Lawvere completeness in Topology

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Abstract

It is known since 1973 that Lawvere's notion of (Cauchy-)complete enriched category is meaningful for metric spaces: it captures exactly Cauchy-complete metric spaces. In this paper we introduce the corresponding notion of Lawvere completeness for $(\mathbb{T},\mathsf{V})$-categories and show that it has an interesting meaning for topological spaces and quasi-uniform spaces: for the former ones means weak sobriety while for the latter means Cauchy completeness. Further, we show that V has a canonical $(\mathbb{T},\mathsf{V})$-category structure which plays a key role: it is Lawvere-complete under reasonable conditions on the setting; permits us to define a Yoneda embedding in the realm of $(\mathbb{T},\mathsf{V})$-categories.

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0 Introduction

Lawvere in his 1973 paper *Metric spaces, generalized logic, and closed categories* formulates a notion of *complete* V-category and shows that for (generalised) metric spaces it means Cauchy completeness. This notion of completeness deserved the attention of the categorical community, and the notion of *Cauchy-complete category*, or *Freyd-Karoubi complete category* is well-known, mostly in the context of Ab-enriched categories. However, it never got the attention of the topological community. In this paper we interpret Lawvere's completeness in topological settings. We extend Lawvere's notion of complete V-category to the (topological) setting of $(\mathbb{T},\mathsf{V})$-categories (for a symmetric and unital quantale V), and show that it encompasses well-known notions in topological categories, meaning *weakly sober space* in the category of topological spaces and continuous maps, *weakly sober approach space* in the category of approach spaces and non-expansive maps, and *Cauchy-completeness* in the category of quasi-uniform spaces and uniformly continuous maps.

We present also a first step towards a possible construction of completion. Indeed, in the setting of V-categories, it is well-known that the completion of a V-category may be built out of the Yoneda embedding $X \to \mathsf{V}^{X^{\mathrm{op}}}$. In the $(\mathbb{T},\mathsf{V})$-setting, we could prove that V has a canonical $(\mathbb{T},\mathsf{V})$-categorical structure and that every $(\mathbb{T},\mathsf{V})$-category X has a canonical dual

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X^{op}. Using this structure and the free Eilenberg-Moore algebra structure $|X|$ on TX, we get two "Yoneda-like" morphisms

$$X \to \mathsf{V}^{X^{\mathrm{op}}} \ \text{ and } \ X \to \mathsf{V}^{|X|}.$$

For the latter one we prove a Yoneda Lemma (see 4.2).

Furthermore, we show that, under suitable conditions, V is a Lawvere-complete $(\mathbb{T},\mathsf{V})$-category, a first step towards a completion construction which will be the subject of a forthcoming paper.

In order to make the presentation of this paper smoother, in Section 1 we recall the notions and properties of V-categories we will generalize throughout. First we introduce V-categories and V-bimodules, and define Lawvere-complete V-categories, for a commutative and unital quantale V. V is then naturally equipped with the V-categorical structure hom. We give a direct proof of Lawvere completeness of the V-category $(V, \hom)$.

In Section 2 we describe our basic setting for the study of $(\mathbb{T},\mathsf{V})$-categories and introduce them. We describe Kleisli composition in the category V-Mat of V-valued matrices and define $(\mathbb{T},\mathsf{V})$-bimodule. Although $(\mathbb{T},\mathsf{V})$-bimodules do not compose in general, one can still formulate and study the notion of Lawvere-complete $(\mathbb{T},\mathsf{V})$-category.

Similarly to what was done in V-categories, we define a canonical $(\mathbb{T},\mathsf{V})$-categorical structure on V, as the composition of hom with the (canonical) $\mathbb{T}$-algebra structure on V described by Manes in [21]. This is the subject of Section 3. In addition we also prove that, under some conditions, the $(\mathbb{T},\mathsf{V})$-category V is Lawvere-complete.

In Section 1 we present the Yoneda embedding for V-categories as a subproduct of the fact that a V-matrix $\psi : X \longrightarrow\!\!\!\!\!\mapsto Y$ between V-categories (X,a) and (Y,b) is a V-bimodule if and only if, as a map $\psi : X^{\mathrm{op}} \otimes Y \to \mathsf{V}$, is a V-functor (Theorem 1.5); then the monoidal-closed structure of V-Cat gives us the *Yoneda Functor* $X \to \mathsf{V}^{X^{\mathrm{op}}}$. In the $(\mathbb{T},\mathsf{V})$-setting this construction becomes more elaborated (see Theorem 3.3): a V-matrix $\psi : TX \longrightarrow\!\!\!\!\!\mapsto Y$ is a $(\mathbb{T},\mathsf{V})$-bimodule $\psi : (X,a) \multimapdotboth (Y,b)$ if and only if both $\psi : |X| \otimes Y \to \mathsf{V}$ and $\psi : X^{\mathrm{op}} \otimes Y \to \mathsf{V}$ are $(\mathbb{T},\mathsf{V})$-functors. Thus, given a $(\mathbb{T},\mathsf{V})$-category (X,a), the $(\mathbb{T},\mathsf{V})$-bimodule $a : X \multimapdotboth X$ gives rise to two *Yoneda* $(\mathbb{T},\mathsf{V})$*-functors* $X \to \mathsf{V}^{X^{\mathrm{op}}}$ and $X \to \mathsf{V}^{|X|}$.

In Section 5 we present the announced topological examples, with the exception of quasi-uniform spaces, which are presented in the Appendix, due to the fact that their presentation as lax algebras does not fit in the $(\mathbb{T},\mathsf{V})$-setting (as shown in [20]).

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1 The category of V-categories

Although the material of this section can be found essentially on [17], we find that its inclusion here may enlighten the corresponding – but more technical – notions and results for $(\mathbb{T},\mathsf{V})$-

categories presented in the forthcoming sections.

1.1 V. Throughout V is a (commutative and unital) *quantale*. In other words, V is a complete lattice equipped with a symmetric and associative tensor product $\otimes$, with unit k, and with right adjoint hom; that is, for each $u, v, w \in \mathsf{V}$,

$$u \otimes v \leq w \iff v \leq \hom(u, w).$$

Considered as a (thin) category, V is said to be *symmetric monoidal-closed*. If k is the bottom element $\perp$ of V, then $\mathsf{V} = 1$ is the trivial lattice. Throughout this paper *we assume that* V *is non-trivial, i.e.* $k \neq \perp$.

Every non-trivial Heyting algebra – with $\otimes = \wedge$ and $k = \top$ the top element – is an example of such a lattice, in particular the two-element chain $2 = \{\text{false} \models \text{true}\}$, with the monoidal structure given by "&" (and) and "true". The complete real half-line $\mathsf{P} = [0, \infty]$, with the categorical structure induced by the relation $\geq$ (i.e., $a \to b$ means $a \geq b$), admits several interesting monoidal structures. First of all, with $\wedge = \max$ it is a Heyting algebra $\mathsf{P}_{\max}$. Another possible choice for $\otimes$ is $+$; we denote P equipped with this tensor by P_+. Note that in this example the right adjoint hom is given by truncated minus: $\hom(u, v) = \max\{v - u, 0\}$.

1.2 V**-**Mat. The category V-Mat of V-*matrices* [3, 10] has sets as objects, and a morphism $r : X \to\!\!\!\!\!\!\mapsto Y$ in V-Mat is a map $r : X \times Y \to \mathsf{V}$. Composition of V-matrices $r : X \to\!\!\!\!\!\!\mapsto Y$ and $s : Y \to\!\!\!\!\!\!\mapsto Z$ is defined as matrix multiplication:

$$s \cdot r(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z).$$

The identity arrow $1_X : X \to\!\!\!\!\!\!\mapsto X$ in V-Mat is the V-matrix which sends all diagonal elements (x, x) to k and all other elements to the bottom element $\perp$ of V. In fact, each Set-map $f : X \to Y$ can be interpreted as the V-matrix

$$f : X \to\!\!\!\!\!\!\mapsto Y,\ f(x, y) = \begin{cases} k & \text{if } f(x) = y, \\ \perp & \text{else.} \end{cases}$$

To keep notation simple, in the sequel we will write $f : X \to Y$ rather then $f : X \to\!\!\!\!\!\!\mapsto Y$ for a V-matrix induced by a map. The formula for matrix composition becomes considerably easier if one of the V-matrices is a Set-map:

$$s \cdot f(x, z) = s(f(x), z), \qquad\qquad g \cdot r(x, z) = \bigvee_{y \in g^{-1}(z)} r(x, y)$$

for maps $f : X \to Y$ and $g : Y \to Z$ and V-matrices $r : X \to\!\!\!\!\!\!\mapsto Y$ and $s : Y \to\!\!\!\!\!\!\mapsto Z$.

The complete order on V induces a complete order on $\mathsf{V}\text{-}\mathsf{Mat}(X, Y) = \mathsf{V}^{X \times Y}$: for V-matrices $r, r' : X \to\!\!\!\!\!\!\mapsto Y$ we define

$$r \leq r' \ :\iff\ \forall x \in X\ \forall y \in Y\ r(x, y) \leq r'(x, y).$$

The *transpose* $r^\circ : Y \to\!\!\!\!\!\!\mapsto X$ of a V-matrix $r : X \to\!\!\!\!\!\!\mapsto Y$ is defined by $r^\circ(y, x) = r(x, y)$. It is easy to see that $(\,)^\circ : V\text{-}\mathsf{Mat}(X, Y) \to V\text{-}\mathsf{Mat}(Y, X)$ is order-preserving, and

$$1_X^\circ = 1_X, \qquad\qquad (s \cdot r)^\circ = r^\circ \cdot s^\circ, \qquad\qquad r^{\circ\circ} = r.$$

For each Set-map $f : X \to Y$ we have $1_X \leq f^\circ \cdot f$ and $f \cdot f^\circ \leq 1_Y$, i.e. f is left adjoint to f° and we write $f \dashv f^\circ$. In general, given V-matrices $r : X {\longrightarrow\!\!\!\!\!\!\mapsto}\, Y$ and $s : Y {\longrightarrow\!\!\!\!\!\!\mapsto}\, X$, we say that r is left adjoint to s (and that s is right adjoint to r) if $1_X \leq s \cdot r$ and $1_Y \geq r \cdot s$.

