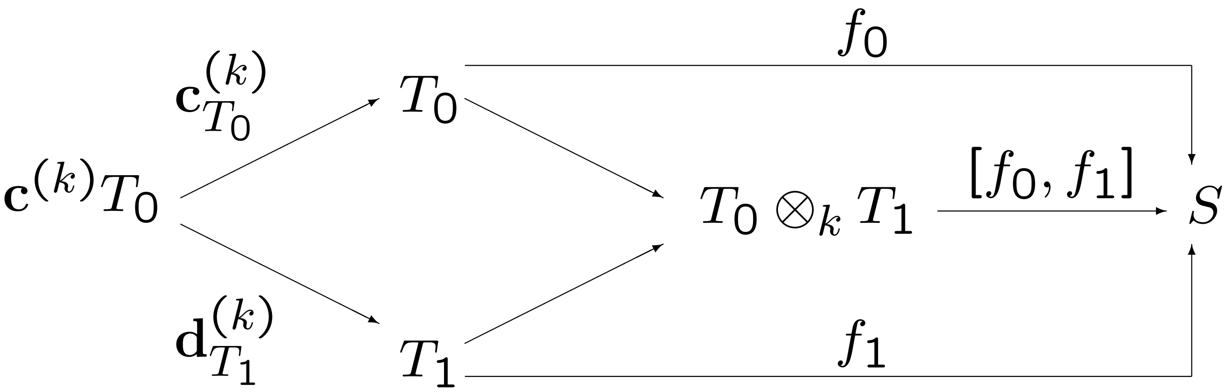
commutes.

domains and codomains S^*:



$d^{(k)}(f) = f \circ \mathbf{d}_T^{(k)}$ and $c^{(k)}(f) = f \circ \mathbf{c}_T^{(k)}$

compositions in S^*: if $c^{(k)}(f_0) = d^{(k)}(f_1)$



then $f_1 \circ_k f_0 = [f_0, f_1]$

$(-)^*$ acts on morphism by compositions.

Theorem.

$$(-)^* : \mathbf{oFs}_{loc} \longrightarrow \mathbf{Comp}^{m/1}$$

induces the functor

$$\mathbf{Comp}^{m/1} \longrightarrow Set^{\mathbf{oFs}_{loc}^{op}}$$

$$C \longmapsto \mathbf{Comp}((-)^*, C)$$

which is full and faithful, and whose essential image consists of functors sending tensor squares in $\mathbf{oFs}_{loc}^{op}$ *to pullbacks in* Set.