Graph-theoretic fibring of logics

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reporting on

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V Portuguese Category Seminar

Context

Logic

- language: typically freely generated from a signature
- deductive system
  - Hilbert calculi
  - tableaux systems
  - Gentzen systems
  - ...
- semantics
  - algebraic
  - relational
  - games
  - ...

Context

Classical propositional logic

- language:
  freely generated from a set of propositional symbols Π_c by using the unary connective $\neg_c$ and the binary connective $\supset_c$
- semantics:
  the class of all valuations from Π_c into $\{0, 1\}$
- deductive system:
  Hilbert system with the following axioms

$$\xi \supset_c (\xi' \supset_c \xi)$$
$$(\xi \supset_c (\xi' \supset_c \xi'')) \supset_c ((\xi \supset_c \xi') \supset_c (\xi \supset_c \xi''))$$
$$((\neg_c \xi) \supset_c (\neg_c \xi')) \supset_c (\xi' \supset_c \xi)$$

and the rule of Modus Ponens stating that from ξ and $\xi \supset_c \xi'$ it is possible to conclude ξ'

Context

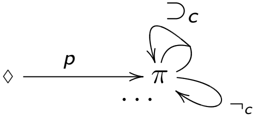
Fibring

- operation on logic systems:
  the fibring of two logic systems is a logic system such that
  - language:
    obtained by interleaving the constructors of both logics in formulas
  - deductive system:
    should be such that the set of consequences are in some sense minimal
  - semantics:
    should be sound and complete with respect to the deductive system

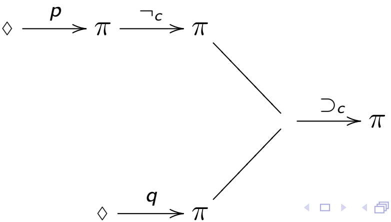
Motivation

Multi-graphs: very appropriate for describing a logic system

- language:
  constructors are seen as edges between sorts:



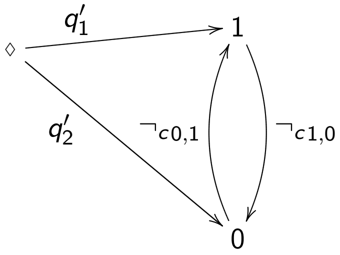
and formulas as paths (or trees) in a multi-graph:



Motivation

Multi-graphs: very appropriate for describing:

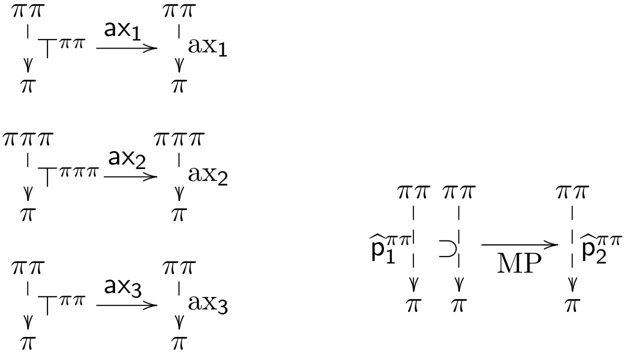
- semantics:
  graph representation of the operations:



Motivation

Multi-graphs: very appropriate for describing:

- deductive systems:
  rules are m-edges between formulas:



Motivation

Multi-graphs: very appropriate for describing fibring:

- offer a natural way to represent interleaving which is at the heart of fibring
- allow the fibring of an interpretation structure of a logic with any interpretation structure of the other logic
- make possible the definition of the semantic aspects of fibring in such a way that the collapsing problem does not appear

Motivation

Moreover, multi-graphs allow the representation of a wide class of logics:

- logics with an algebraic semantics
- substructural logics
- logics with a partial semantics
- logics endowed with a nondeterministic semantics
- ...

Multi-graphs

By a *multi-graph* (in short, a *m-graph*) we mean a tuple

$$G = (V, E, \mathrm{src}, \mathrm{trg})$$

where:

- V is a set (of *vertexes* or *nodes*);
- E is a set (of *m-edges*);
- $\mathrm{src} : E \to V^+$;
- $\mathrm{trg} : E \to V$.

