Descent for compact 0-dimensional spaces

Manuela Sobral

(joint work with George Janelidze)

V Portuguese Category Seminar

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Monadicity/Descent

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- a descent morphism if p^* is premonadic.

Exact/Regular

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- finite limits;
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Examples of regular categories where regular epis fail to be e.d.m. appear in *Facets of descent* I, by G. Janelidze and W. Tholen (1994).

General spaces/Stone spaces

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In the subcategories $\mathcal{CH}aus$ and $\mathcal{S}tone$ (= compact, T_0 and 0-dimensional spaces) of the category of compact 0-dimensional spaces e.d.m. have an easy description. What are the e.d.m. in the category of compact 0-dimensional spaces?

Compact 0-dimensional spaces

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Let $p : E \longrightarrow B$ be surjective. If p fulfills that requirement then for each pair of inseparable points $p(e)$ and b in B there exists $x \in E$ such that $p(x) = b$ with e and x inseparable points in E. (“inseparable"= have the same closure)

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We call such maps *fibrations*.

Compact 0-dimensional spaces categorically

Let $\mathcal{S} = \mathcal{S}et$, $\mathcal{X} = \mathcal{S}tone$, $U : \mathcal{S}tone \to \mathcal{S}et$ be the forgetful functor and $\mathbb{E}$ be the class of surjective maps.

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Let $\mathcal{C}[\mathcal{X}, \mathcal{S}, U, \mathbb{E}]$ be the full subcategory of $\mathcal{S} \downarrow U$ with objects all triples (A_1, e_A, A_0), in which $e_A : A_1 \longrightarrow U(A_0)$ is a surjection.

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The category $\mathcal{C}$ of compact 0-dimensional spaces is equivalent to $\mathcal{C}[\mathcal{X}, \mathcal{S}, U, \mathbb{E}]$.
Under this equivalence a space A corresponds the the triple (A_1, e_A, A_0) where A_1 is the underlying set, A_0 is the T_0-reflection of A and e_A is the canonical map.

What are the fibrations?

A morphism $f \in \mathcal{C}$ is a fibration if and only if in

$$\begin{array}{ccc} A_1 & \xrightarrow{e_A} & U(A_0) \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle Uf_0} \\ B_1 & \xrightarrow[e_B]{} & U(B_0) \end{array}$$

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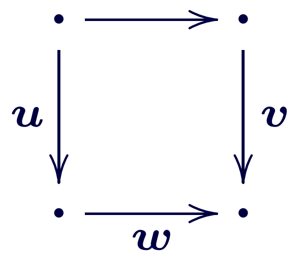
Morphisms $f = (f_1, f_0)$ in $\mathcal{S} \downarrow U$ for which $< f_1, e_A >$ is in $\mathbb{E}$ will be called *fibrations* and its class will be denoted by $\mathbb{F}$.

What do we get from this description?

Consider $\mathcal{X}$ and $\mathcal{S}$ categories with pullbacks, $U : \mathcal{X} \longrightarrow \mathcal{C}$ a pullback preserving functor, and $\mathbb{E}$ a class of morph. in $\mathcal{S}$ which

- contains all isomorphisms;
- is pullback stable;
- is closed under composition;
- forms a *stack*: for each pullback with w e.d.m.



then $u \in \mathbb{E} \Rightarrow v \in \mathbb{E}$.

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If $\mathcal{S}$ has coequalizers of equivalence relations and $p = (p_1, p_0)$ is a morphism in $\mathcal{C}$, for which p_1 and p_0 are e.d.m. in $\mathcal{S}$ and in $\mathcal{X}$ respectively, then

- p is an effective $\mathbb{F}$-descent morphism in $\mathcal{C}$.
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Theorem. The following conditions on a morphism $p : E \longrightarrow B$ in $\mathcal{C}$ are equivalent:
(a) p is an effective descent morphism;
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