Weak Galois Extensions

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V Portuguese Category Seminar

Universidade de Coimbra. October 17, 2008

Introduction

Let H be a Hopf algebra over a commutative base ring R, and A a right H-comódule algebra with comodule structure $\rho_A : A \longrightarrow A \otimes H$, $\rho_A(a) = a_{(0)} \otimes a_{(1)}$.
The extension $B \subset A$, where $B = A^{coH} = \{a \in A \; ; \; \rho_A(a) = a \otimes 1\}$ is the subalgebra of coinvariants elements, is said to be an H-Galois extension if the Galois map (canonical map)

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defined by

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A relevant question is the following:

When does surjectivity of β_A already imply bijectivity?

The Kreimer-Takeuchi Theorem (**Indiana Univ. Math. J.** (1981)) says that if β_A is surjective and H is finite, then β_A is bijective and A is a projective B-module.

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Theorem. Let (A, C, ψ) be an entwining structure. Assume that (A, μ_A, ρ_A) belongs to $\mathcal{M}_A^C(\psi)$ and put

$$B = A^{coC} = \{a \in A \ ; \ \rho_A(a) = a\rho_A(1)\}$$

Consider the following statements:

(i) The morphism $\beta'_A : A \otimes A \to A \otimes C$ is surjective and splits as a C-comodule map.

(ii)

(ii-1) The morphism $\beta_A : A \otimes_B A \to A \otimes C$ is an isomorphism.

(ii-2) A is relative projective as right B-module.

Then (ii) implies (i). If ψ is bijective and the map $s_A : A \otimes B \to (A \otimes A)^{coC}$ (for example if A is R-flat) is an isomorphism, then (i) implies (ii).

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- $A \otimes A^{coC} \approx (A \otimes A)^{coC}$.
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The previous arguments can be formulated almost without modification in terms of entwining structures over non-commutative algebras with bijective entwining map, i.e. entwining structures over $A - A$-Bimod where A is an algebra in R-Mod, to prove a generalization of Kreimer-Takeuchi Theorem for Hopf algebroids Böhm (**Ann. Univ. Ferrara-Sez. VII-Sc. Mat.** (2005)).

Preliminaries

Let $\mathcal{C} = (\mathcal{C}, \otimes, K)$ be a strict monoidal category with equalizers and coequalizers.

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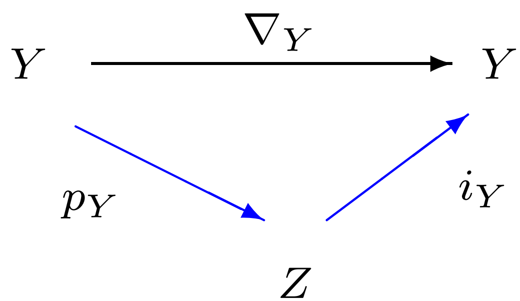

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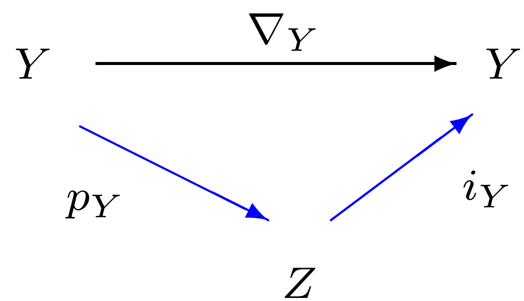

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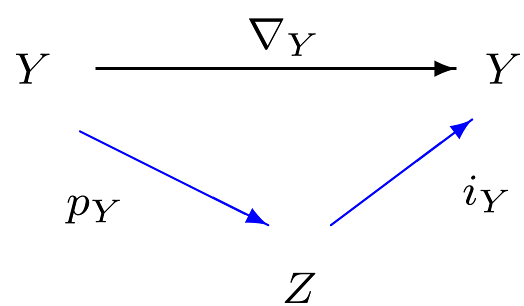

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We assume that the algebras are associative with unit and the coalgebras are coassociative with counit. If A is an algebra and C a coalgebra:

$$\eta_A : K \to A, \quad \mu_A : A \otimes A \to A, \quad \varepsilon_C : C \to K, \quad \delta_C : C \to C \otimes C$$

denotes the unit, the product, the counit and the coproduct respectively.