Multi-graphs

and by a *m-graph morphism* $h : G_1 \longrightarrow G_2$ we mean a pair of maps

$$\begin{cases} h^{\mathsf{v}} : V_1 \longrightarrow V_2 \\ h^{\mathsf{e}} : E_1 \longrightarrow E_2 \end{cases}$$

such that:

- $\mathrm{src}_2 \circ h^{\mathsf{e}} = h^{\mathsf{v}} \circ \mathrm{src}_1$;
- $\mathrm{trg}_2 \circ h^{\mathsf{e}} = h^{\mathsf{v}} \circ \mathrm{trg}_1$.

We denote by **mGraph** the category of m-graphs and their morphisms.

Generation of a category with binary products out of a m-graph

1. from a m-graph G to a graph $G^\dagger$ where:
   - the set of vertexes of $G^\dagger$ is V^+
   - the edges of $G^\dagger$ are the edges of G plus edges for projections and tuples
2. from the graph $G^\dagger$ to the category $G^\ddagger$ freely generated by $G^\dagger$
3. from the category $G^\ddagger$ to the category G^+ with binary products: quotient over the class of morphisms ensuring that projections and tuples have the required universal properties

Language

A *language signature* or, simply, a *signature* is a tuple

$$\Sigma = (G, \pi, \diamond)$$

where $G = (V, E, \text{src}, \text{trg})$ is a m-graph, and π and $\diamond$ are in V.

Example The *propositional signature* Σ_{Π_c} where Π_c is a set of propositional symbols, is a m-graph with sorts π and $\diamond$ and the following m-edges:

- $p : \diamond \longrightarrow \pi$ for each p in Π_c;
- $\neg_c : \pi \longrightarrow \pi$;
- $\supset_c : \pi\pi \longrightarrow \pi$.

Language

Given a signature $\Sigma = (G, \pi, \diamond)$

- the objects of G^+ are the finite and non-empty sequences of sorts in Σ
- the morphisms of G^+ play the role of *expressions* (schema formulas, schema terms, whatever) over Σ.

The morphisms of G^+ constitute the *language* generated by the signature, denoted by $L(\Sigma)$.

Language

For instance, the schema formula

$$(\xi_1 \supset_c (\xi_1 \supset_c \xi_1)) \supset_c \xi_2$$

is represented by the morphism

$$\supset_c \circ \langle \supset_c \circ \langle \xi_1, \supset_c \circ \langle \xi_1, \xi_1 \rangle \rangle, \xi_2 \rangle : \pi\pi \rightarrow \pi$$

where ξ_i is $\widehat{\mathsf{p}}_i^{\pi\pi}$, for $i = 1, 2$.

Semantics

An *interpretation structure* I *over a signature* $(G, \pi, \diamond)$ is a tuple

$$(G', \alpha, D, \blacklozenge)$$

such that

- G' is a m-graph (*operations graph*)
- $\alpha : G' \to G$ is a m-graph morphism (*abstraction morphism*)
- D is a non-empty set contained in $(\alpha^{\mathsf{v}})^{-1}(\pi)$
- $\blacklozenge$ is an element of $(\alpha^{\mathsf{v}})^{-1}(\diamond)$.

Semantics

concretization vs abstraction

- concretization: from syntax to semantics, that is, a syntactic constructor gets a concrete meaning when interpreted for instance as an operation in a algebra
- abstraction: from semantics to syntax, that is, each semantic component is abstracted to its syntactic counterpart

Semantics

Example

The interpretation structure $(G', \alpha, D, \blacklozenge)$ for the signature Σ_{Π_c} where $\Pi_c = \{q_1, q_2, q_3\}$ over a valuation $v : \{q_1, q_2, q_3\} \to \{0, 1\}$ such that $v(q_1) = 1$ and $v(q_2) = v(q_3) = 0$, is defined as follows:

- G' is such that:

  $V' = \{0, 1\} \cup \{\blacklozenge\}$;
  $E' = \{q_1', q_2', q_3', \neg_0, \neg_1, \supset_{00}, \supset_{01}, \supset_{10}, \supset_{11}\}$;
  src′ and trg′ are such that:

  $q_1' : \blacklozenge \to 1$;
  $q_i' : \blacklozenge \to 0$ for $i = 2, 3$;
  $\neg_{v'} : v' \to (1 - v')$ for each v' in V'_π;
  $\supset_{v_1' v_2'} : v_1' \, v_2' \to ((1 - v_1') + v_2')$ for each v_1' and v_2' in V'_π.

Semantics

- $\alpha : G' \to G$ is such that:

  $\alpha^{\mathsf{v}}(0) = \pi;$

  $\alpha^{\mathsf{v}}(1) = \pi;$

  $\alpha^{\mathsf{v}}(\blacklozenge) = \diamond;$

  $\alpha^{\mathsf{e}}(q_i') = q_i$ for $i = 1, 2, 3;$

  $\alpha^{\mathsf{e}}(\neg_{v'}) = \neg$ for each v' in $V_\pi';$

  $\alpha^{\mathsf{e}}(\supset_{v_1' v_2'}) = \supset$ for each v_1' and v_2' in V_π'.
- $D = \{1\}$.

Deductive systems

A *deductive signature* or *meta-signature* is a tuple

$$\Phi = (\Sigma, \top, \mathrm{R})$$

where $\Sigma = (G, \pi, \diamond)$ is a language signature such that

$$G^\Phi = (V^\Phi, E^\Phi, \mathrm{src}^\Phi, \mathrm{trg}^\Phi)$$

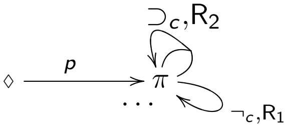
is a m-graph extending G with

- $V^\Phi = V$;
- $E^\Phi = E \cup \mathrm{R}$ where $\mathrm{R} = \{\mathrm{R}_n : \overbrace{\pi \ldots \pi}^{n} \to \pi\}_{n>0}$;

and $\top$ is a set $\{\top^s : s \to \pi\}_{s \in V^+}$.

Deductive systems

Example Graphical representation of part of the m-graph G^{ϕ} based on Σ_{Π_c}:



Deductive systems

A *deductive system over a meta-signature* Φ is a pair

$$(G'', \beta)$$

where G'' is a m-graph such that

- V'' is the class of morphisms of $G^+_\top$ whose target is in V;
- $E''(\widehat{w}_1 : s \to v_1 \ldots \widehat{w}_n : s \to v_n, \widehat{w} : s \to v)$, for $\widehat{w}$ in G^+, contains, among others, the m-edges $e : v_1 \ldots v_n \to v$ of E such that $\widehat{w} = e \circ \langle \widehat{w}_1, \ldots, \widehat{w}_n \rangle$ in G^+;
- $E''(\widehat{w}_1 : s_1 \to v_1 \ldots \widehat{w}_n : s_n \to v_n, \widehat{w} : s \to v) = \emptyset$ whenever $\widehat{w}$ is not in G^+ or $s_i \neq s$ for some $i = 1, \ldots, n$, or $\widehat{w}_i$ is not in G^+ and $n \neq 1$;

Deductive systems

and β is a m-graph morphism from G'' to G such that

- $\beta^{\mathsf{v}}(\widehat{w} : s \to v) = v$;
- $\beta^{\mathsf{e}}(e : (\widehat{w}_1 : s \to v_1 \ldots \widehat{w}_n : s \to v_n) \to (\widehat{w} : s \to v)) = e$ if e is in E and $\widehat{w} = e \circ \langle \widehat{w}_1, \ldots, \widehat{w}_n \rangle$;
- $\beta^{\mathsf{e}}(f') \in \mathrm{R}$ otherwise.