Lemma. *Let* V *be a quantale and* $r, r' : X {\longrightarrow\!\!\!\!\!\!\mapsto}\, Y$ *and* $s, s' : Y {\longrightarrow\!\!\!\!\!\!\mapsto}\, X$ *be* V*-matrices such that* $r \dashv s$ *and* $r' \dashv s'$*. Then* $r \leq r'$ *if and only if* $s' \leq s$*. Consequently, if* $r \leq r'$ *and* $s \leq s'$*, then* $r = r'$ *and* $s = s'$*.*

As another consequence of the lemma above we have that left and right adjoints are uniquely determined by s (respectively r). Therefore we say that r is left adjoint if it has a right adjoint s, and likewise, s is right adjoint if it has a left adjoint r. In pointwise notation, we have $r \dashv s$ if and only if

$$\forall x \in X \bigvee_{y \in Y} r(x, y) \otimes s(y, x) \geq k,$$

$$\forall x \in X\ \forall y, y' \in Y\ s(y, x) \otimes r(x, y') \leq \begin{cases} \bot & \text{if } y \neq y', \\ k & \text{if } y = y' \end{cases}$$

which, by symmetry of $\otimes$, is equivalent to

$$\forall x \in X \bigvee_{y \in Y} r(x, y) \otimes s(y, x) = k,$$

$$\forall x \in X\ \forall y, y' \in Y\ (y \neq y' \ \Rightarrow\ s(y, x) \otimes r(x, y') = \bot).$$

Our next example shows that there exist indeed left adjoint V-matrices which are not induced by Set-maps.

Example. Consider a set X and the Boolean algebra $\mathsf{V} = PX$ the powerset of X. Define a V-matrix $r : 1 {\longrightarrow\!\!\!\!\!\!\mapsto}\, X$ by putting $r(\star, x) = \{x\}$ for $x \in X$. Then

$$r^\circ \cdot r(\star, \star) = \bigcup_{x \in X} \{x\} = X \qquad \text{and} \qquad r \cdot r^\circ(x, y) = \{x\} \cap \{y\} = \begin{cases} \emptyset & \text{if } x \neq y, \\ \{x\} & \text{if } x = y, \end{cases}$$

hence $r \dashv r^\circ$. But r is not a Set-map unless X has at most one element.

We wish to characterise those quantales V where the class of left adjoint V-matrices coincides with the class of Set-maps. In order to do so we introduce some notation. Let $u, v \in \mathsf{V}$. We say that v is a $\otimes$-*complement* of u if

$$u \vee v = k \qquad \text{and} \qquad u \otimes v = \bot.$$

Clearly, each $u \in \mathsf{V}$ has at most one $\otimes$-complement. Moreover, if u is $\otimes$-complemented (i.e. has a $\otimes$-complement v), then

$$u = u \otimes k = u \otimes (u \vee v) = (u \otimes u) \vee (u \otimes v) = u \otimes u,$$

that is, u is idempotent. Our next result generalises [14, 2.14].

Proposition. *Let* V *be a quantale. Each left adjoint* V*-matrix is a* Set*-map if and only if* k *and* $\perp$ *are the only* $\otimes$*-complemented elements of* V *and*

$$\forall u, v \in \mathsf{V}\ (u \otimes v = k \quad \Rightarrow \quad u = k = v).$$

Proof. Assume first that each left adjoint V-matrix is a Set-map. Let $u, v \in \mathsf{V}$. If $u \otimes v = k$, then $u \dashv v$, and we have $u = v = k$. Suppose that $u \vee v = k$ and $u \otimes v = \perp$. Let $X = \{u, v\}$ and define $r : 1 \longrightarrow\!\!\!\!\!\!\mapsto X$ with $r(\star, u) = u$ and $r(\star, v) = v$. Then $r \dashv r^\circ$ and, by assumption, $u = k$ or $v = k$.

Let $r : X \longrightarrow\!\!\!\!\!\!\mapsto Y$ and $s : Y \longrightarrow\!\!\!\!\!\!\mapsto X$ be V-matrices such that $r \dashv s$. Let $x \in X$. There is some $y \in Y$ such that $r(x, y) \otimes s(y, x) > \perp$[1]. Then

$$k = (r(x, y) \otimes s(y, x)) \vee \bigvee_{y' \neq y} (r(x, y') \otimes s(y', x))$$

and

$$r(x, y) \otimes s(y, x) \otimes \bigvee_{y' \neq y} r(x, y') \otimes s(y', x) = \bigvee_{y' \neq y} r(x, y) \otimes s(y, x) \otimes r(x, y') \otimes s(y', x) = \perp.$$

Hence, by assumption, $r(x, y) = k = s(y, x)$ and $r(x, y') \otimes s(y', x) = \perp$ for all $y' \neq y$. We have shown that, for each $x \in X$, there exists exactly one $y \in Y$ with $r(x, y) = k = s(y, x)$. Consider now $f : X \to Y$ which assigns to x this unique y. Clearly, $f \leq r$, but also $f^\circ \leq s$ since

$$f^\circ(y, x) = k \quad \Rightarrow \quad f(x) = y \quad \Rightarrow \quad s(y, x) = k.$$

The assertion follows now from the previous lemma. $\square$

1.3 V-categories. V-enriched categories were introduced and studied in [11, 17] in the more general context of symmetric monoidal-closed categories. For a very nice presentation of this material we refer to [18]. In the next subsections we recall some well-known facts about V-categories, which will serve as a guideline for our study of $(\mathbb{T}, \mathsf{V})$-categories.

A V-*enriched category* (or simply V-*category*) is a pair (X, a) with X a set and $a : X \longrightarrow\!\!\!\!\!\!\mapsto X$ a V-matrix such that

$$1_X \leq a \cdot a \quad \text{and } a \cdot a \leq a;$$

equivalently, the map $a : X \times X \to \mathsf{V}$ satisfies the following conditions:

(R) for each $x \in X$, $k \leq a(x, x)$;

(T) for each $x, x', x'' \in X$, $a(x, x') \otimes a(x', x'') \leq a(x, x'')$.

Given V-categories (X, a) and (Y, b), a V-*functor* $f : (X, a) \to (Y, b)$ is a map $f : X \to Y$ such that, for each $x, x' \in X$, $a(x, x') \leq b(f(x), f(x'))$. V-categories and V-functors are the objects and morphisms of the category $\mathsf{V}\text{-}\mathsf{Cat}$. Finally, given a V-category $X = (X, a)$, the *dual category* X^{op} of X is defined by $X^{\mathrm{op}} = (X, a^\circ)$.

We remark that $\mathsf{V}\text{-}\mathsf{Cat}$ is actually a *closed category* since the tensor product on V can be naturally transported to $\mathsf{V}\text{-}\mathsf{Cat}$. More precisely, for V-categories $X = (X, a)$ and $Y = (Y, b)$, we

[1]Since $\perp < k$. The assertion of the proposition is trivially true if $k = \perp$

put $X\otimes Y=(X\times Y,a\otimes b)$ where $a\otimes b((x,y),(x',y'))=a(x,x')\otimes b(y,y')$ for all $x,x'\in X$ and $y,y'\in Y$. Then, for each V-category $X=(X,a)$, the functor $X\otimes\_:\mathsf{V}\text{-}\mathsf{Cat}\to\mathsf{V}\text{-}\mathsf{Cat}$ has a right adjoint $\_^X$ defined by $Y^X=(\mathsf{V}\text{-}\mathsf{Cat}(X,Y),d)$ with $d(f,g)=\bigwedge_{x\in X}a(f(x),g(x))$.

Being monoidal-closed, V has a natural structure as V-category:

$$\hom:\mathsf{V}\to\!\!\!\!\!\!\mapsto\mathsf{V}.$$

Indeed, for $u,v,w\in\mathsf{V}$,

$$k\otimes v=v\ \Rightarrow\ k\leq\hom(v,v),$$

$$u\otimes(\hom(u,v)\otimes\hom(v,w))\leq v\otimes\hom(v,w)\leq w\ \Rightarrow\ \hom(u,v)\otimes\hom(v,w)\leq\hom(u,w),$$

that is, $1_{\mathsf{V}}\leq\hom$ and $\hom\cdot\hom\leq\hom$.

For $\mathsf{V}=\mathsf{2}$, with the usual notation $x\leq x':\iff a(x,x')=\text{true}$, axioms (R) and (T) read as

$$\forall x\in X\ \text{true}\models x\leq x\qquad\text{and}\qquad\forall x,x',x''\in X\ x\leq x'\ \&\ x'\leq x''\models x\leq x'',$$

that is, $(X,\leq)$ is an ordered set[2]. A $\mathsf{2}$-functor is a map $f:(X,\leq)\to(Y,\leq)$ between ordered sets such that

$$\forall x,x'\in X\ x\leq x'\ \models\ f(x)\leq f(x');$$

that is, f is a monotone map. Hence $\mathsf{2}\text{-}\mathsf{Cat}$ is equivalent to the category Ord of ordered sets and monotone maps.

A P_+-category is a set X endowed with a map $a:X\times X\to\mathsf{P}_+$ such that

$$\forall x\in X\ 0\geq a(x,x)\qquad\text{and}\qquad\forall x,x',x''\in X\ a(x,x')+a(x',x'')\geq a(x,x'');$$

that is, $a:X\times X\to\mathsf{P}_+$ is a (generalised) metric on X. A P_+-functor is a map $f:(X,a)\to(Y,b)$ between metric spaces satisfying the following inequality:

$$\forall x,x'\in X\ a(x,x')\geq b(f(x),f(x')),$$

which means precisely that f is a non-expansive map. Therefore the category $\mathsf{P}_+\text{-}\mathsf{Cat}$ coincides with the category Met of metric spaces and non-expansive maps. (For more details, see [18, 10].)