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If A is an algebra, B is a coalgebra and $\alpha : B \longrightarrow A$, $\beta : B \longrightarrow A$ are morphisms in $\mathcal{C}$,

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For M, N, P objects in $\mathcal{C}$ and $f : M \to N$ a morphism in $\mathcal{C}$, we write

$P \otimes f$ by $id_P \otimes f$ and $f \otimes P$ by $f \otimes id_P$,

where id_P denotes the identity morphism of P.

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Let A be an algebra in $\mathcal{C}$.

We will say that (M, φ_M) is a left A-module if M is an object in $\mathcal{C}$ and $\varphi_M : A \otimes M \to M$ is a morphism in $\mathcal{C}$ such that:

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If (M, φ_M) and (N, φ_N) are left A-modules, $f : M \rightarrow N$ is a morphism of left A-modules if $\varphi_N \circ (A \otimes f) = f \circ \varphi_M$. We denote the category by ${}_A\mathcal{M}$ ($\mathcal{M}_A$).

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If (M, ϱ_M) and (N, ϱ_N) are morphism of left C-comodules, $f : M \to N$ is a morphism of left C-comodules if $\varrho_N \circ f = (C \otimes f) \circ \varrho_M$. We denote the category by ${}^C\mathcal{M}$ ($\mathcal{M}^C$).

Weak entwining structures

Definition. A right-right weak entwining structure over $\mathcal{C}$ is a triple (A, C, ψ_R), where A is an algebra, C is a coalgebra, and

$$\psi_R : C \otimes A \to A \otimes C$$

is a morphism such that

(1) $\psi_R \circ (C \otimes \mu_A) = (\mu_A \otimes C) \circ (A \otimes \psi_R) \circ (\psi_R \otimes A)$,

(2) $(A \otimes \delta_C) \circ \psi_R = (\psi_R \otimes C) \circ (C \otimes \psi_R) \circ (\delta_C \otimes A)$,

(3) $\psi_R \circ (C \otimes \eta_A) = (e_{RR} \otimes C) \circ \delta_C$,

(4) $(A \otimes \varepsilon_C) \circ \psi_R = \mu_A \circ (e_{RR} \otimes A)$,

where $e_{RR} : C \to A$ is the morphism

$$e_{RR} = (A \otimes \varepsilon_C) \circ \psi_R \circ (C \otimes \eta_A).$$

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(3) If A is an algebra in R-Mod and $\mathcal{C} = A - A$-Bimod we have the definition of entwining structure over a non-commutative algebra introduced by Böhm (**Contemp. Math.** (2005)).

Weak entwining structures

Definition. Let (A, C, ψ_R) be a right-right entwining structure in $\mathcal{C}$. By $\mathcal{M}_A^C(\psi_R)$ we denote the category whose objects are triples (M, ϕ_M, ρ_M), where (M, ϕ_M) is a right A-module, (M, ρ_M) is a right C-comodule and

$$\rho_M \circ \phi_M = (\phi_M \otimes C) \circ (M \otimes \psi_R) \circ (\rho_M \otimes A).$$

The morphisms in $\mathcal{M}_A^C(\psi_R)$ are the obvious, i.e., morphisms of right A-modules and right C-comodules.
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If $(A, \mu_A, \rho_A) \in \mathcal{M}_A^C(\psi_R)$, the morphism

$$\Delta_{A\otimes C}^R = (\mu_A \otimes C) \circ (A \otimes \psi_R) \circ (A \otimes C \otimes \eta_A) : A \otimes C \to A \otimes C$$

is idempotent and as a consequence, there exist an object $A \square C$ and morphisms $i_{A\otimes C}^R : A \square C \to A \otimes C$, $p_{A\otimes C}^R : A \otimes C \to A \square C$ such that $\Delta_{A\otimes C}^R = i_{A\otimes C}^R \circ p_{A\otimes C}^R$, $id_{A\square C} = p_{A\otimes C}^R \circ i_{A\otimes C}^R$.