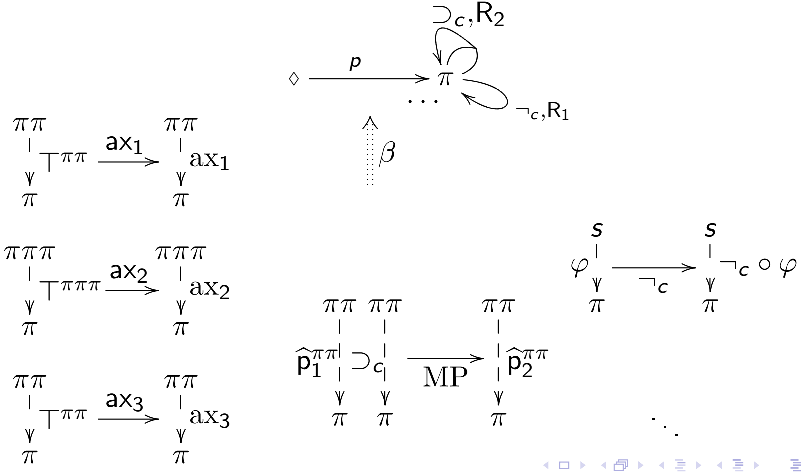
Deductive systems

Example Deductive system $(\Phi_{\Pi_c}, G'', \beta)$ for classical logic:

- Φ_{Π_c} is the meta-signature $(\Sigma_{\Pi_c}, \top, \mathsf{R})$;
- G'' has the mandatory m-edges for connectives, and
  - m-edge $\mathsf{ax}_1 : \top^{\pi\pi} \to \mathrm{ax}_1$ where ax_1 is $(\xi \supset_c (\xi' \supset_c \xi)) : \pi\pi \to \pi$;
  - m-edge $\mathsf{ax}_2 : \top^{\pi\pi\pi} \to \mathrm{ax}_2$ where ax_2 is $((\xi \supset_c (\xi' \supset_c \xi'')) \supset_c ((\xi \supset_c \xi') \supset_c (\xi \supset_c \xi''))) : \pi\pi\pi \to \pi$;
  - m-edge $\mathsf{ax}_3 : \top^{\pi\pi} \to \mathrm{ax}_3$ such that ax_3 is $(((\neg_c \xi) \supset_c (\neg_c \xi')) \supset_c (\xi' \supset_c \xi)) : \pi\pi \to \pi$;
  - m-edge $\mathrm{MP} : \widehat{\mathsf{p}}_1^{\pi\pi} \supset_c \to \widehat{\mathsf{p}}_2^{\pi\pi}$;
- $\beta : G'' \to G^{\Phi_\Pi}$ is such that:
  - $\beta^{\mathsf{e}}(\mathsf{ax}_i) = \mathsf{R}_1$ for $i = 1, 2, 3$;
  - $\beta^{\mathsf{e}}(\mathrm{MP}) = \mathsf{R}_2$.

Deductive systems

Example Graphical representation of part of the m-graph of the deductive system for classical propositional logic:



Deductive systems

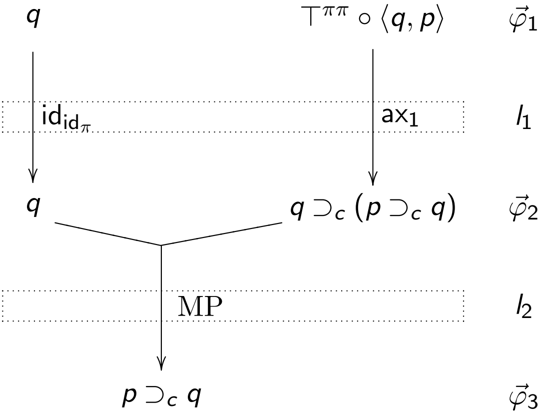
Derivations are seen as a sequence of *derivation steps*, also called *derivation levels*, where in each level one or several rules may be applied to different schema formulas coming from the preceding level.

The morphism $\mathrm{id}_{\mathrm{id}_\pi}$ is applied in a level to a schema formula when no rule is applied to it in that level.

