Insertion in biframes

Maria João Ferreira

Centre of Mathematics of the University of Coimbra

Joint work with Javier Gutiérrez García and Jorge Picado

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The Category of Biframes

[Banaschewski, Brümmer, Hardie-1983]

Biframes

(L_0, L_1, L_2)

- L_0 frame
- L_1, L_2 subframes of L_0
- $\forall a \in L_0$,
$a = \bigvee_i (x_i \wedge y_i),\ x_i \in L_1,\ y_i \in L_2$

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Biframe maps

$$f : (L_0, L_1, L_2) \to (M_0, M_1, M_2)$$

- $f : L_0 \to M_0$ frame homomorphism
- $f(L_i) \subseteq M_i,\ i = 1, 2$

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$(X, \mathfrak{T}_1, \mathfrak{T}_2)$ bitopological space $\Rightarrow (\mathfrak{T}_1 \vee \mathfrak{T}_2, \mathfrak{T}_1, \mathfrak{T}_2)$ biframe

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$$\mathfrak{c}_L(0) = \mathfrak{o}_L(1) = 0 \qquad \mathfrak{c}_L(1) = \mathfrak{o}_L(0) = 1$$

$$a \leq b \Rightarrow \mathfrak{c}_L(a) \leq \mathfrak{c}_L(b) \qquad a \leq b \Rightarrow \mathfrak{o}_L(b) \leq \mathfrak{o}(a)$$

$$\bigvee_{i \in J} \mathfrak{c}_L(a_i) = \mathfrak{c}_L(\bigvee_{i \in J} a_i) \qquad \mathfrak{c}_L(a) \wedge \mathfrak{c}_L(b) = \mathfrak{c}_L(a \wedge b)$$

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Sublocale biframe $(\mathfrak{S}L, \mathfrak{c}L, \mathfrak{o}L)$

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$$(-, q) := \bigvee_{p \in \mathbb{Q}} (p, q) \qquad (p, -) := \bigvee_{q \in \mathbb{Q}} (p, q)$$

$$\mathfrak{L}_l(\mathbb{R}) := \langle (-, r) \mid r \in \mathbb{Q} \rangle \qquad \mathfrak{L}_u(\mathbb{R}) := \langle (r, -) \mid r \in \mathbb{Q} \rangle$$

Real functions: a general localic setting

[Gutiérrez García, Kubiak, Picado-J. Pure Appl. Alg, to appear]

L frame

$\mathsf{F}(L) \;:=\; \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathfrak{S}L)$ Real functions

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$f, g \in \mathsf{F}(L)$

$$f \leq g \quad \Leftrightarrow \quad f(r, -) \leq g(r, -) \quad \text{for all } r \in \mathbb{Q}$$
$$\Leftrightarrow \quad g(-, r) \leq f(-, r) \quad \text{for all } r \in \mathbb{Q}.$$

Real functions: a general localic setting

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L frame

S complemented sublocale of L

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The characteristic map $\chi_S : \mathfrak{L}(\mathbb{R}) \to \mathfrak{S}L$ is defined by

$$\chi_S(r,-) = \begin{cases} 1 & \text{if } r < 0, \\ S^* & \text{if } 0 \le r < 1, \\ 0 & \text{if } r \ge 1, \end{cases} \quad \text{and} \quad \chi_S(-,r) = \begin{cases} 0 & \text{if } r \le 0, \\ S & \text{if } 0 < r \le 1, \\ 1 & \text{if } r > 1, \end{cases}$$

for each $r \in \mathbb{Q}$

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for each $r \in \mathbb{Q}$

1. $\chi_S \in \mathrm{LSC}(L)$ if and only if S is open,
2. $\chi_S \in \mathrm{USC}(L)$ if and only if S is closed,
3. $\chi_S \in \mathrm{C}(L)$ if and only if S is clopen.

Semicontinuity in biframes

(L_0, L_1, L_2) biframe

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- L_i-usc if $f(\mathfrak{L}_l(\mathbb{R})) \subseteq (\mathfrak{c}L_0)_i$

f is L_1-usc and L_2-lsc if and only if $f \in \mathrm{C}(L_0)$ and

$$\mathfrak{c}_{L_0}^{-1} \circ f : (\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})) \to (L_0, L_1, L_2)$$

is a biframe homomorphism

Normal and extremally disconnected biframes

(L_0, L_1, L_2) biframe

(L_0, L_1, L_2) is normal if

$$a \vee b = 1, a \in L_i, b \in L_j \Rightarrow \exists u \in L_j, v \in L_i : u \wedge v = 0, a \vee u = 1 = b \vee v$$

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$$a \in L_i,\ a^{\bullet} := \bigvee \{v \in L_j : u \wedge v = 0\}$$

Normal and extremally disconnected biframes

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$a \wedge b = 0, a \in L_i, b \in L_j \Rightarrow \exists u \in L_j, v \in L_i : u \vee v = 1, a \wedge u = 0 = b \wedge v$

