Cover Relations on Categories

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Part I

Relations $\sqsubset$ arising from factorization systems

Factorization systems

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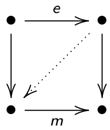

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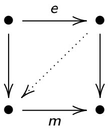
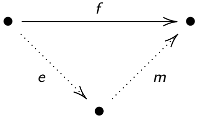
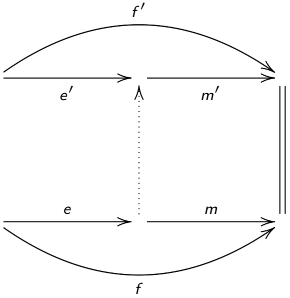
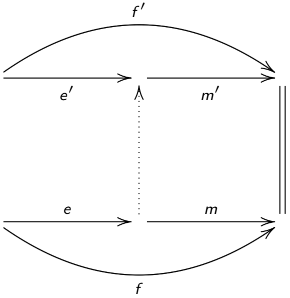


- Existence of $(\mathcal{E}, \mathcal{M})$-factorizations:





f'
e'
m'
e
m
f



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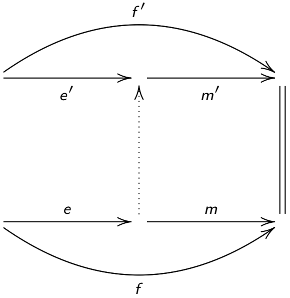

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The answer is *yes* if every morphism in $\mathcal{M}$ is a monomorphism.

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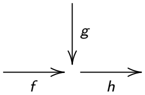

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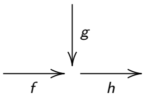

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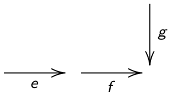

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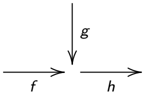

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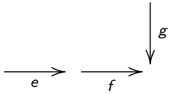

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F3. Every morphism f has a $\sqsubset$-image m, i.e. a morphism m such that $m \sqsubset f$ and m is universal with this property

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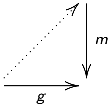

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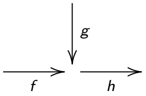

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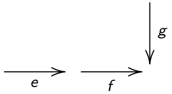

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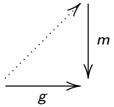

F4. Every $\sqsubset$-image h is a $\sqsubset$-reflecting morphism, i.e. for every two morphisms f, g (as in the display in Axiom 1) we have $hf \sqsubset hg \;\Rightarrow\; f \sqsubset g$.

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- What happens if we remove Axiom F0?

Part II

Relations $\sqsubset$ *arising from certain monoidal structures*

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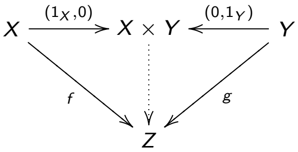

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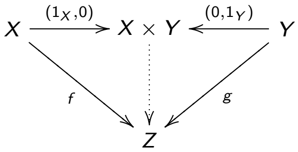

Unital categories in the sense of D. Bourn: $f \sqsubset g$ iff f and g cooperate.

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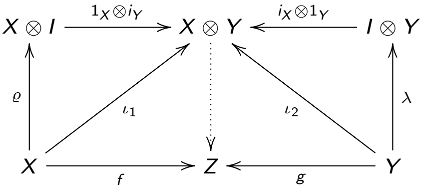

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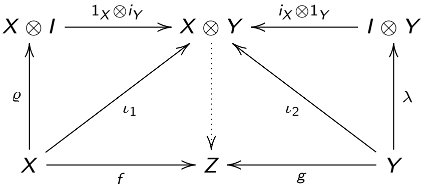

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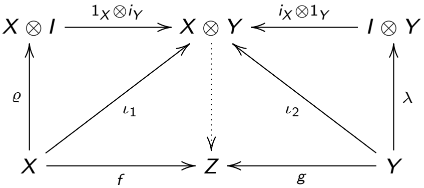

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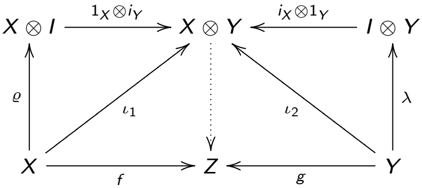

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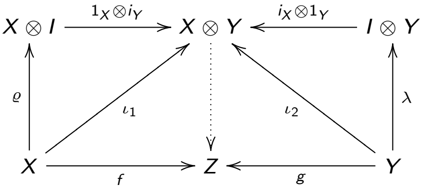

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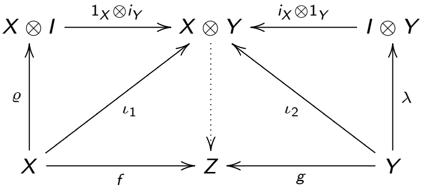