Weak entwining structures

The triple $(A\Box C, \phi_{A\Box C}, \rho_{A\Box C})$ is a right entwined module, where the action and the coaction are defined by

$$\phi_{A\Box C} = p^R_{A\otimes C} \circ (\mu_A \otimes C) \circ (A \otimes \psi_R) \circ (i^R_{A\otimes C} \otimes A), \quad \rho_{A\Box C} = (p^R_{A\otimes C} \otimes C) \circ (A \otimes \delta_C) \circ i^R_{A\otimes C}$$

respectively.

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respectively.

If $(M, \phi_M, \rho_M) \in \mathcal{M}^C_A(\psi_R)$, with M^R_C we denote the subobject of right coinvariants of M, i.e., M^R_C is the equalizer of ρ_M and

$$\zeta^R_M = (\phi_M \otimes C) \circ (M \otimes (\rho_A \circ \eta_A)).$$

$$M^R_C \xrightarrow{\ i^R_M\ } M \underset{\zeta^R_M}{\overset{\rho_M}{\rightrightarrows}} M\otimes C.$$

Weak entwining structures

If $M = A$ the triple $(A_C^R, \eta_{A_C^R}, \mu_{A_C^R})$ is an algebra in $\mathcal{C}$, where

$$\eta_{A_C^R} : K \to A_C^R, \qquad \mu_{A_C^R} : A_C^R \otimes A_C^R \to A_C^R$$

are the factorization through the equalizer i_A^R of η_A and $\mu_A \circ (i_A^R \otimes i_A^R) : A_C^R \otimes A_C^R \to A$, respectively. This is the subalgebra of the right coinvariants in this setting

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is the factorization of

$$\phi_M \circ (i_M^R \otimes i_A^R) : M_C^R \otimes A_C^R \to M$$

through the equalizer i_M^R.

Weak entwining structures

The morphism

$$r_A^R = p^R_{A\otimes C} \circ (\mu_A \otimes C) \circ (A \otimes \rho_A) : A \otimes A \longrightarrow A \Box C$$

admits a factorization through the coequalizer q_A^R of

$$A \otimes (\mu_A \circ (i_A^R \otimes A)) : A \otimes A_C^R \otimes A \longrightarrow A \otimes A, \quad (\mu_A \circ (i_A^R \otimes A)) \otimes A : A \otimes A_C^R \otimes A \longrightarrow A \otimes A.$$

Then there exists a morphism called right canonical morphism

$$\beta_A^R : A \otimes_{A_C^R} A \longrightarrow A \Box C$$

such that $\beta_A^R \circ q_A^R = r_A^R$.

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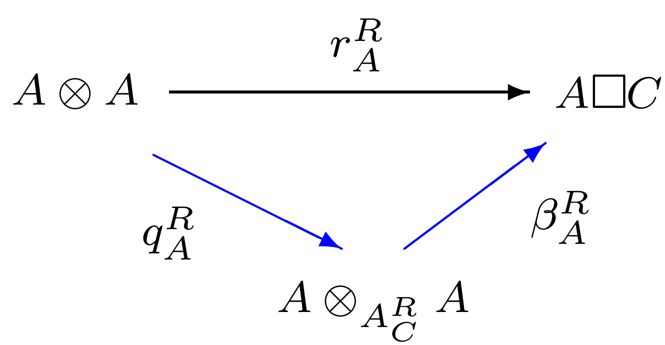

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The morphisms r_A^R and β_A^R are morphisms of right C-comodules where $\rho_{A\otimes A} = A \otimes \rho_A$ and $\rho_{A\otimes_{A_C^R} A}$ is the factorization of $(q_A^R \otimes C) \circ (A \otimes \rho_A)$ through the coequalizer q_A^R.
If the functor $A \otimes -$ preserves coequalizers, β_A^R is a morphism of left A-modules where $\varphi_{A\otimes_{A_C^R} A}$ is the factorization of $q_A^R \circ (\mu_A \otimes A)$ through the coequalizer of $A \otimes q_A^R$ and $\varphi_{A\Box C} = p_{A\otimes C}^R \circ (\mu_A \otimes C) \circ (A \otimes i_{A\otimes C}^R)$.