$\Leftrightarrow \forall a \in L_1, a^{\bullet\bullet} \vee a^{\bullet} = 1 \Leftrightarrow \forall b \in L_2, b^{\bullet\bullet} \vee b^{\bullet} = 1$

Normal biframes

(L_0, L_1, L_2) biframe

Characterization of normality

(L_0, L_1, L_2) is normal iff for any $\{a_k\}_{k\in\mathbb{N}} \subseteq L_1$ and $\{b_k\}_{k\in\mathbb{N}} \subseteq L_2$ s.t.

$$\left.\begin{array}{l} \bigwedge_{k\in\mathbb{N}} a_k \in L_1 \\ \bigwedge_{k\in\mathbb{N}} b_k \in L_2 \\ a_k \vee (\bigwedge_{\ell\in\mathbb{N}} b_\ell) = 1 \\ b_k \vee (\bigwedge_{\ell\in\mathbb{N}} a_\ell) = 1 \end{array}\right\} \Rightarrow \exists u \in L_2 : \forall k \in \mathbb{N},\ a_k \vee u = 1 = b_k \vee u^\bullet$$

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$\{\alpha_k \mid k \in \mathbb{N}\}$ enumeration of $\mathbb{Q}$

Construction of the insertion function

$$\left.\begin{array}{l} (L_0, L_1, L_2) \text{ is normal} \\ g \text{ is } L_1\text{-usc} \\ f \text{ is } L_2\text{-lsc} \\ g \leq f \end{array}\right\} \Rightarrow \begin{array}{l} \exists \{u_{\alpha_k}\}_{k\in\mathbb{N}} \subseteq L_2 : \\ \quad q > \alpha_k \Rightarrow g(-, q) \vee \mathfrak{c}_0(u_{\alpha_k}) = 1 \\ \quad p < \alpha_k \Rightarrow f(p, -) \vee \mathfrak{c}_0(u^\bullet_{\alpha_k}) = 1 \\ \quad \alpha_{k_1} < \alpha_{k_2} \Rightarrow u_{\alpha_{k_1}} \vee u^\bullet_{\alpha_{k_2}} = 1 \end{array}$$

Insertion theorems

Katětov-Tong-type insertion theorem for biframes

TFAE for a biframe (L_0, L_1, L_2):

(i) (L_0, L_1, L_2) is normal

(ii) $\left.\begin{array}{l} \forall f : \mathfrak{L}(\mathbb{R}) \to \mathfrak{S}L_0 \;\; L_2\text{-lsc} \\ \forall g : \mathfrak{L}(\mathbb{R}) \to \mathfrak{S}L_0 \;\; L_1\text{-usc} \end{array}\right\} g \leq f \Rightarrow g \leq h \leq f$

for some L_1-usc and L_2-lsc $h : \mathfrak{L}(\mathbb{R}) \to \mathfrak{S}L_0$

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Stone-type insertion theorem for biframes

TFAE for a biframe (L_0, L_1, L_2):

(i) (L_0, L_1, L_2) is extremally disconnected

(ii) $$\left.\begin{array}{l} \forall f : \mathfrak{L}(\mathbb{R}) \to \mathfrak{S}L_0 \;\; L_2\text{-lsc} \\ \forall g : \mathfrak{L}(\mathbb{R}) \to \mathfrak{S}L_0 \;\; L_1\text{-usc} \end{array}\right\} f \leq g \Rightarrow f \leq h \leq g$$
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Consequences: biframes

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$a \in L_1,\ b \in L_2 : a \vee b = 1$

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(L_0, L_1, L_2) biframe

$a \in L_1,\ b \in L_2 : a \vee b = 1 \quad \Longrightarrow \quad \chi_{\mathfrak{c}_{L_0}(a)} \le \chi_{\mathfrak{o}_{L_0}(b)}$

L_1-usc $\quad$ L_2-lsc

Consequences: biframes

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[Schauert-Ph.D.Thesis,1992]

Urysohn-type lemma

A biframe (L_0, L_1, L_2) is normal iff, whenever $a \vee b = 1$, $a \in L_1$, $b \in L_2$, there exists $h : (\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})) \to (L_0, L_1, L_2)$ such that $h(-, 1) \leq a$ and $h(0, -) \leq b$

Consequences: biframes

(L_0, L_1, L_2) biframe

$a \in L_1,\ b \in L_2 : a \wedge b = 0$

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Urysohn-type lemma for extremally disconnected biframes

A biframe (L_0, L_1, L_2) is extremally disconnected iff whenever $a \wedge b = 0$, $a \in L_1$, $b \in L_2$, there exists $h : (\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})) \to (L_0, L_1, L_2)$ such that $a \wedge h(0, -) = 0$ and $b \vee h(-, 1) = 1$.