Axioms are seen as unary rules whose antecedent is a verum schema formula.

Deductive systems

Examples Derivation of $p \supset_c q$ from q:



Deductive systems

In order to define derivations we consider a new category with binary products, $G''^{\star}$, obtained from G''^{+} by adding the morphisms

- $f_1 \otimes \cdots \otimes f_n : (\widehat{a}_{11} \ldots \widehat{a}_{1m_1}) \circ \widehat{\mathrm{p}}^{s_1 \ldots s_n}_{s_1} \ldots (\widehat{a}_{n1} \ldots \widehat{a}_{nm_n}) \circ \widehat{\mathrm{p}}^{s_1 \ldots s_n}_{s_n} \longrightarrow (\widehat{c}_1 \circ \widehat{\mathrm{p}}^{s_1 \ldots s_n}_{s_1} \ldots \widehat{c}_n \circ \widehat{\mathrm{p}}^{s_1 \ldots s_n}_{s_n})$ where $f_i : \widehat{a}_{i1} \ldots \widehat{a}_{im_i} \longrightarrow \widehat{c}_i$ is $\mathrm{id}_{\mathrm{id}_\pi}$ or is in $(\beta^{\mathrm{e}})^{-1}(\mathrm{R})$ and $\mathrm{src}(c_i) = s_i$;
- $\ell \odot \widehat{u} : (\widehat{a}_1 \ldots \widehat{a}_m) \circ \widehat{u} \longrightarrow (\widehat{c}_1 \ldots \widehat{c}_n) \circ \widehat{u}$ if $\widehat{u}$ in G^{+} is composable with $\widehat{c}_1$ and $\ell : \widehat{a}_1 \ldots \widehat{a}_m \longrightarrow \widehat{c}_1 \ldots \widehat{c}_n$ is of the form $f_1 \otimes \cdots \otimes f_n$;

while imposing:

- $\mathrm{id}_{\mathrm{id}_\pi} \odot \widehat{u} = \mathrm{id}_{\widehat{u}}$;
- $\ell \odot \mathrm{id}_s = \ell$;
- $(\ell \odot \widehat{u}_2) \odot \widehat{u}_1 = \ell \odot (\widehat{u}_2 \circ \widehat{u}_1)$;
- $(f_1 \otimes \cdots \otimes f_n) \odot \widehat{u} = (f_1 \odot (\widehat{\mathrm{p}}^{s_1 \ldots s_n}_{s_1} \circ \widehat{u})) \otimes \cdots \otimes (f_n \odot (\widehat{\mathrm{p}}^{s_1 \ldots s_n}_{s_n} \circ \widehat{u}))$.

Deductive systems

We write $\ell \star \vec{\varphi}$ whenever there is a substitution $\widehat{u}$ (a morphism in G^+) with $\vec{\varphi} = \mathrm{ANT}(\ell) \circ \widehat{u}$. In this case we define $\ell \star \vec{\varphi}$ as being equal to $\ell \odot \widehat{u}$.

By a *derivation* we mean a pair

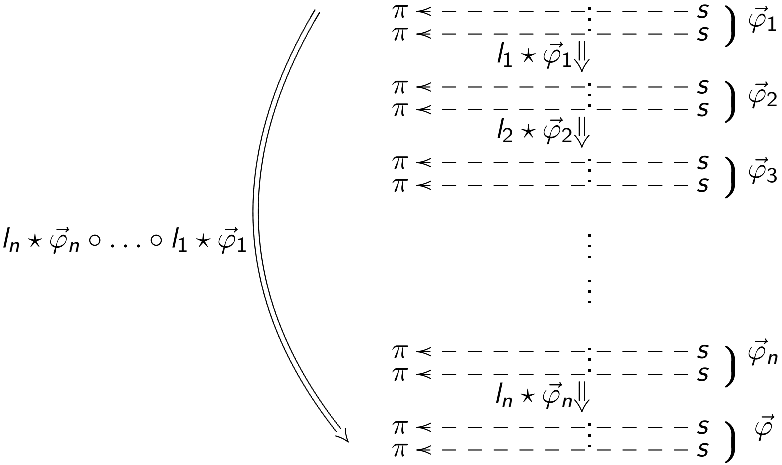
$$d = \ell_1, \ldots, \ell_n; \vec{\varphi}_1$$