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Definition. Let (A, C, ψ_R) a right-right weak entwining structure in $\mathcal{C}$ such that $A \otimes -$ preserves coequalizers and there exist a coaction ρ_A satisfying that (A, μ_A, ρ_A) belongs to $\mathcal{M}_A^C(\psi_R)$. We will say that $A_C^R \hookrightarrow A$ is a right weak C-Galois extension if the morphism β_A^R is an isomorphism.

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We have similar results and definitions for the left side.

Weak entwining structures

(1) (A, C, ψ_L), $\quad \psi_L : A \otimes C \to C \otimes A$, $\quad e_{LL} = (\varepsilon_C \otimes A) \circ \psi_L \circ (\eta_A \otimes C)$.

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(3) $\Delta^L_{C \otimes A} = (C \otimes \mu_A) \circ (\psi_L \otimes A) \circ (\eta_A \otimes C \otimes A) : C \otimes A \to C \otimes A$

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(1) (A, C, ψ_L), $\psi_L : A \otimes C \to C \otimes A$, $e_{LL} = (\varepsilon_C \otimes A) \circ \psi_L \circ (\eta_A \otimes C)$.

(2) ${}^C_A\mathcal{M}(\psi_L)$, $(M, \varphi_M, \varrho_M)$.

(3) $\Delta^L_{C\otimes A} = (C \otimes \mu_A) \circ (\psi_L \otimes A) \circ (\eta_A \otimes C \otimes A) : C \otimes A \to C \otimes A$

(4) $\Delta^L_{C\otimes A} = i^L_{C\otimes A} \circ p^L_{C\otimes A}$, $Im(\Delta^L_{C\otimes A}) = C\square A$, $p^L_{C\otimes A} \circ i^L_{C\otimes A} = id_{C\square A}$.

Weak entwining structures

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(5) If $(A, \mu_A, \varrho_A) \in {}^C_A\mathcal{M}(\psi_L)$ then $(C\Box A, \varphi_{C\Box A}, \varrho_{C\Box A}) \in {}^C_A\mathcal{M}(\psi_L)$.

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(5) If $(A, \mu_A, \varrho_A) \in {}^{C}_{A}\mathcal{M}(\psi_L)$ then $(C\Box A, \varphi_{C\Box A}, \varrho_{C\Box A}) \in {}^{C}_{A}\mathcal{M}(\psi_L)$.

(6) If $(M, \varphi_M, \varrho_M) \in {}^{C}_{A}\mathcal{M}(\psi_L)$ we define M^L_C as the equalizer of the morphisms ϱ_M and $\zeta^L_M = (C \otimes \varphi_M) \circ ((\varrho_A \circ \eta_A) \otimes M)$.

$$M^L_C \xrightarrow{\;i^L_M\;} M \underset{\zeta^L_M}{\overset{\varrho_M}{\rightrightarrows}} C \otimes M.$$

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$$M^L_C \xrightarrow{i^L_M} M \underset{\zeta^L_M}{\overset{\varrho_M}{\rightrightarrows}} C \otimes M.$$

(7) If $M = A$, the triple $(A^L_C, \eta_{A^L_C}, \mu_{A^L_C})$ is an algebra in $\mathcal{C}$.

Weak entwining structures

(1) (A, C, ψ_L), $\psi_L : A \otimes C \to C \otimes A$, $e_{LL} = (\varepsilon_C \otimes A) \circ \psi_L \circ (\eta_A \otimes C)$.

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(3) $\Delta_{C\otimes A}^L = (C \otimes \mu_A) \circ (\psi_L \otimes A) \circ (\eta_A \otimes C \otimes A) : C \otimes A \to C \otimes A$

(4) $\Delta_{C\otimes A}^L = i_{C\otimes A}^L \circ p_{C\otimes A}^L$, $Im(\Delta_{C\otimes A}^L) = C\square A$, $p_{C\otimes A}^L \circ i_{C\otimes A}^L = id_{C\square A}$.

