From pretorsion to torsion theory for preorders

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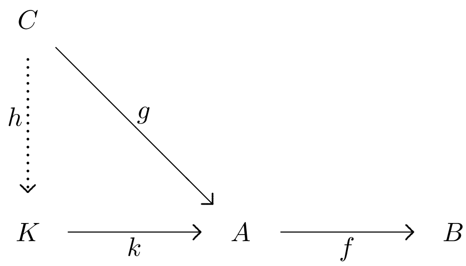
In a category $\mathcal{C}$ provided with an ideal $\mathcal{Z}$ of morphisms, $k\colon K \longrightarrow A$ is the $\mathcal{Z}$-kernel of $f\colon A \longrightarrow B$, when $fk \in \mathcal{Z}$ and if $fg \in \mathcal{Z}$, there exists a unique h such that $kh = g$.



When the category $\mathcal{C}$ has a zero object and $\mathcal{Z}$ is the ideal of zero morphisms, we recapture the classical notion of kernel. Of course there is a dual notion of $\mathcal{Z}$-cokernel as well as a corresponding notion of short $\mathcal{Z}$-exact sequence: a pair fk of composable morphisms such that k is the $\mathcal{Z}$-kernel of f and f is the $\mathcal{Z}$-cokernel of k.

The notion of torsion theory can be adapted in the same spirit, in the absence of a zero object. A pretorsion theory on a category $\mathcal{C}$ consists in giving two full sub-categories $(\mathcal{T}, \mathcal{F})$, whose objects are respectively called the "torsion" and "torsion free" objects. The objects in $\mathcal{T} \cap \mathcal{F}$ are called "trivial" (they are the zero objects in the case of a torsion theory) and the ideal $\mathcal{Z}$ of trivial morphisms is that of those morphisms factoring through a trivial object. The two pretorsion axioms are then, as expected

1. every morphism $T \longrightarrow F$, with $T \in \mathcal{T}$ and $F \in \mathcal{F}$, is trivial;
2. for every object A, there exists a short $\mathcal{Z}$-exact sequence $T \longrightarrow A \longrightarrow F$ with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

In the category of preordered sets, the pair $(\mathsf{Eq}, \mathsf{ParOrd})$ of equivalence relations and partial orders constitutes such a pretorsion theory. The trivial objects are the discrete ones. The corresponding "stable" category should be the universal solution to the problem of turning this situation into a pointed category provided with an actual torsion theory. The study of this pretorsion theory, together with the construction of a possible stable category, in the case of non-empty preordered sets, is due to Facchini and Finocchiaro.

We propose an alternative approach and prove its universal property, in the more general case of the category of preordered objects in a pretopos, without any restriction on the nature of the objects. These results are part of a joint work with Marino Gran and Federico Campanini.

Francis Borceux, Federico Campanini, Marino Gran, *The stable category of preorders in a pretopos I: General tTheory*, Journal of Pure and Applied Algebra, 106997, 2021
https://arxiv.org/abs/2201.05992

Francis Borceux, Federico Campanini, Marino Gran, *The stable category of preorders in a pretopos II: the universal property*, submitted for publication
https://arxiv.org/abs/2201.08016