where each ℓ_i is a derivation step and $\vec{\varphi}_1$ is a sequence of morphisms in V'' such that the sequence given by $\vec{\varphi}_{i+1} = \mathrm{CONC}(\ell_i \star \vec{\varphi}_i)$, for $i = 1, \ldots, n$, is well defined, and so there exists the composite morphism

$$(\ell_n \star \vec{\varphi}_n) \circ \ldots \circ (\ell_1 \star \vec{\varphi}_1)$$

in $G''^{\star}$.

Deductive systems



Deductive systems

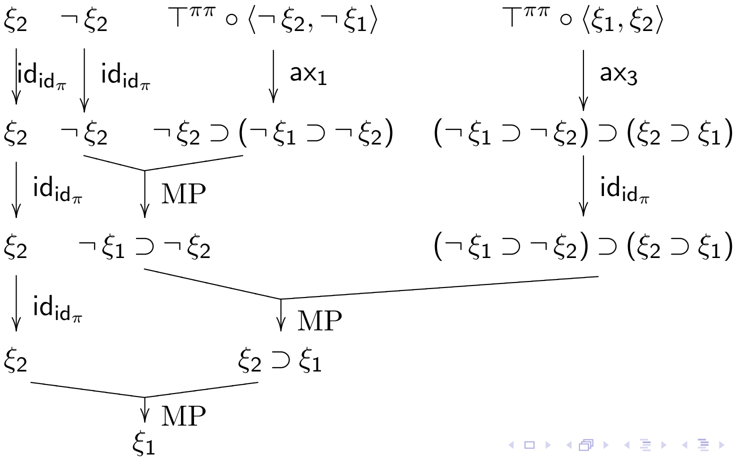
Example Derivation of ξ_1 from ξ_2 and $\neg\xi_2$:

in the Hilbert calculus for classical logic, ξ_1 is derived from ξ_2 and $\neg\,\xi_2$ as follows:

1.	ξ_2	Hyp
2.	$\neg\,\xi_2$	Hyp
3.	$(\neg\,\xi_2) \supset ((\neg\,\xi_1) \supset (\neg\,\xi_2))$	ax_1
4.	$(\neg\,\xi_1) \supset (\neg\,\xi_2)$	MP $2, 3$
5.	$((\neg\,\xi_1) \supset (\neg\,\xi_2)) \supset (\xi_2 \supset \xi_1)$	ax_3
6.	$\xi_2 \supset \xi_1$	MP $4, 5$
7.	ξ_1	MP $1, 6$

Deductive systems

which can be depicted graphically by the following picture:



Deductive systems

in our setting this corresponds to the derivation given by

$$\mathrm{MP}, (\mathsf{id}_{\mathsf{id}_\pi} \otimes \mathrm{MP}), (\mathsf{id}_{\mathsf{id}_\pi} \otimes \mathrm{MP} \otimes \mathsf{id}_{\mathsf{id}_\pi}), (\mathsf{id}_{\mathsf{id}_\pi} \otimes \mathsf{id}_{\mathsf{id}_\pi} \otimes \mathsf{ax}_1 \otimes \mathsf{ax}_3); \vec{\varphi}_1$$

where $\vec{\varphi}_1$ is the sequence $\xi_2, \neg\,\xi_2, \top^{\pi\pi} \circ \langle \neg\,\xi_2, \neg\,\xi_1 \rangle, \top^{\pi\pi} \circ \langle \xi_1, \xi_2 \rangle$

Logic system

A *logic system* is a triple

$$\mathcal{L} = (\Sigma, \mathcal{I}, \mathcal{D})$$

such that:

- $\mathcal{I} = (\Sigma, \mathfrak{I})$ is an interpretation system;
- $\mathcal{D} = (\Phi, G'', \beta)$ is a deductive system where Φ is a meta-signature over Σ.