For $\mathsf{V}=\mathsf{P}_{\max}$, the transitivity axiom (T) reads as

$$\max\{a(x,x'),a(x',x'')\}\geq a(x,x''),$$

hence the category $\mathsf{P}_{\max}\text{-}\mathsf{Cat}$ coincides with the category UMet of *(generalised) ultrametric spaces and non-expansive maps.*

1.4 V-bimodules. Given V-categories (X,a) and (Y,b), a *bimodule*[3] $\psi:(X,a)\to\!\!\!\!\!\circ\!\!\!\to(Y,b)$ is a V-matrix $\psi:X\to\!\!\!\!\!\!\mapsto Y$ such that $\psi\cdot a\leq\psi$ and $b\cdot\psi\leq\psi$; that is, for each $x,x'\in X$ and $y,y'\in Y$,

$$a(x,x')\otimes\psi(x',y)\leq\psi(x,y)\qquad\text{and}\qquad\psi(x,y')\otimes b(y',y)\leq\psi(x,y).$$

It is easy to verify that bimodules compose and that V-categorical structures are themselves bimodules. In fact, they are the identities for the composition of bimodules, that is, for any bimodule $\psi:(X,a)\to\!\!\!\!\!\circ\!\!\!\to(Y,b)$, $\psi\cdot a=\psi$ and $b\cdot\psi=\psi$. Therefore, V-categories and V-bimodules constitute a category, which we will denote by $\mathsf{V}\text{-}\mathsf{Mod}$. The category $\mathsf{V}\text{-}\mathsf{Mod}$ inherits the bicategorical structure of $\mathsf{V}\text{-}\mathsf{Mat}$ via the forgetful functor $\mathsf{V}\text{-}\mathsf{Mod}\to\mathsf{V}\text{-}\mathsf{Mat}$.

[2]Note that we do *not* require $\leq$ to be anti-symmetric.

[3]Also known as *profunctor* or *distributor* (see [2, 5, 26]).

1.5 V-functors as V-bimodules. Any V-functor $f:(X,a)\to(Y,b)$ defines a pair of matrices $f_*:(X,a){\longrightarrow\!\!\!\!\!\!\mapsto}(Y,b)$ and $f^*:(Y,b){\longrightarrow\!\!\!\!\!\!\mapsto}(X,a)$, with $f_*=b\cdot f$ and $f^*=f^\circ\cdot b$, that is $f_*(x,y)=b(f(x),y)$ and $f^*(y,x)=b(y,f(x))$, which are in fact bimodules: for every $x,x'\in X$ and $y,y'\in Y$,

$$a(x,x')\otimes f_*(x',y)=a(x,x')\otimes b(f(x'),y)\le b(f(x),f(x'))\otimes b(f(x'),y)\le b(f(x),y),$$
$$f_*(x,y')\otimes b(y'y)=b(f(x),y')\otimes b(y',y)\le b(f(x),y),$$

and similarly for f^*.

Moreover, the bimodules f_* and f^* form an adjunction, as we show next. We recall first that, given bimodules $\varphi:(X,a){\multimap\!\!\!\rightarrow}(Y,b)$ and $\psi:(Y,b){\multimap\!\!\!\rightarrow}(X,a)$, φ is left adjoint to ψ, $\varphi\dashv\psi$, if $1_{(X,a)}\le\psi\cdot\varphi$ and $\varphi\cdot\psi\le 1_{(Y,b)}$, i.e. $a\le\psi\cdot\varphi$ and $\varphi\cdot\psi\le b$. It is now straightforward to check that $f_*\dashv f^*$, since, for $x,x'\in X$ and $y,y'\in Y$, the inequality

$$a(x,x')\le\bigvee_{y\in Y}f_*(x,y)\otimes f^*(y,x')=\bigvee_{y\in Y}b(f(x),y)\otimes b(y,f(x'))=b(f(x),f(x'))$$

follows from V-functoriality of f, while

$$\bigvee_{x\in X}f^*(y,x)\otimes f_*(x,y')=\bigvee_{x\in X}b(y,f(x))\otimes b(f(x),y')\le b(y,y')$$

follows from the associativity axiom for V-categories. A quite different connection between functors and bimodules offers the following

Theorem. *For V-categories $X=(X,a)$ and $Y=(Y,b)$ and a V-matrix $\psi:X{\longrightarrow\!\!\!\!\!\!\mapsto}Y$, the following conditions are equivalent:*

(i) $\psi:X{\multimap\!\!\!\rightarrow}Y$ *is a bimodule;*

(ii) $\psi:X^{\mathrm{op}}\otimes Y\to\mathsf{V}$ *is a V-functor.*

Proof. (i) $\Rightarrow$ (ii): For $x,x'\in X$ and $y,y'\in Y$,

$$\begin{aligned}\psi(x,y)\otimes a^\circ(x,x')\otimes b(y,y') &= a(x',x)\otimes\psi(x,y)\otimes b(y,y')\\ &\le \psi(x',y)\otimes b(y,y')\\ &\le \psi(x',y'),\end{aligned}$$

hence

$$a^\circ(x,x')\otimes b(y,y')\le\hom(\psi(x,y),\psi(x',y')).$$

(ii) $\Rightarrow$ (i): For $x,x'\in X$ and $y,y'\in Y$,

$$\begin{aligned}a(x,x')\otimes\psi(x',y) &\le \psi(x',y)\otimes a^\circ(x',x)\otimes b(y,y)\\ &\le \psi(x,y),\end{aligned}$$

that is $a\cdot\psi\le\psi$, and

$$\begin{aligned}\psi(x,y')\otimes b(y',y) &\le \psi(x,y')\otimes a^\circ(x,x)\otimes b(y',y)\\ &\le \psi(x,y),\end{aligned}$$

that is $\psi\cdot b\le\psi$. $\square$

Corollary. *There is a* V*-functor* $\ulcorner a \urcorner : X \to \mathsf{V}^{X^{\mathrm{op}}}$*. Moreover, for each* $x \in X$ *and* $f \in \mathsf{V}^{X^{\mathrm{op}}}$*, we have*

$$d(a(-,x),f) = f(x).$$

Proof. Note that $d(a(-,x),f) = \bigwedge_y \hom(a(y,x),f(y)) \leq f(x)$. On the other hand, for each $y \in Y$,

$$\begin{aligned} a(y,x) \leq \hom(f(x),f(y)) &\iff f(x) \otimes a(y,x) \leq f(y) \\ &\iff f(x) \leq \hom(a(y,x),f(y)). \end{aligned}$$

□

1.6 Lawvere-complete V-categories.

Definition. A V-category (X,a) is said to be *Lawvere-complete* if, for any V-category (Y,b), for every pair of adjoint bimodules

$$\begin{tikzcd} Y \arrow[loop left, "b"] \arrow[r, bend left, "\varphi"] \arrow[r, phantom, "\perp"] & X \arrow[l, bend left, "\psi"] \arrow[loop right, "a"] \end{tikzcd}$$

φ is in the image of $(\ )_* : \mathsf{V}\text{-}\mathsf{Cat} \to \mathsf{V}\text{-}\mathsf{Mod}$, i.e. there exists a V-functor $f : (X,a) \to (Y,b)$ such that $f_* = \varphi$ and $f^* = \psi$.

It is interesting to notice that, in order to check Lawvere completeness, we can restrict to the case (Y,b) is the V-category $(1,p)$, where $1 = \{\star\}$ is a singleton and $p(\star,\star) = k$.

Proposition. *For a* V*-category* (X,a)*, the following conditions are equivalent:*

(i) (X,a) *is Lawvere-complete;*

(ii) *for each pair of adjoint bimodules* $(\varphi : (1,p) \multimap (X,a)) \dashv (\psi : (X,a) \multimap (1,p))$*, there exists a* V*-functor* $f : (1,p) \to (X,a)$ *such that* $\varphi = f_*$ *and* $\psi = f^*$ *(in this situation we say that* $f(\star)$ *represents the adjunction* $\varphi \dashv \psi$*).*

Proof. It is a special case of Proposition 2.7. We omit the proof here because it follows, step by step, the proof of Proposition 2.7. □

Theorem. *The* V*-category* $(\mathsf{V},\hom)$ *is Lawvere-complete.*

Proof. Although this fact can be deduced from more general categorical results, we prefer to give here a direct proof, which provides guidance for the more general corresponding result for the $(\mathbb{T},\mathsf{V})$-categorical structure of V we will study later.

Consider

$$\begin{tikzcd} 1 \arrow[loop left, "p"] \arrow[r, bend left, "\varphi"] \arrow[r, phantom, "\perp"] & V \arrow[l, bend left, "\psi"] \arrow[loop right, "\hom"] \end{tikzcd}$$

From the above theorem it follows that

(1) $$\begin{aligned} \varphi \text{ is a bimodule} &\iff \varphi : (\mathsf{V},\hom) \to (\mathsf{V},\hom) \text{ is a } \mathsf{V}\text{-functor} \\ &\iff \forall u,v \in \mathsf{V} \ \hom(u,v) \leq \hom(\varphi(u),\varphi(v)); \end{aligned}$$

(2) $$\psi \text{ is a bimodule} \iff \psi : (\mathsf{V}, \hom^{\mathrm{op}}) \to (\mathsf{V}, \hom) \text{ is a } \mathsf{V}\text{-functor}$$
$$\iff \forall u, v \in \mathsf{V}\ \hom(u,v) \le \hom(\psi(v), \psi(u));$$

the conditions for the adjunction read as:

(3) $$\varphi \dashv \psi \iff \forall u, v \in \mathsf{V} \quad \psi(u) \otimes \varphi(v) \le \hom(u,v) \quad \& \quad k \le \bigvee_{u \in V} \varphi(u) \otimes \psi(u)$$

We will show that the adjunction $\varphi \dashv \psi$ is represented by $\psi(k)$, i.e, $\varphi(v) = \hom(\psi(k), v)$ and $\psi(v) = \hom(v, \psi(k))$, for every $v \in \mathsf{V}$. First we notice that from (3) it follows that $\psi(k) \otimes \varphi(v) \le \hom(k, v) = v$, hence $\varphi(v) \le \hom(\psi(k), v)$. Now the proof consists of checking three equalities:

(1st) $\psi(k) = \bigvee_{u \in \mathsf{V}} \psi(u) \otimes u$:

It is immediate that $\psi(k) = \psi(k) \otimes k \le \bigvee_{u \in \mathsf{V}} \psi(u) \otimes u$, and, moreover, for every $v \in \mathsf{V}$,

$$\begin{aligned} \psi(u) \otimes u = \psi(u) \otimes \hom(k,u) &\le \psi(u) \otimes \hom(\psi(u), \psi(k)) && \text{(by (2))} \\ &\le \psi(k). \end{aligned}$$

(2nd) $\forall v \in \mathsf{V}\ \hom(v, \psi(k)) = \bigvee_{u \in \mathsf{V}} \hom(v,u) \otimes \psi(u)$:

To show "$\ge$" we just observe that

$$v \otimes (\hom(v,u) \otimes \psi(u)) \le u \otimes \psi(u) \le \psi(k);$$

for "$\le$", we have

$$\begin{aligned} \hom(v, \psi(k)) &\le \hom(v, \psi(k)) \otimes \bigvee_{u \in V} \varphi(u) \otimes \psi(u) && \text{(by (3))} \\ &= \bigvee_{u \in V} \hom(v, \psi(k)) \otimes \varphi(u) \otimes \psi(u) \\ &\le \bigvee_{u \in V} \hom(v, \psi(k)) \otimes \hom(\psi(k), u) \otimes \psi(u) && (\text{since } \varphi \le \hom(\psi(k), -)) \\ &\le \bigvee_{u \in V} \hom(v,u) \otimes \psi(u). \end{aligned}$$

(3rd) Since $\psi = \psi \cdot \hom$ we have $\forall v \in \mathsf{V}\ \psi(v) = \bigvee_{u \in V} \hom(v,u) \otimes \psi(u)$. $\square$

A new insight on Lawvere completeness for V-categories may be found in [25].

2 Basic properties of $(\mathbb{T}, \mathsf{V})$-categories

In the first part of this section we present the setting for the study of $(\mathbb{T}, \mathsf{V})$-categories, or (Eilenberg-Moore) lax algebras, that can be studied in more detail in [7, 10, 8].

2.1 $\mathbb{T}$ and its extension. Recall that a *monad* $\mathbb{T} = (T, e, m)$ on Set consists of a functor $T : \mathsf{Set} \to \mathsf{Set}$ together with natural transformations $e : \mathrm{Id}_{\mathsf{Set}} \to T$ (unit) and $m : TT \to T$ (multiplication) such that

$$m \cdot Tm = m \cdot m_T \qquad \text{and} \qquad m \cdot Te = 1_T = m \cdot e_T.$$

There are two *trivial monads* on Set, one sending all sets X to the terminal set 1, and the other with $T\varnothing = \varnothing$ and $TX = 1$ for $X \neq \varnothing$. Any other monad is called *non-trivial.*

By a *lax extension* of a Set-monad $\mathbb{T} = (T, e, m)$ to V-Mat we mean an extension of the endofunctor $T : \mathsf{Set} \to \mathsf{Set}$ to V-matrices acting on Set-maps as T and satisfying

(a) $(Ta)^\circ = T(a^\circ)$ (and we write Ta°),

(b) $Tb \cdot Ta \leq T(b \cdot a)$,

(c) $a \leq a' \;\Rightarrow\; Ta \leq Ta'$,

(d) $e_Y \cdot a \leq Ta \cdot e_X$,

(e) $m_Y \cdot T^2a \leq Ta \cdot m_X$,

for all $a, a' : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ and $b : Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z$ [4]. Note that we have automatically equality in (b) if $a = f$ is a Set-map. A Set-monad $\mathbb{T} = (T, e, m)$ admitting a lax extension to V-Mat is called V-*admissible.* Although $\mathbb{T}$ may have many lax extensions to V-Mat, in the sequel we usually have a fixed extension in mind when talking about a V-admissible monad. Trivially, the identity monad $\mathbb{1} = (\mathrm{Id}, 1, 1)$ on Set can be extended to the identity monad on V-Mat. In [1] M. Barr shows how to extend Set-monads to $\mathsf{Rel} = \mathsf{2}$-$\mathsf{Mat}$: first observe that each relation $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ can be written as $r = p \cdot q^\circ$ where $q : R \to X$ and $p : R \to Y$ are the projection maps, then put $Tr = Tp \cdot Tq^\circ$. All conditions above but the second one are satisfied, and this extension satisfies (b) if and only if the Set-functor T has (BC) (that is, sends pullbacks to weak pullbacks). In [8] we showed how to make the step from Rel to V-Mat, provided that in addition V is *constructively completely distributive (ccd)*[5]. Given a monad $\mathbb{T} = (T, e, m)$ and a V-matrix $a : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$, we define relations $a_v : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ $(v \in \mathsf{V})$ by $a_v(x, y) = \text{true} \iff a(x, y) \geq v$ and put, for $\mathfrak{x} \in TX$ and $\mathfrak{y} \in TY$,

$$Ta(\mathfrak{x}, \mathfrak{y}) = \bigvee\{v \in \mathsf{V} \mid Ta_v(\mathfrak{x}, \mathfrak{y}) = \text{true}\}.$$

Then the formula above defines an extension of $T : \mathsf{Set} \to \mathsf{Set}$ provided that either $k = \top$ or $T\varnothing = \varnothing$. Moreover, all five conditions above are satisfied. In addition we have

(f) $Tb \cdot Ta = T(b \cdot a)$ provided that $\otimes = \wedge$,

(g) $Tg \cdot Ta = T(g \cdot a)$,

for all V-matrices $a : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ and $b : Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z$ and all maps $g : Y \to Z$. In some occasions we will need that the (Set-based) natural transformation $m : TT \to T$ has (BC) (that is, each naturality square is a weak pullback); this guarantees that m is also a (strict) natural transformation for the extension of T to V-Mat described above.

[4] The conditions for our extension are stronger than Seal's in [24].

[5] Recall that a lattice Y is (ccd) if $\bigvee : 2^{Y^{\mathrm{op}}} \to Y$ has a left adjoint; for more details see [27].

2.2 $(\mathbb{T},\mathsf{V})$-categories. Let $\mathbb{T}=(T,e,m)$ be a V-admissible monad. A $(\mathbb{T},\mathsf{V})$-*category* is a pair (X,a) consisting of a set X and a V-matrix $a:TX \to\!\!\!\!\!\mapsto X$ such that:

$$1_X \leq a\cdot e_X \quad \text{and } a\cdot Ta\leq a\cdot m_X;$$

that is, the map $a:TX\times X\to\mathsf{V}$ satisfies the conditions:

(R) for each $x\in X$, $\;k\leq a(e_X(x),x)$;

(T) for each $\mathfrak{X}\in T^2X$, $\mathfrak{x}\in TX$, $x\in X$, $\;Ta(\mathfrak{X},\mathfrak{x})\otimes a(\mathfrak{x},x)\leq a(m_X(\mathfrak{X}),x)$.

Given $(\mathbb{T},\mathsf{V})$-categories (X,a) and (Y,b), a $(\mathbb{T},\mathsf{V})$-*functor* $f:(X,a)\to(Y,b)$ is a map $f:X\to Y$ such that, for each $\mathfrak{x}\in TX$ and $x\in X$, $a(\mathfrak{x},x)\leq b(Tf(\mathfrak{x}),f(x))$. $(\mathbb{T},\mathsf{V})$-categories and $(\mathbb{T},\mathsf{V})$-functors are the objects and morphisms of the category $(\mathbb{T},\mathsf{V})$-Cat.

Note that each Eilenberg-Moore algebra for $\mathbb{T}$ can be viewed as a $(\mathbb{T},\mathsf{V})$-category; in fact, we have an embedding

$$\mathsf{Set}^{\mathbb{T}}\hookrightarrow(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}.$$

In particular, for each set X we have the $(\mathbb{T},\mathsf{V})$-category (TX,m_X) which we denote by $|X|$.

Obviously, each V-category is a $(\mathbb{T},\mathsf{V})$-category for $\mathbb{T}=\mathbb{1}$ the identity monad "identically" extended to V-Mat. A further class of interesting examples involves the ultrafilter monad $\mathbb{U}=(U,e,m)$. The extension of $U:\mathsf{Set}\to\mathsf{Set}$ to V-Mat of 2.1 can be equivalently described by

$$Ur(\mathfrak{x},\mathfrak{y})=\bigwedge_{(A\in\mathfrak{x},B\in\mathfrak{y})}\bigvee_{(x\in A,y\in B)} r(x,y),$$

for all $r:X\to\!\!\!\!\!\mapsto Y$ in V-Mat, $\mathfrak{x}\in TX$ and $\mathfrak{y}\in TY$. The main result of [1] states that $(\mathbb{U},\mathsf{2})$-$\mathsf{Cat}\cong\mathsf{Top}$. In [7] it is shown that $(\mathbb{U},\mathsf{P}_{+})$-$\mathsf{Cat}\cong\mathsf{App}$, the category of approach spaces and non-expansive maps (see [19] for details.)

