Categories with involution-rigid monomorphisms

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In this talk we introduce a new exactness property, which can be seen as a natural strengthening of the well-known Mal'tsev property, and which may be interesting for study in categorical and in universal algebra.

Call a substructure M of a mathematical structure X *rigid* under an involution i of X (by an involution we mean an endomorphism $i : X \to X$ such that $i^2 = 1_X$), when the image of M under i is contained in any extension of M which contains all the fixed points of i. This condition can also be expressed internally in any category $\mathbb{C}$, giving rise to a notion of rigidness of a morphism $m : M \to X$ with respect to an involution $i : X \to X$ in $\mathbb{C}$. We are then led to a new exactness property of a category asserting that any morphism $m : M \to X$ is rigid under any involution $i : X \to X$, for any object X in the category. Any commutative monoid, regarded as a single-object category, has this exactness property, as does any groupoid and any additive category. A finitely complete category having this exactness property is a Mal'tsev category, but the converse is not true. In fact, already the category of groups doe s not have this property, although the category of rings does, as do all varieties of abelian groups with operations. We give a characterization of varieties of universal algebras having the involution-rigidness property via term identities which are in some sense complementary to those that characterize protomodular varieties.

A variety of universal algebras has this exactness property if and only if its algebraic theory contains binary terms $a_1, \ldots, a_n$ and an $(n+1)$-ary term d satisfying $d(a_1(x,y), \ldots, a_n(x,y), y) = x$, and $a_j(x,y) = a_j(y,x)$ for each $j \in \{1, \ldots, n\}$.

(Joint work with Zurab Janelidze.)