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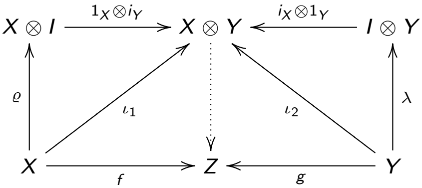

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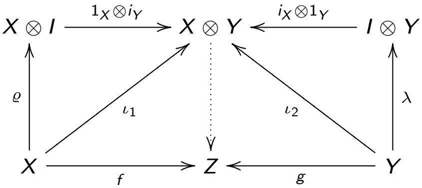

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with the property $f \sqsubset g$, has an initial object

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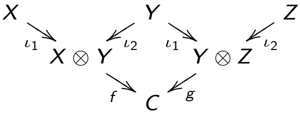

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M5. There exists an object I such that for any morphism $f : X \to Y$, we have $g \sqsubset f$, for exactly one morphism $g : I \to Y$.

Part III

Cover relations

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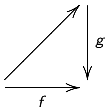

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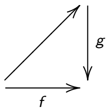

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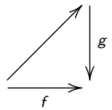

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A strange example

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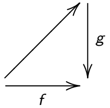

- For a pullback-stable class $\mathcal{M}$ of morphisms the cover relation $\sqsubset^{\mathcal{M}}$ is defined as follows:

$$f \sqsubset^{\mathcal{M}} g \quad \Leftrightarrow \quad \forall_{m \in \mathcal{M}} (g \angle m \Rightarrow f \angle m).$$

A strange example

$\mathbb{C} = \mathbf{Top}$: $f \sqsubset g \;\Leftrightarrow\; \forall_{x \in \mathrm{Im}(f)} \forall_{y \in \mathrm{Im}(g)} (x \rightsquigarrow y)$.

All axioms are satisfied except F1 and M3.

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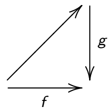

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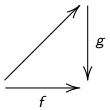

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Just one — the codiscrete $\sqsubset$.

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Lemma

Let $(\mathcal{E}, \mathcal{M})$ be a factorization system such that kernel pairs of morphisms from $\mathcal{M}$ exist. Then the following conditions are equivalent to each other:

- *Every morphism in $\mathcal{M}$ is a monomorphism.*
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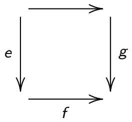
Theorem

A class $\mathcal{E}$ of morphisms is the class of $\sqsubset$-coverings for some reflexive and transitive cover relation $\sqsubset$ if and only if $\mathcal{E}$ has the following properties:

- *$\mathcal{E}$ contains identity morphisms.*
- *$\mathcal{E}$ is closed under composition.*
- *For any morphism f and for any $e \in \mathcal{E}$, if $e \angle f$ then $f \in \mathcal{E}$.*

Relations $\sqsubset$ induced by classes of morphisms

For a class $\mathcal{E}$ of morphisms by $\sqsubset_{\mathcal{E}}$ we denote the relation defined as follows: $f \sqsubset_{\mathcal{E}} g$ if and only if f and g are part of a commutative square



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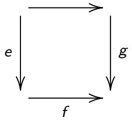
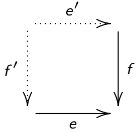


where $e \in \mathcal{E}$.

Lemma

$\sqsubset_{\mathcal{E}}$ has left preservation property and it is a cover relation if and only if every morphism $e \in \mathcal{E}$ is a $\sqsubset_{\mathcal{E}}$-covering, i.e. for every pair f, e of solid arrows in the diagram



with $e \in \mathcal{E}$, there exist the dotted arrows e', f', with $e' \in \mathcal{E}$, making the square commute (in other words, $\mathcal{E}$ admits lifts).

Theorem

Let $\mathcal{E}$ be a class of morphisms such that for any morphism f and for any $e \in \mathcal{E}$, if $e \angle f$ then $f \in \mathcal{E}$. Then the following conditions are equivalent to each other:

- *$\sqsubset_{\mathcal{E}}$ is a reflexive and transitive cover relation.*
- *$\mathcal{E}$ contains identity morphisms, is closed under composition, and admits lifts.*

Theorem

For a factorization system $(\mathcal{E}, \mathcal{M})$ the following conditions are equivalent:

- *$\mathcal{E}$ admits lifts.*
- *The relation $\sqsubset_{\mathcal{E}}$ coincides with the relation $\sqsubset^{\mathcal{M}}$.*

In particular, if every morphism from $\mathcal{M}$ is a monomorphism and pullbacks of morphisms from $\mathcal{E}$ exist, then the above conditions are equivalent to $\mathcal{E}$ being stable under pullbacks.

Part IV

Motivation from Logic and Universal Algebra

Closedness properties of internal relations

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- Closedness properties of internal relations V: Linear Mal'tsev conditions, *Algebra Universalis* (to appear).

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$$\begin{array}{ccc}
\bullet & \longrightarrow & R^m \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle r^m} \\
A^l & \longrightarrow & (A^n)^m
\end{array}$$

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