2.3 The dual $(\mathbb{T},\mathsf{V})$-category. We have the canonical forgetful functor

$$\begin{aligned}E:(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}&\to\mathsf{V}\text{-}\mathsf{Cat},\\ (X,a)&\mapsto(X,a\cdot e_X)\end{aligned}$$

with left adjoint

$$\begin{aligned}E^\circ:\mathsf{V}\text{-}\mathsf{Cat}&\to(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}.\\ (X,a)&\mapsto(X,e_X^\circ\cdot Ta)\end{aligned}$$

Furthermore, (the extension of) T induces an endofunctor

$$\begin{aligned}T:V\text{-}\mathsf{Cat}&\to V\text{-}\mathsf{Cat}.\\ (X,a)&\mapsto(TX,Ta)\end{aligned}$$

If m is a (strict) natural transformation, we can represent this functor as the composite

$$\begin{tikzcd} & (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat} \arrow[dr, "M^\circ"] & \\ \mathsf{V}\text{-}\mathsf{Cat} \arrow[ur, "E^\circ"] \arrow[rr, "T"'] & & \mathsf{V}\text{-}\mathsf{Cat}, \end{tikzcd}$$

where $M^\circ : (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat} \to \mathsf{V}\text{-}\mathsf{Cat}$ is given by $(X,a) \mapsto (TX, Ta \cdot m_X^\circ)$. In fact, given a V-category (X,a), we have

$$T(e_X^\circ \cdot Ta) \cdot m_X^\circ = Te_X^\circ \cdot T^2 a \cdot m_X^\circ = Te_X^\circ \cdot m_X^\circ \cdot Ta = Ta.$$

The functors M° and E° are the keys to define the *dual* $(\mathbb{T},\mathsf{V})$*-category* X^{op} of a $(\mathbb{T},\mathsf{V})$-category $X = (X,a)$: we put $X^{\mathrm{op}} = E^\circ(M^\circ(X)^{\mathrm{op}})$. We point out that if X is a V-category interpreted as a $(\mathbb{T},\mathsf{V})$-category, i.e. $X = (X, e_X^\circ \cdot Ta)$ for a given V-category structure $a : X \!-\!\!\!\!\!\to\!\!\!\!\!\!\!\to X$, then

$$X^{\mathrm{op}} = E^\circ(M^\circ(E^\circ(X,a))^{\mathrm{op}}) = E^\circ((TX,Ta)^{\mathrm{op}}),$$

that is, X^{op} is the dual – as a V-category – of $T(X,a)$.

Our Theorem 3.3 shows that this is indeed a reasonable definition.

Finally, for later use we record the following

Lemma. *Let (X,a) be a V-category and (X,α) be a $\mathbb{T}$-algebra. Then $(X, a\cdot\alpha)$ is a $(\mathbb{T},\mathsf{V})$-category if and only if $\alpha : (TX,Ta) \to (X,a)$ is a V-functor.*

Proof. First we remark that from $1_X \le a$ and $1_X = \alpha \cdot e_X$ it follows that $1_X \le (a\cdot\alpha)\cdot e_X$, that is $a \cdot \alpha$ always fulfils the reflexivity axiom. Now, if α is a V-functor, i.e. $\alpha \cdot Ta \le a \cdot \alpha$, then

$$(a\cdot\alpha)\cdot T(a\cdot\alpha) = a\cdot\alpha\cdot Ta\cdot T\alpha \le a\cdot a\cdot\alpha\cdot T\alpha \le (a\cdot\alpha)\cdot m_X.$$

Conversely, if $a\cdot\alpha$ is a $(\mathbb{T},\mathsf{V})$-categorical structure, then

$$\alpha\cdot Ta = \alpha\cdot Ta\cdot T\alpha\cdot Te_X \le a\cdot\alpha\cdot Ta\cdot T\alpha\cdot Te_X \le a\cdot\alpha\cdot m_X\cdot Te_X = a\cdot\alpha. \qquad \square$$

2.4 Kleisli composition. Many notions and techniques can be transported from $\mathsf{V}\text{-}\mathsf{Cat}$ to $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$ by formally replacing composition of V-matrices by *Kleisli composition* (see [15]) defined as

$$b * a := b\cdot Ta\cdot m_X^\circ,$$

$$\begin{tikzcd}
TX \arrow[dd, "a"] & TY \arrow[dd, "b"] & & TX \arrow[r, "m_X^\circ"] \arrow[ddr, "b*a"'] & TTX \arrow[d, "Ta"] \\
 & & \longmapsto & & TY \arrow[d, "b"] \\
Y & Z & & & Z
\end{tikzcd}$$

for all $a : TX \!-\!\!\!\!\!\to\!\!\!\!\!\!\!\to Y$ and $b : TY \!-\!\!\!\!\!\to\!\!\!\!\!\!\!\to Z$ in $\mathsf{V}\text{-}\mathsf{Mat}$. The matrix $e_X^\circ : TX \!-\!\!\!\!\!\to\!\!\!\!\!\!\!\to X$ acts as a lax identity for this composition, in the following sense:

$$a * e_X^\circ = a \qquad\qquad \text{and} \qquad\qquad e_X^\circ * b \ge b,$$

for $a : TX \!-\!\!\!\!\!\to\!\!\!\!\!\!\!\to Y$ and $b : TY \!-\!\!\!\!\!\to\!\!\!\!\!\!\!\to X$. Moreover,

$$c * (b * a) \le (c * b) * a$$

if $T : \mathsf{V}\text{-}\mathsf{Mat} \to \mathsf{V}\text{-}\mathsf{Mat}$ preserves composition, and

$$c * (b * a) \ge (c * b) * a$$

if $m : TT \to T$ is a (strict) natural transformation.

2.5 $(\mathbb{T},\mathsf{V})$**-bimodules.** Given $(\mathbb{T},\mathsf{V})$-categories (X,a) and (Y,b), a $(\mathbb{T},\mathsf{V})$-*bimodule* (or simply a *bimodule*) $\psi:(X,a)\,{-\!\circ\!\!\to}\,(Y,b)$ is a V-matrix $\psi:TX\,{-\!\!\!\!\!\mapsto}\,Y$ such that $\psi*a\le\psi$ and $b*\psi\le\psi$. This means that $\psi\cdot Ta\cdot m_X^\circ\le\psi$ and $b\cdot T\psi\cdot m_X^\circ\le\psi$; that is, for $\mathfrak{X}\in T^2X$, $\mathfrak{x}\in TX$, $\mathfrak{y}\in TY$ and $y\in Y$,

$$Ta(\mathfrak{X},\mathfrak{x})\otimes\psi(\mathfrak{x},y)\le\psi(m_X(\mathfrak{X}),y),$$

$$T\psi(\mathfrak{X},\mathfrak{y})\otimes b(\mathfrak{y},y)\le\psi(m_X(\mathfrak{X}),y).$$

Whenever the Kleisli composition is associative (in particular if $T:\mathsf{V}\text{-}\mathsf{Mat}\to\mathsf{V}\text{-}\mathsf{Mat}$ is a functor and m is a natural transformation: see [15]), bimodules compose. The identities for the composition law are again the $(\mathbb{T},\mathsf{V})$-categorical structures, and we can consider the category $(\mathbb{T},\mathsf{V})$-Mod of $(\mathbb{T},\mathsf{V})$-categories and $(\mathbb{T},\mathsf{V})$-bimodules.

2.6 $(\mathbb{T},\mathsf{V})$**-functors as** $(\mathbb{T},\mathsf{V})$**-bimodules.** Analogously to the situation in V-categories, each $(\mathbb{T},\mathsf{V})$-functor $f:(X,a)\to(Y,b)$ defines a pair of bimodules $f_*:(X,a)\,{-\!\circ\!\!\to}\,(Y,b)$ and $f^*:(Y,b)\,{-\!\circ\!\!\to}\,(X,a)$ as indicated in the following diagram

$$\begin{tikzcd} TX \arrow[r, "Tf"] \arrow[d, "a"'] \arrow[dr, bend left=80, "f_*"] & TY \arrow[d, "b"] \arrow[dl, bend right=80, "f^*"'] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

that is, $f_*:=b\cdot Tf$ and $f^*:=f^\circ\cdot b$. In fact, the following assertions are equivalent for $(\mathbb{T},\mathsf{V})$-categories (X,a) and (Y,b) and a function $f:X\to Y$.

(i) $f:(X,a)\to(Y,b)$ is a $(\mathbb{T},\mathsf{V})$-functor.

(ii) $f_*:(X,a)\,{-\!\circ\!\!\to}\,(Y,b)$ is a $(\mathbb{T},\mathsf{V})$-bimodule.

(iii) $f^*:(Y,b)\,{-\!\circ\!\!\to}\,(X,a)$ is a $(\mathbb{T},\mathsf{V})$-bimodule.

We point out that, although in general bimodules do not compose, if $f:(X,a)\to(Y,b)$ is a functor, then, for any bimodules $\varphi:(Y,b)\,{-\!\circ\!\!\to}\,(Z,c)$ and $\psi:(Z,c)\,{-\!\circ\!\!\to}\,(Y,b)$,

$$\begin{tikzcd} X \arrow[loop left, "a"] \arrow[r, bend left, "f_*"] \arrow[r, "f" description] & Y \arrow[loop above, "b"] \arrow[l, bend left, "f^*"] \arrow[r, bend left, "\varphi"] & Z \arrow[loop right, "c"] \arrow[l, bend left, "\psi"] \end{tikzcd}$$

$\varphi*f_*$ and $f^**\psi$ are bimodules, as we show next. First note that

$$\varphi*f_*=\varphi\cdot Tf\qquad\text{and}\qquad f^**\psi=f^\circ\cdot\psi.$$

The latter equality follows from

$$f^**\psi=f^\circ\cdot b\cdot T\psi\cdot m_Z^\circ=f^\circ\cdot\psi,$$

and V-functoriality of f implies

$$\begin{aligned}
\varphi * f_* = \varphi * (b \cdot Tf) = \varphi \cdot T(b \cdot Tf) \cdot m_X^\circ & \\
\geq \varphi \cdot T(f \cdot a) \cdot m_X^\circ & \qquad \text{(by functoriality of } f\text{)}\\
\geq \varphi \cdot Tf \cdot Ta \cdot m_X^\circ & \\
\geq \varphi \cdot Tf \cdot Ta \cdot Te_X & \qquad \text{(since } m_X^\circ \geq Te_X\text{)}\\
\geq \varphi \cdot Tf, & \qquad \text{(since } a \cdot e_X \geq 1_X\text{)}
\end{aligned}$$

whereby φ bimodule gives us

$$\varphi * f_* = \varphi * (b \cdot Tf) = \varphi \cdot Tb \cdot T^2 f \cdot m_X^\circ \leq \varphi \cdot Tb \cdot m_Y^\circ \cdot Tf = \varphi \cdot Tf.$$