(5) If $(A, \mu_A, \varrho_A) \in {}_A^C\mathcal{M}(\psi_L)$ then $(C\square A, \varphi_{C\square A}, \varrho_{C\square A}) \in {}_A^C\mathcal{M}(\psi_L)$.

(6) If $(M, \varphi_M, \varrho_M) \in {}_A^C\mathcal{M}(\psi_L)$ we define M_C^L as the equalizer of the morphisms ϱ_M and $\zeta_M^L = (C \otimes \varphi_M) \circ ((\varrho_A \circ \eta_A) \otimes M)$.

$$M_C^L \xrightarrow{\;i_M^L\;} M \underset{\zeta_M^L}{\overset{\varrho_M}{\rightrightarrows}} C \otimes M.$$

(7) If $M = A$, the triple $(A_C^L, \eta_{A_C^L}, \mu_{A_C^L})$ is an algebra in $\mathcal{C}$.

(8) $(M_C^L, \varphi_{M_C^L})$ is a left A_C^L-module.

Weak entwining structures

For the morphism

$$r_A^L = p_{A\otimes C}^L \circ (C \otimes \mu_A) \circ (\varrho_A \otimes A) : A \otimes A \to C \Box A$$

there exists a factorization through the coequalizer q_A^L of

$$(\mu_A \circ (A \otimes i_A^L)) \otimes A : A \otimes A_C^L \otimes A \to A \otimes A, \quad A \otimes (\mu_A \circ (i_A^L \otimes A)) : A \otimes A_C^L \otimes A \to A \otimes A.$$

As a consequence, there exists a morphism called left canonical morphism

$$\beta_A^L : A \otimes_{A_C^L} A \to C \Box A$$

such that $\beta_A^L \circ q_A^L = r_A^L$.

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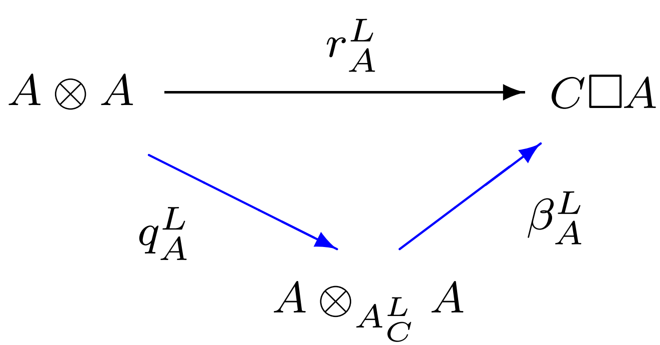

Weak entwining structures

The morphisms r_A^L and β_A^L are morphisms of left C-comodules where $\varrho_{A\otimes A} = \varrho_A \otimes A$ and $\varrho_{A\otimes_{A_C^L} A}$ is the factorization of $(C \otimes q_A^L) \circ (\varrho_A \otimes A)$ through the coequalizer q_A^L.
If the functor $- \otimes A$ preserves equalizers, then β_A^L is a morphism of right A-modules where $\phi_{A\otimes_{A_C^L} A}$ is the factorization of $q_A^L \circ (A \otimes \mu_A)$ through the coequalizer $q_A^L \otimes A$ and $\phi_{C\Box A} = p_{C\otimes A}^L \circ (C \otimes \mu_A) \circ (i_{A\otimes C}^L \otimes A)$.

Weak entwining structures

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Definition. Let (A, C, ψ_L) left-left weak entwining structure in $\mathcal{C}$ such that $- \otimes A$ preserve coequalizers and there exists a coaction ϱ_A satisfying that (A, μ_A, ρ_A) belongs to ${}_A^C\mathcal{M}(\psi_L)$.
We will say that $A_C^L \hookrightarrow A$ is a left weak C-Galois extension if β_A^L is an isomorphism.