Consequences: frames

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[Gutiérrez García,Picado-J. Pure Appl. Alg., 2007]

Katětov-Tong-type insertion theorem

TFAE for a frame L:

(i) L is normal

(ii) $$\left.\begin{array}{l}\forall f : \mathfrak{L}(\mathbb{R}) \to \mathfrak{S}L \;\; lsc \\ \forall g : \mathfrak{L}(\mathbb{R}) \to \mathfrak{S}L \;\; usc\end{array}\right\} g \le f \Rightarrow g \le h \le f$$

for some continuous $h : \mathfrak{L}(\mathbb{R}) \to \mathfrak{S}L$

Consequences: frames

$L_0 = L_1 = L_2 = L$

[Y.M. Li-Z.H. Li-Alg. Univ., 2000]
[Gutiérrez García,Kubiak,Picado-Alg. Univ., to appear]

Katětov-Tong-type insertion theorem

TFAE for a frame L:

(i) L is extremally disconnected

(ii) $\left.\begin{array}{l} \forall f : \mathfrak{L}(\mathbb{R}) \to \mathfrak{S}L \;\; lsc \\ \forall g : \mathfrak{L}(\mathbb{R}) \to \mathfrak{S}L \;\; usc \end{array}\right\} f \leq g \Rightarrow f \leq h \leq g$

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Consequences: bitopological spaces

$(X, \mathfrak{T}_1, \mathfrak{T}_2)$ bitopological space

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[H. Priestley-J. London Math. Soc., 1971]

Katětov-Tong-type insertion theorem

TFAE for a bitopological space $(X, \mathfrak{T}_1, \mathfrak{T}_2)$:

(i) $(X, \mathfrak{T}_1, \mathfrak{T}_2)$ is normal

(ii) $\left.\begin{array}{l} \forall f : X \to \mathbb{R} \ \mathfrak{T}_2\text{-lsc} \\ \forall g : X \to \mathbb{R} \ \mathfrak{T}_1\text{-usc} \end{array}\right\} g \leq f \Rightarrow g \leq h \leq f$

for some $\mathfrak{T}_1$-usc and $\mathfrak{T}_2$-lsc $h : X \to \mathbb{R}$

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Consequences: ordered topological spaces

$(X, \mathfrak{T}, \leq)$ ordered topological space is

- normal if $$F_1 \in \downarrow \mathfrak{F},\ F_2 \in \uparrow \mathfrak{F},\ F_1 \cap F_2 = \emptyset$$ $$\Downarrow$$ $$\exists G_1 \in \downarrow \mathfrak{T},\ G_2 \in \uparrow \mathfrak{T},\ F_r \subseteq G_r, r = 1, 2$$

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- N_d-space if
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$$\Downarrow$$
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$(X, \mathfrak{T}, \leq)$ ordered topological space

$L_0 = \uparrow \mathfrak{T} \vee \downarrow \mathfrak{T}$, $L_1 = \uparrow \mathfrak{T}$, $L_2 = \downarrow \mathfrak{T}$

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[H. Priestley-J. London Math. Soc., 1971]

Katětov-Tong-type insertion theorem

TFAE for an order topological space $(X, \mathfrak{T}, \leq)$:

(i) $(X, \mathfrak{T}, \leq)$ is normal

(ii) $\left.\begin{array}{l} \forall f : X \to \mathbb{R} \text{ lsc,monotone} \\ \forall g : X \to \mathbb{R} \text{ usc,monotone} \end{array}\right\} g \leq f \Rightarrow g \leq h \leq f$

for some continuous monotone $h : X \to \mathbb{R}$

Consequences: ordered topological spaces

$(X, \mathfrak{T}, \leq)$ ordered topological space

$L_0 = \mathfrak{T} \vee \downarrow \mathfrak{T} = \mathfrak{T}$, $L_1 = \mathfrak{T}$, $L_2 = \downarrow \mathfrak{T}$

[H. Priestley-J. London Math. Soc., 1971]

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(i) $(X, \mathfrak{T}, \leq)$ is N_i

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Consequences: ordered topological spaces

$(X, \mathfrak{T}, \leq)$ ordered topological space

$L_0 = \uparrow \mathfrak{T} \vee \mathfrak{T} = \mathfrak{T}$, $L_1 = \uparrow \mathfrak{T}$, $L_2 = \mathfrak{T}$

[H. Priestley-J. London Math. Soc., 1971]

Katětov-Tong-type insertion theorem

TFAE for an order topological space $(X, \mathfrak{T}, \leq)$:

(i) $(X, \mathfrak{T}, \leq)$ is N_d

(ii) $\left.\begin{array}{l} \forall f : X \to \mathbb{R} \text{ lsc, monotone} \\ \forall g : X \to \mathbb{R} \text{ usc} \end{array}\right\} g \leq f \Rightarrow g \leq h \leq f$

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The frame of reals

$$\mathfrak{L}(\mathbb{R}) = \langle (p, q), p, q \in \mathbb{Q} \mid$$

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