The bimodule properties of $\varphi * f_*$ and $f^* * \psi$ follow now from

$$\begin{aligned}
& c * (\varphi * f_*) = c * (\varphi \cdot Tf) \leq (c * \varphi) \cdot Tf = \varphi * f_*,\\
& (\varphi * f_*) * a = \varphi \cdot Tf \cdot Ta \cdot m_X^\circ \leq \varphi \cdot Tb \cdot T^2 f \cdot m_X^\circ \leq \varphi \cdot Tb \cdot m_Y^\circ \cdot Tf = \varphi * f_*,\\
& a * (f^* * \psi) = a \cdot T(f^\circ \cdot \psi) \cdot m_Z^\circ = a \cdot Tf^\circ \cdot T\psi \cdot m_Z^\circ \leq f^\circ \cdot b \cdot T\psi \cdot m_Z^\circ = f^\circ \cdot \psi = f^* * \psi,\\
& (f^* * \psi) * c = f^\circ \cdot \psi \cdot Tc \cdot m_Z^\circ = f^\circ \cdot (\psi * c) = f^\circ \cdot \psi = f^* * \psi.
\end{aligned}$$

Therefore we can define the "whiskering" functors

$$\begin{aligned}
- * f_* : (\mathbb{T}, \mathsf{V})\text{-}\mathsf{Mod}(Y, Z) & \longrightarrow (\mathbb{T}, \mathsf{V})\text{-}\mathsf{Mod}(X, Z), \text{ and}\\
\varphi & \longmapsto \varphi \cdot Tf\\[1ex]
f^* * - : (\mathbb{T}, \mathsf{V})\text{-}\mathsf{Mod}(Z, Y) & \longrightarrow (\mathbb{T}, \mathsf{V})\text{-}\mathsf{Mod}(Z, X)\\
\psi & \longmapsto f^\circ \cdot \psi.
\end{aligned}$$

Moreover, given a pair of adjoint bimodules $(\varphi : (Y, b) \!-\!\!\circ\!\!\rightarrow (Z, c)) \dashv (\psi : (Z, c) \!-\!\!\circ\!\!\rightarrow (Y, b))$, we have

$$\varphi * f_* \dashv f^* * \psi,$$

provided that the diagram

$$\begin{tikzcd}
T^2X \arrow[r, "m_X"] \arrow[d, "T^2 f"'] & TX \arrow[d, "Tf"] \\
T^2Y \arrow[r, "m_Y"] & TY
\end{tikzcd}$$

satisfies (BC): $(\varphi * f_*) * (f^* * \psi) \leq c$ is always true, since

$$(\varphi \cdot Tf) * (f^\circ \cdot \psi) = \varphi \cdot Tf \cdot Tf^\circ \cdot T\psi \cdot m_Z^\circ \leq \varphi \cdot T\psi \cdot m_Z^\circ = \varphi * \psi \leq c,$$

while to conclude that $a \leq (f^* * \psi) * (\varphi * f_*)$ we need the condition above:

$$a \leq f^\circ \cdot b \cdot Tf \leq f^\circ \cdot \psi \cdot T\varphi \cdot m_Y^\circ \cdot Tf = f^\circ \cdot \psi \cdot T\varphi \cdot T^2 f \cdot m_X^\circ = (f^\circ \cdot \psi) * (\varphi \cdot Tf).$$

2.7 Lawvere-complete $(\mathbb{T}, \mathsf{V})$-categories.

Definition. A $(\mathbb{T},\mathsf{V})$-category (X,a) is called *Lawvere-complete* if, for each $(\mathbb{T},\mathsf{V})$-category (Y,b) and each pair of adjoint bimodules

$$\begin{tikzcd} Y \arrow[loop left, "b"] \arrow[r, bend left, "\varphi"] \arrow[r, phantom, "\perp"] & X \arrow[l, bend left, "\psi"] \arrow[loop right, "a"] \end{tikzcd}$$

there exists a functor $f:(Y,b)\to(X,a)$ such that $f_*=\varphi$ and $f^*=\psi$.

Analogously to the V-categorical case, Lawvere completeness is fully tested by left adjoint bimodules with domain $(1,p)$, where $p=e_1^\circ$, hence $p(\overset{\bullet}{\star},\star)=k$ and $p(\mathfrak{x},\star)=\bot$ for $\mathfrak{x}\neq\overset{\bullet}{\star}$ in $T1$.

Proposition. *Assume that either $T1=1$ or that the (*Set*-based) natural transformation m satisfies* (BC)*. Then, for a $(\mathbb{T},\mathsf{V})$-category (X,a), the following conditions are equivalent:*

(i) (X,a) *is Lawvere-complete;*

(ii) *each pair of adjoint bimodules* $\begin{tikzcd} (1,p) \arrow[r, bend left, "\varphi"] \arrow[r, phantom, "\perp"] & (X,a) \arrow[l, bend left, "\psi"] \end{tikzcd}$ *is induced by a functor $f:(1,p)\to(X,a)$;*

(iii) *each pair of adjoint bimodules* $\begin{tikzcd} (1,p) \arrow[r, bend left, "\varphi"] \arrow[r, phantom, "\perp"] & (X,a) \arrow[l, bend left, "\psi"] \end{tikzcd}$ *is induced by a map $f:1\to X$ (so that $\varphi=a\cdot Tf$ and $\psi=f^\circ\cdot a$).*

Proof. (iii) $\Rightarrow$ (i): Let $\begin{tikzcd} (Y,b) \arrow[r, bend left, "\varphi"] \arrow[r, phantom, "\perp"] & (X,a) \arrow[l, bend left, "\psi"] \end{tikzcd}$ be a pair of adjoint bimodules. For each $y\in Y$, let $g_y:(1,p)\to(Y,b)$ be the functor that picks y. This functor induces a pair of adjoint bimodules $(g_y)_*\dashv(g_y)^*$, whence we have

$$\begin{tikzcd} 1 \arrow[loop left, "p"] \arrow[r, bend left, "(g_y)_*"] & Y \arrow[l, bend left, "(g_y)^*"] \arrow[loop above, "b"] \arrow[r, bend left, "\varphi"] \arrow[r, phantom, "\perp"] & X \arrow[l, bend left, "\psi"] \arrow[loop right, "a"] \end{tikzcd}$$

If m satisfies (BC), we know already that $\varphi_y\dashv\psi_y$, where $\varphi_y=\varphi*(g_y)_*=\varphi\cdot Tg_y$ and $\psi_y=(g_y)^**\psi=g_y^\circ\cdot\psi$. The same happens whenever $T1=1$, as it is easily checked. By hypothesis, there exists a map $f_y:1\to X$ such that $\varphi_y=b\cdot Tf_y$ and $\psi_y=f_y^\circ\cdot b$. Gluing together the maps $(f_y)_{y\in Y}$ we obtain a map $f:Y\to X$. Then, for $\mathfrak{x}\in TX$ and $y\in Y$,

$$\psi(\mathfrak{x},y)=\psi_y(\mathfrak{x},\star)=f_y^\circ\cdot a(\mathfrak{x},\star)=a(\mathfrak{x},f_y(\star))=a(\mathfrak{x},f(y)),$$

that is, $\psi=f^*=f^\circ\cdot a$. We can show now that f is necessarily a functor:

$$b\cdot Tf^\circ\le b\cdot Tf^\circ\cdot Ta\cdot Te_X\le b\cdot Tf^\circ\cdot Ta\cdot m_X^\circ=b*\psi\le\psi=f^\circ\cdot a.$$

This concludes the proof since, by unicity of adjoints, φ is necessarily f_*. $\square$

3 V as a $(\mathbb{T},\mathsf{V})$-category

3.1 The $\mathbb{T}$-algebra structure of V. Our next goal is to explore the notions introduced in the previous section. In particular we are aiming for results which extend known facts about V-categories (as Theorem 1.5 or Theorem 1.6). To do so, from now on *we will always assume that the extension $T : \mathsf{V}\text{-}\mathsf{Mat} \to \mathsf{V}\text{-}\mathsf{Mat}$ is constructed as in [8] and consequently we assume V to be constructively completely distributive. Furthermore, we assume that $\mathbb{T} = (T, e, m)$ is non-trivial and that T and m satisfy (BC).*

Under these conditions, as Manes essentially showed in [21],

$$\begin{aligned} \xi : T\mathsf{V} &\longrightarrow \mathsf{V} \\ \mathfrak{x} &\longmapsto \bigvee\{v \in \mathsf{V} \mid \mathfrak{x} \in T(\uparrow v)\} \end{aligned}$$

is a $\mathbb{T}$-algebra structure on V, where $\uparrow v = \{u \in \mathsf{V} \mid v \leq u\}$.

There is an interesting link between this $\mathbb{T}$-algebra structure and the image under the lax functor $T : \mathsf{V}\text{-}\mathsf{Mat} \to \mathsf{V}\text{-}\mathsf{Mat}$ of the identity $1_{\mathsf{V}} : \mathsf{V} \to \mathsf{V}$ considered as a matrix $i : 1 \relbar\joinrel\mapsto \mathsf{V}$, with $i(\star, v) = v$. Let us compute $Ti : T1 \relbar\joinrel\mapsto T\mathsf{V}$. We consider, for each $v \in \mathsf{V}$, the relation

$$\begin{aligned} i_v : 1 \times V &\longrightarrow 2 \\ (\star, u) &\longmapsto \begin{cases} \text{true} & \text{if } v \leq u, \\ \text{false} & \text{elsewhere,} \end{cases} \end{aligned}$$

hence the diagram

$$\begin{tikzcd} 1 \arrow[rr, "i_v", "|"] \arrow[dr, "q_v^\circ"'] & & V, \\ & \uparrow v \arrow[ur, "p_v"'] & \end{tikzcd}$$

commutes where p_v and q_v are the projections. Now, for each $\mathfrak{x} \in T1$ and $\mathfrak{y} \in T\mathsf{V}$,

$$\begin{aligned} Ti(\mathfrak{x},\mathfrak{y}) &= \bigvee\{v \in \mathsf{V} \mid T(i_v)(\mathfrak{x},\mathfrak{y}) = \text{true}\} \\ &= \bigvee\{v \in \mathsf{V} \mid Tp_v Tq_v^\circ(\mathfrak{x},\mathfrak{y}) = \text{true}\} \\ &= \bigvee\{v \in \mathsf{V} \mid \exists \mathfrak{z} \in \uparrow v \,:\, Tq_v(\mathfrak{z}) = \mathfrak{x} \text{ and } Tp_v(\mathfrak{z}) = \mathfrak{y}\}, \end{aligned}$$

hence, since T preserves injections and considering Tp_v as an inclusion, we can write

$$Ti(\mathfrak{x},\mathfrak{y}) = \bigvee\{v \in \mathsf{V} \mid \mathfrak{y} \in T(\uparrow v) \text{ and } Tq_v(\mathfrak{y}) = \mathfrak{x}\} \leq \xi(\mathfrak{y}),$$

by definition of ξ. In particular, if $\mathfrak{x} = Tq(\mathfrak{y})$, for $q : \mathsf{V} \to 1$, then $Ti(\mathfrak{x},\mathfrak{y}) = \xi(\mathfrak{y})$. Whenever $T1 = 1$, $Tq(\mathfrak{y}) = \overset{\bullet}{\star}$ for every $\mathfrak{y} \in T\mathsf{V}$, whence

$$Ti(\overset{\bullet}{\star}, \mathfrak{y}) = \bigvee\{v \in \mathsf{V} \mid \mathfrak{y} \in T(\uparrow v)\} = \xi(\mathfrak{y}).$$