Invertible weak entwining structures

Definition. (**Brzeziński, Turner y Wrightson**, **Comm. in Algebra., 2006**, $\mathcal{C} = R$-Mod)
If (A, C, ψ_R) is a right-right weak entwining structure in $\mathcal{C}$ and (A, C, ψ_L) is a left-left weak entwining structure in $\mathcal{C}$, we will say that (A, C, ψ_R, ψ_L) is an invertible weak entwining structure in $\mathcal{C}$, if $\psi_R \circ \psi_L = \Delta^R_{A\otimes C}$ and $\psi_L \circ \psi_R = \Delta^L_{C\otimes A}$.

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Nota.

(1) If (A, C, ψ_R, ψ_L) is an invertible weak entwining structure in $\mathcal{C}$ and $\psi_R \circ \psi_L = id_{A\otimes C}$ y $\psi_L \circ \psi_R = id_{C\otimes A}$, then (A, C, ψ_R) is an entwining structure.

(2) As a consequence of this definition we have $e_{RR} = e_{LL}$.

Invertible weak entwining structures

Definition.(Alonso, Fernández y González, **J. of Algebra, 2008**)

A weak Hopf algebra in a strict braided monoidal category $\mathcal{C}$, with braiding c, is an algebra (D, η_D, μ_D), coalgebra $(D, \varepsilon_D, \delta_D)$, satisfying the following:

(1) $\delta_D \circ \mu_D = (\mu_D \otimes \mu_D) \circ (D \otimes c_{D,D} \otimes D) \circ (\delta_D \otimes \delta_D)$.

(2) $\varepsilon_D \circ \mu_D \circ (\mu_D \otimes D) = ((\varepsilon_D \circ \mu_D) \otimes (\varepsilon_D \circ \mu_D)) \circ (D \otimes \delta_D \otimes D)$

$= ((\varepsilon_D \circ \mu_D) \otimes (\varepsilon_D \circ \mu_D)) \circ (D \otimes (c^{-1}_{D,D} \circ \delta_D) \otimes D)$.

(3) $(\delta_D \otimes D) \circ \delta_D \circ \eta_D = (D \otimes \mu_D \otimes D) \circ ((\delta_D \circ \eta_D) \otimes (\delta_D \circ \eta_D))$

$= (D \otimes (\mu_D \circ c^{-1}_{D,D}) \otimes D) \circ ((\delta_D \circ \eta_D) \otimes (\delta_D \circ \eta_D))$.

(4) There exists a morphism $\lambda_D : D \to D$ in $\mathcal{C}$ (called the antipode of D) such that:

(4-1) $id_D \wedge \lambda_D = ((\varepsilon_D \circ \mu_D) \otimes D) \circ (D \otimes c_{D,D}) \circ ((\delta_D \circ \eta_D) \otimes D)$,

(4-2) $\lambda_D \wedge id_D = (D \otimes (\varepsilon_D \circ \mu_D)) \circ (c_{D,D} \otimes D) \circ (D \otimes (\delta_D \circ \eta_D))$,

(4-3) $\lambda_D \wedge id_D \wedge \lambda_D = \lambda_D$.

Invertible weak entwining structures

Definition. Let D be a weak Hopf algebra in a strict braided monoidal category $\mathcal{C}$. Let A be an algebra with structure of right D-comodule $\rho_A : A \to A \otimes D$ and such that $\mu_{A\otimes D} \circ (\rho_A \otimes \rho_A) = \rho_A \circ \mu_A$ where $\mu_{A\otimes D} = (\mu_A \otimes \mu_D) \circ (A \otimes c_{D,A} \otimes D)$. We will say that A is a right D-comodule algebra if any of the following equivalent conditions holds:

(i) $(\rho_A \otimes D) \circ \rho_A \circ \eta_A = (A \otimes (\mu_D \circ c^{-1}_{D,D}) \otimes D) \circ (\rho_A \otimes \delta_D) \circ (\eta_A \otimes \eta_D)$,

(ii) $(\rho_A \otimes D) \circ \rho_A \circ \eta_A = (A \otimes \mu_D \otimes D) \circ (\rho_A \otimes \delta_D) \circ (\eta_A \otimes \eta_D)$,