Soundness

A logic system $\mathcal{L}$ is said to be *sound* if $\Gamma \vDash_{\mathcal{I}} \varphi$ whenever $\Gamma \vdash_{\mathcal{D}} \varphi$, where φ is a formula and Γ is a set of formulas of G^+.

An interpretation structure I in $\mathfrak{I}$ is said to be *sound for a deductive rule* r in $\mathcal{D}$, if $I, \rho \Vdash \mathrm{CONC}(r)$ whenever $I, \rho \Vdash \mathrm{proper}(\mathrm{ANT}(r))$ for every assignment ρ over I.

A logic system $\mathcal{L}$ is said to be *sound for a deductive rule* r in $\mathcal{D}$, if all its interpretation structures over its signature are sound for r.

Soundness

Theorem A logic system is sound if it is sound for its deductive rules.

Completeness

The *canonical interpretation structure* $S^{\Gamma}(\mathcal{D}) = (\Sigma, (G', \alpha, D, \blacklozenge))$ generated by $\mathcal{D}$ and Γ, is such that:

- $G' = (V', E', \mathrm{src}', \mathrm{trg}')$ where
  - V' are the morphisms of G^+ whose target is an element of V
  - $E'(\widehat{w}_1 \ldots \widehat{w}_n, \widehat{w})$ is composed by all the m-edges e of E such that $\widehat{w} = \widehat{e} \circ \langle \widehat{w}_1, \ldots, \widehat{w}_n \rangle$ in G^+
- $\alpha^{\mathsf{v}}(\widehat{w} : s \to v) = v$ and $\alpha^{\mathsf{e}}(e) = e$
- $D = \{\widehat{w} \in V' : \Gamma \vdash_{\mathcal{D}} \widehat{w}\}$
- $\blacklozenge$ is the morphism $\mathrm{id}_{\lozenge}$ in G^+.

Completeness

Proposition For every deductive rule r in $\mathcal{D}$, set of formulas Γ, and assignment ρ over $S^{\Gamma}(\mathcal{D})$, then $S^{\Gamma}(\mathcal{D}), \rho \Vdash \mathsf{CONC}(r)$ whenever $S^{\Gamma}(\mathcal{D}), \rho \Vdash \mathsf{proper}(\mathsf{ANT}(r))$.

Completeness

In order for completeness to hold in a logic system it is not necessary to impose as sufficient condition that its interpretation system contains canonical structures.

It is enough to guarantee that its interpretation system contains structures that share with canonical structures some characteristics.

Completeness

A logic system contains *a representative of the canonical structure over a set* Γ when it contains an interpretation structure I_Γ such that

- $I_\Gamma \Vdash \varphi$ implies $S^\Gamma(\mathcal{D}) \Vdash \varphi$;
- $I_\Gamma \Vdash \Gamma$;

for every formula φ and set of formulas Γ in G^+.

Completeness

Theorem

A logic system with representatives of the canonical structures over all sets of formulas is complete.

Completeness

Theorem

A logic system is weakly complete if it contains a representative of the canonical structure over the empty set.

Corollary

A logic system is (weakly) complete whenever it contains all the interpretation structures that are sound with respect to the rules.

Completeness

Some logics to which our completeness results apply:

- classical propositional logic;
- classical propositional modal logic **T**;
- intuitionistic propositional logic;
- relevance logic **R**;
- **mbC** paraconsistent logic;
- one-sorted equational logic;
- ...

Conclusions and future work

- Graph-theoretic account of logics with provisos and quantification
- Graph-theoretic account of deductive systems like sequents and labelled deduction
- Study in this context other preservation results like cut elimination, interpolation, quantifier elimination, decidability
- Graph-theoretic account of fibring of protoalgebraic and weakly algebraizable logics
- Extension of fusion to the graph-theoretic setting

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Thank you!!