This link between the extension of T and the $\mathbb{T}$-algebra structure ξ is more general. Whenever necessary, in the sequel we denote the Set-endofunctor T by T_o, and keep T for its extension to $\mathsf{V}\text{-}\mathsf{Mat}$. Each V-matrix $r : X \relbar\joinrel\mapsto Y$ can be considered also as a map $r : X \times Y \to \mathsf{V}$. The interplay between $T_\mathrm{o} r$ and Tr is stated in the following result, whose proof is straightforward.

Proposition. *For any* V*-matrix* $r : X \longrightarrow\!\!\!\!\!\!\mapsto Y$*, each* $\mathfrak{x} \in TX$ *and* $\mathfrak{y} \in TY$*,*

$$Tr(\mathfrak{x},\mathfrak{y}) = \bigvee_{\mathfrak{w}:\ \substack{T_\circ\pi_X(\mathfrak{w})=\mathfrak{x}\\ T_\circ\pi_Y(\mathfrak{w})=\mathfrak{y}}} \xi \cdot T_\circ r(\mathfrak{w}),$$

that is the following diagram

$$\begin{tikzcd} T(X\times Y) \arrow[r, "T_\circ r"] & T\mathsf{V} \arrow[d, "\xi"] \\ TX\times TY \arrow[u, "<T_\circ\pi_X{,}T_\circ\pi_Y>", "\circ"'] \arrow[r, "Tr"] & \mathsf{V} \end{tikzcd}$$

commutes.

Remark. Besides being the structure map of an Eilenberg-Moore algebra, $\xi : T\mathsf{V} \to \mathsf{V}$ satisfies also the inequalities

$$\otimes \cdot \langle \xi \cdot T_\circ\pi_1, \xi \cdot T_\circ\pi_2\rangle \le \xi \cdot T_\circ(\otimes) \qquad \text{and} \qquad k\cdot ! \le \xi \cdot T_\circ k.$$

$$\begin{tikzcd} T(\mathsf{V}\times\mathsf{V}) \arrow[r, "T_\circ(\otimes)"] \arrow[d, "\langle\xi\cdot T_\circ\pi_1{,}\xi\cdot T_\circ\pi_2\rangle"'] \arrow[dr, phantom, "\le"] & T\mathsf{V} \arrow[d, "\xi"] \\ \mathsf{V}\times\mathsf{V} \arrow[r, "\otimes"'] & \mathsf{V} \end{tikzcd} \qquad \begin{tikzcd} T1 \arrow[r, "T_\circ k"] \arrow[d, "!"'] \arrow[dr, phantom, "\le"] & T\mathsf{V} \arrow[d, "\xi"] \\ 1 \arrow[r, "k"'] & \mathsf{V} \end{tikzcd}$$

Recall that we assume $Tf = T_\circ f$ for each Set-map $f : X \to Y$; this condition requires and implies equality in the latter inequality (see [16]).

3.2 The canonical $(\mathbb{T},\mathsf{V})$-categorical structure of V. The composition of the natural V-categorical and $\mathbb{T}$-algebra structures of V defines an interesting structure, $\hom_\xi$, of a $(\mathbb{T},\mathsf{V})$-category on V

$$T\mathsf{V} \overset{\hom_\xi}{\longrightarrow\!\!\!\!\!\circ\;} \mathsf{V} = (T\mathsf{V} \overset{\xi}{\to} \mathsf{V} \overset{\hom}{\longrightarrow\!\!\!\!\!\circ\;} \mathsf{V}),$$

as we show next.

Proposition. $\xi : (T\mathsf{V}, T\hom) \to (\mathsf{V},\hom)$ *is a* V*-functor.*

Proof. We have to show that $\xi \cdot T\hom \le \hom\cdot\xi$, or, equivalently, $T\hom \le \xi^\circ \cdot \hom\cdot\xi$. This means that, for $\mathfrak{x},\mathfrak{y} \in T\mathsf{V}$,

$$T\hom(\mathfrak{x},\mathfrak{y}) \le \hom(\xi(\mathfrak{x}),\xi(\mathfrak{y})).$$

We consider again the matrix $i : 1 \longrightarrow\!\!\!\!\!\!\mapsto V$, and compute $1 \overset{i}{\to} \mathsf{V} \overset{\hom}{\longrightarrow\!\!\!\!\!\circ\;} \mathsf{V}$:

$$\hom\cdot i(\star, v) = \bigvee_{u\in V} i(\star,u)\otimes\hom(u,v) = \bigvee_{u\in\mathsf{V}} u\otimes\hom(u,v) \le v;$$

that is $\hom\cdot i \le i$. Hence $T\hom\cdot Ti \le T(\hom\cdot i) \le Ti$, and so, for $\mathfrak{x},\mathfrak{y} \in T\mathsf{V}$ and $\mathfrak{z} = Tq(\mathfrak{x})$ as in Section 3.1, we have

$$\xi(\mathfrak{x}) \otimes T\hom(\mathfrak{x},\mathfrak{y}) = Ti(\mathfrak{z},\mathfrak{x}) \otimes T\hom(\mathfrak{x},\mathfrak{y}) \le Ti(\mathfrak{z},\mathfrak{y}) \le \xi(\mathfrak{y}),$$

and therefore

$$T\hom(\mathfrak{x},\mathfrak{y}) \le \hom(\xi(\mathfrak{x}),\xi(\mathfrak{y}))$$

as claimed. $\square$

Corollary. $(\mathsf{V}, \hom_\xi)$ *is a* $(\mathbb{T}, \mathsf{V})$*-category.*

Proof. Follows from the proposition above and Lemma 2.2. $\square$

3.3 The tensor product. The tensor product in V defines in a natural way a (not necessarily closed) product structure in $(\mathbb{T}, \mathsf{V})$-Cat. Given $(\mathbb{T}, \mathsf{V})$-categories $X = (X, a)$ and $Y = (Y, b)$, we put $X \otimes Y = (X \times Y, a \otimes b)$ where $a \otimes b(\mathfrak{w}, (x, y)) = a(T\pi_X(\mathfrak{w}), x) \otimes b(T\pi_Y(\mathfrak{w}), y)$ for all $\mathfrak{w} \in T(X \times Y)$, $x \in X$ and $y \in Y$. One easily verifies reflexivity of $a \otimes b$, while transitivity holds if and only if $\otimes \cdot \langle \xi \cdot T_\circ \pi_1, \xi \cdot T_\circ \pi_2 \rangle = \xi \cdot T_\circ(\otimes)$ (see Remark 3.1 and [16]) *which we assume from now on.* We remark that this condition guarantees that $\mathbb{T}$ is a *(lax) Hopf monad* on V-Mat (see [22]) where the tensor product in V is naturally extended to V-Mat. However, we will not develop this aspect here.

It is well-known that in general the functor $X \otimes \_ : (\mathbb{T}, \mathsf{V})\text{-}\mathsf{Cat} \to (\mathbb{T}, \mathsf{V})\text{-}\mathsf{Cat}$ has no right adjoint as, for example, Top is not Cartesian closed. The problem of characterising those $(\mathbb{T}, \mathsf{V})$-categories $X = (X, a)$ such that tensoring with X has a right adjoint is studied in [16].

Theorem. *Let m be a natural transformation. For $(\mathbb{T}, \mathsf{V})$-categories (X, a) and (Y, b) and a V-matrix $\psi : TX \to\!\!\!\!\!\!\!\shortmid\;\; Y$, the following assertions are equivalent.*

(i) $\psi : (X, a) \circ\!\!\!\rightarrow (Y, b)$ *is a* $(\mathbb{T}, \mathsf{V})$*-bimodule.*

(ii) *Both* $\psi : |X| \otimes Y \to \mathsf{V}$ *and* $\psi : X^{\mathrm{op}} \otimes Y \to \mathsf{V}$ *are* $(\mathbb{T}, \mathsf{V})$*-functors.*

Proof. Assume that $\psi : (X, a) \circ\!\!\!\rightarrow (Y, b)$ is a $(\mathbb{T}, \mathsf{V})$-bimodule. First observe that, for $\mathfrak{W} \in T(TX \times Y)$,

$$\xi \cdot T_\circ \psi(\mathfrak{W}) \le T\psi(T_\circ \pi_{TX}(\mathfrak{W}), T_\circ \pi_Y(\mathfrak{W})).$$

Let $\mathfrak{W} \in T(TX \times Y)$, $\mathfrak{x} \in TX$ and $y \in Y$. To see that $\psi : |X| \otimes Y \to \mathsf{V}$ is a $(\mathbb{T}, \mathsf{V})$-functor, note that the structure c on $|X| \otimes Y$ is given by

$$c(\mathfrak{W}, (\mathfrak{x}, y)) = \begin{cases} \bot & \text{if } \mathfrak{x} \ne m_X(T_\circ \pi_{TX}(\mathfrak{W})), \\ b(T\pi_Y(\mathfrak{W}), y) & \text{if } \mathfrak{x} = m_X(T_\circ \pi_{TX}(\mathfrak{W})). \end{cases}$$