(iii) $(A \otimes \overline{\Pi}^R_D) \circ \rho_A = (\mu_A \otimes D) \circ (A \otimes \rho_A) \circ (A \otimes \eta_A)$,

(iv) $(A \otimes \Pi^L_D) \circ \rho_A = ((\mu_A \circ c^{-1}_{A,A}) \otimes D) \circ (A \otimes \rho_A) \circ (A \otimes \eta_A)$,

(v) $(A \otimes \overline{\Pi}^R_D) \circ \rho_A \circ \eta_A = \rho_A \circ \eta_A$,

(vi) $(A \otimes \Pi^L_D) \circ \rho_A \circ \eta_A = \rho_A \circ \eta_A$.

Invertible weak entwining structures

Proposition. Let D be a weak Hopf algebra in a strict braided monoidal category $\mathcal{C}$ such that λ_D is an isomorphism. Let A be a D-comodule algebra. Then (A, D, ψ_R, ψ_L) with

$$\psi_R = (A \otimes \mu_D) \circ (c_{D,A} \otimes D) \circ (D \otimes \rho_A) : D \otimes A \to A \otimes D$$

and

$$\psi_L = c^{-1}_{D,A} \circ (A \otimes (\mu_D \circ c^{-1}_{D,D} \circ (\lambda_D^{-1} \otimes D))) \circ (\rho_A \otimes D) : A \otimes D \to D \otimes A$$

is an invertible weak entwining structure.

Invertible weak entwining structures

Proposition. Let (A, C, ψ_R, ψ_L) be an invertible weak entwining structure in $\mathcal{C}$.
(1) $\Phi_{A,C} = p^L_{C\otimes A} \circ \psi_L \circ i^R_{A\otimes C} : A\Box C \to C\Box A$ is an isomorphism with inverse

$$\Phi^{-1}_{A,C} = p^R_{A\otimes C} \circ \psi_R \circ i^L_{C\otimes A}.$$

(2) If (A, μ_A, ρ_A) is a right entwined module, $(A, \mu_A, \rho^L_A = \psi_L \circ \zeta^R_A)$ is a left entwined module. Also, if $(A, \mu_A, \varrho_A) \in {}^C_A\mathcal{M}(\psi_L)$ then $(A, \mu_A, \varrho^R_A = \psi_R \circ \zeta^L_A) \in \mathcal{M}^C_A(\psi_R)$. Moreover, $\rho^{LR}_A = \rho_A$ and $\varrho^{RL}_A = \varrho_A$
(3) If $(A, \mu_A, \rho_A) \in \mathcal{M}^C_A(\psi_R)$ there exists an algebra isomorphism $a_A : A^L_C \to A^R_C$ (a_A is the unique morphism such that $i^R_A \circ a_A = i^L_A$).
(4) If $(A, \mu_A, \rho_A) \in \mathcal{M}^C_A(\psi_R)$, then $\beta^L_A = \Phi_{A,C} \circ \beta^R_A \circ h_A$ where

$$h_A : A \otimes_{A^L_C} A \to A \otimes_{A^R_C} A$$

is an isomorphism and the unique morphism such that $h_A \circ q^L_A = q^R_A$.

Invertible weak entwining structures

Proposition. Let (A, C, ψ_R, ψ_L) be an invertible weak entwining structure. Then, if the functors $A \otimes -$ and $- \otimes A$ preserve coequalizers, $A_C^R \hookrightarrow A$ is a right weak C-Galois extension iff, $A_C^L \hookrightarrow A$ is a left weak C-Galois extension.

Proposition. Let (A, C, ψ_R, ψ_L) be an invertible weak entwining structure in $\mathcal{C}$. Let (M, ϕ_M, ρ_M) be an object in $\mathcal{M}_A^C(\psi_R)$. There exists a bijective map

$$\Theta_M^R : Hom_{\mathcal{M}^C}(C, M) \to Hom_{\mathcal{M}_A^C}(A\Box C, M).$$

defined by

$$\Theta_M^R(f) = \phi_M \circ (f \otimes A) \circ \psi_L \circ i_{A\otimes C}^R$$

and with inverse

$$(\Theta_M^R)^{-1}(g) = g \circ p_{A\otimes C}^R \circ (\eta_A \otimes C).$$

Moreover, each morphism in $Hom_{\mathcal{M}_A^C}(M, A\Box C)$ that splits in $\mathcal{M}^C$, splits also in $\mathcal{M}_A^C$.

Projectivity and weak Galois extensions

Theorem. Let (A, C, ψ_R) be a right-right weak entwining structure in $\mathcal{C}$. Suppose that (A, μ_A, ρ_A) belongs to $\mathcal{M}_A^C(\psi_R)$.
Consider the following statements:

(i) The morphism $r_A^R : A \otimes A \to A \Box C$ splits in $\mathcal{M}^C$.

(ii)

(ii-1) The morphism $\beta_A^R : A \otimes_{A_C^R} A \to A \Box C$ is an isomorphism.

(ii-2) $(A, \phi_A = \mu_A \circ (A \otimes i_A^R))$ is relative projective in $\mathcal{M}_{A_C^R}$,i.e., $\phi_A : A \otimes A_C^R \to A$ splits as morphism of right A_C^R-modules.

Then (ii) implies (i). If (A, C, ψ_R, ψ_L) is an invertible weak entwining structure in $\mathcal{C}$ and $s_A^R : A \otimes A_C^R \to (A \otimes A)_C^R$ is an isomorphism then (i) implies (ii).

Projectivity and weak Galois extensions

Particular cases.

(1) The Schauenburg-Schneider Theorem (**J. of Pure and Appl. Algebra** (2005)).

Projectivity and weak Galois extensions

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Projectivity and weak Galois extensions

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(2) The Brzeziński-Turner-Wrightson Theorem (weak version of Kreimer-Takeuchi Theorem)(**Comm. in Algebra** (2006)).

(3) The Böhm Theorem for algebroids (algebroid version of Kreimer-Takeuchi Theorem)(**Ann. Univ. Ferrara-Sez. VII-Sc. Mat.** (2005)).

Projectivity and weak Galois extensions

Corollary. Let D be a weak Hopf algebra in a strict braided monoidal category $\mathcal{C}$ with equalizers and coequalizers. Suppose that the antipode λ_D is an isomorphism. Let (A, ρ_A) be a right D-comodule algebra.
Consider the following statements:

(i) The morphism $r_A^R : A \otimes A \to A \square D$ splits in $\mathcal{M}^D$.

(ii)

(ii-1) The morphism $\beta_A^R : A \otimes_{A_D^R} A \to A \square D$ is an isomorphism.

(ii-2) $(A, \phi_A = \mu_A \circ (A \otimes i_D^R))$ is relative projective in $\mathcal{M}_{A_D^R}$.

Then (ii) implies (i). If $s_A^R : A \otimes A_D^R \to (A \otimes A)_D^R$ is an isomorphism, then (i) implies (ii).

References

(1) Alonso Álvarez, J.N., Fernández Vilaboa, J.M., González Rodríguez, R. y Rodríguez Raposo A. B.: Invertible weak entwining structures and weak C-cleft extensions. Applied Categorical Structures 14, N. 5-6 (special issue Categorical methods in Hopf algebras), 411-419 (2006).

(2) Alonso Álvarez, J.N., Fernández Vilaboa, J.M., González Rodríguez, R. y López López M. P.: A characterization of projective weak Galois extensions, Israel J. of Math. (in press) (2008).

(3) Alonso Álvarez, J.N., Fernández Vilaboa, J.M., González Rodríguez, R.: Strong connections and invertible weak entwining structures, preprint (2008).

(4) Alonso Álvarez, J.N., Fernández Vilaboa, J.M., González Rodríguez, R.: Weak Hopf algebras and weak Yang-Baxter operators, J. of Algebra 320, N.6, 2101-2143 (2008).

(5) Alonso Álvarez, J.N., Fernández Vilaboa, J.M., González Rodríguez, R.: Weak braided Hopf algebras, Indiana University Mathematics Journal (2008) (in press: http://www.iumj.indiana.edu/IUMJ/forthcoming.php).