Assume $\mathfrak{x} = m_X(T_\circ \pi_{TX}(\mathfrak{W}))$. Since

$$b(T_\circ \pi_Y(\mathfrak{W}), y) \le \hom(\xi \cdot T_\circ \psi(\mathfrak{W}), \psi(\mathfrak{x}, y))$$

is equivalent to

$$\xi \cdot T_\circ \psi(\mathfrak{W}) \otimes b(T_\circ \pi_Y(\mathfrak{W}), y) \le \psi(\mathfrak{x}, y),$$

the assertion follows at once. We show now that $\psi : X^{\mathrm{op}} \otimes Y \to \mathsf{V}$ is a $(\mathbb{T}, \mathsf{V})$-functor. As above we have that (with $a^{\mathrm{op}} = e_{TX}^\circ \cdot Tm_X \cdot T^2 a^\circ$ the structure on X^{op})

$$a^{\mathrm{op}}(T_\circ \pi_{TX}(\mathfrak{W}), \mathfrak{x}) \otimes b(T_\circ \pi_Y(\mathfrak{W}), y) \le \hom(\xi \cdot T_\circ \psi(\mathfrak{W}), \psi(\mathfrak{x}, y))$$

is equivalent to

$$\xi \cdot T_\circ \psi(\mathfrak{W}) \otimes a^{\mathrm{op}}(T_\circ \pi_{TX}(\mathfrak{W}), \mathfrak{x}) \otimes b(T_\circ \pi_Y(\mathfrak{W}), y) \le \psi(\mathfrak{x}, y).$$

Now

$$\begin{aligned}
\xi\cdot T_{\mathrm{o}}\psi(\mathfrak{W})\otimes a^{\mathrm{op}}(T_{\mathrm{o}}\pi_{TX}(\mathfrak{W}),\mathfrak{x})\otimes b(T_{\mathrm{o}}\pi_Y(\mathfrak{W}),y)&\\
&\le T^2a\cdot Tm_X^\circ\cdot e_{TX}(\mathfrak{x},T_{\mathrm{o}}\pi_{TX}(\mathfrak{W}))\otimes T\psi(T_{\mathrm{o}}\pi_{TX}(\mathfrak{W}),T_{\mathrm{o}}\pi_Y(\mathfrak{W}))\otimes b(T_{\mathrm{o}}\pi_Y(\mathfrak{W}),y)\\
&\le b\cdot T\psi\cdot T^2a\cdot Tm_X^\circ\cdot m_X^\circ(\mathfrak{x},y)\\
&\le b\cdot T\psi\cdot m_X\cdot Ta\cdot m_X^\circ(\mathfrak{x},y)\\
&= \psi\cdot Ta\cdot m_X^\circ(\mathfrak{x},y)=\psi(\mathfrak{x},y).
\end{aligned}$$

Now assume that $\psi:|X|\otimes Y\to\mathsf{V}$ and $\psi:X^{\mathrm{op}}\otimes Y\to\mathsf{V}$ are $(\mathbb{T},\mathsf{V})$-functors. Functoriality of $\psi:|X|\otimes Y\to\mathsf{V}$ implies, for all $\mathfrak{x}\in TX$ and $y\in Y$,

$$\begin{aligned}
\psi(\mathfrak{x},y)&\ge\bigvee_{\substack{\mathfrak{X}\in TTX:\\ m_X(\mathfrak{X})=\mathfrak{x};\\ \mathfrak{y}\in TY}}\bigvee\left\{\xi\cdot T_{\mathrm{o}}\psi(\mathfrak{W})\otimes b(\mathfrak{y},y)\;\middle|\;\mathfrak{W}\in T(TX\times Y):\mathfrak{W}\mapsto\mathfrak{X},\mathfrak{W}\mapsto\mathfrak{y}\right\}\\
&=\bigvee_{\substack{\mathfrak{X}\in TTX:\\ m_X(\mathfrak{X})=\mathfrak{x};\\ \mathfrak{y}\in TY}} T\psi(\mathfrak{X},\mathfrak{y})\otimes b(\mathfrak{y},y)\\
&=\bigvee_{\substack{\mathfrak{X}\in TTX:\\ m_X(\mathfrak{X})=\mathfrak{x}}} b\cdot T\psi(\mathfrak{X},y)\\
&=b\cdot T\psi\cdot m_X^\circ(\mathfrak{x},y).
\end{aligned}$$

On the other hand, by functoriality of $\psi:X^{\mathrm{op}}\otimes Y\to\mathsf{V}$, for all $\mathfrak{x}\in TX$ and $y\in Y$ we have

$$\begin{aligned}
\psi(\mathfrak{x},y)\ge&\bigvee_{\substack{\mathfrak{X}\in TTX,\\ \mathfrak{y}\in TY}}\bigvee\left\{\xi\cdot T_{\mathrm{o}}\psi(\mathfrak{W})\otimes b(\mathfrak{y},y)\otimes a^{\mathrm{op}}(\mathfrak{X},\mathfrak{x})\;\middle|\;\mathfrak{W}\in T(TX\times Y):\mathfrak{W}\mapsto\mathfrak{X},\mathfrak{W}\mapsto\mathfrak{y}\right\}\\
&=\bigvee_{\substack{\mathfrak{X}\in TTX,\\ \mathfrak{y}\in TY}} T\psi(\mathfrak{X},\mathfrak{y})\otimes b(\mathfrak{y},y)\otimes T^2a\cdot Tm_X^\circ\cdot e_{TX}(\mathfrak{x},\mathfrak{X})\\
&=b\cdot T\psi\cdot T^2a\cdot Tm_X^\circ\cdot e_{TX}(\mathfrak{x},\mathfrak{X})\\
&\ge b\cdot e_Y\cdot\psi\cdot Ta\cdot m_X^\circ(\mathfrak{x},y)\\
&\ge\psi\cdot Ta\cdot m_X^\circ(\mathfrak{x},y).
\end{aligned}$$

□

3.4 V is Lawvere-complete.

Theorem. *Assume that $T1=1$. Then $(V,\hom_\xi)$ is a Lawvere-complete $(\mathbb{T},\mathsf{V})$-category provided that $a:=M^\circ(\hom_\xi)=\xi^\circ\cdot\hom\cdot\xi$ (i.e., for $\mathfrak{v},\mathfrak{w}\in T\mathsf{V}$, $a(\mathfrak{v},\mathfrak{w})=\hom(\xi(\mathfrak{v}),\xi(\mathfrak{w}))$).*

Proof. Let

$$\begin{tikzcd}
1 \arrow[loop left, "p"] \arrow[r, bend left, "\varphi"] \arrow[r, phantom, "\perp"] & V \arrow[l, bend left, "\psi"] \arrow[loop right, "\hom_\xi"]
\end{tikzcd}$$

be a pair of adjoint bimodules. By the previous theorem we know that:

$$\begin{aligned}
(4)\qquad \varphi\text{ bimodule}&\iff\varphi:(\mathsf{V},\hom_\xi)\to(\mathsf{V},\hom_\xi)\text{ is a }(\mathbb{T},\mathsf{V})\text{-functor}\\
&\iff\forall\mathfrak{v}\in T\mathsf{V}\ \ \forall v\in\mathsf{V}\ \ \hom(\xi(\mathfrak{v}),v)\le\hom(\xi\cdot T\varphi(\mathfrak{v}),\varphi(v)).
\end{aligned}$$

In particular, for every $\mathfrak{v} \in T\mathsf{V}$, $k \leq \hom(\xi(\mathfrak{v}), \xi(\mathfrak{v})) \leq \hom(\xi \cdot T\varphi(\mathfrak{v}), \varphi \cdot \xi(\mathfrak{v}))$, hence $\xi \cdot T\varphi(\mathfrak{v}) \leq \varphi \cdot \xi(\mathfrak{v})$.

$$\begin{aligned}(5) \qquad \psi \text{ bimodule } &\iff \psi : (T\mathsf{V}, a^\circ) \to (V, \hom) \text{ is a } \mathsf{V}\text{-functor} \\ &\iff \forall \mathfrak{v}, \mathfrak{w} \in T\mathsf{V}\ a(\mathfrak{v}, \mathfrak{w}) \leq \hom(\psi(\mathfrak{w}), \psi(\mathfrak{v})).\end{aligned}$$

Finally,

$$(6) \qquad \varphi \dashv \psi \iff \begin{cases} (a) \quad \varphi * \psi \leq \hom \cdot \xi \iff \forall \mathfrak{v} \in T\mathsf{V}\, \forall v \in \mathsf{V}\ \psi(\mathfrak{v}) \otimes \varphi(v) \leq \hom(\xi(\mathfrak{v}), v), \\ (b) \quad p \leq \psi * \varphi \iff k \leq \displaystyle\bigvee_{\mathfrak{u} \in TV} \psi(\mathfrak{u}) \otimes \xi(T_\circ\varphi(\mathfrak{u})). \end{cases}$$

We will show that the adjunction $\varphi \dashv \psi$ is represented by $\psi(\overset{\bullet}{k}\,)$, where $\overset{\bullet}{k} = e_{\mathsf{V}}(k)$. Similarly to the proof of Theorem 1.6, we split our argument in three steps:

(1st) $\psi(\overset{\bullet}{k}\,) = \displaystyle\bigvee_{\mathfrak{v} \in TV} \psi(\mathfrak{v}) \otimes \xi(\mathfrak{v})$:

"$\leq$" is immediate; for "$\geq$" we argue as follows:

$$\begin{aligned} \psi(\mathfrak{v}) \otimes \xi(\mathfrak{v}) &= \psi(\mathfrak{v}) \otimes \hom(\xi(\overset{\bullet}{k}\,), \xi(\mathfrak{v})) \\ &= \psi(\mathfrak{v}) \otimes a(\overset{\bullet}{k}\,, \mathfrak{v}) && \text{(by hypothesis)} \\ &\leq \psi(\mathfrak{v}) \otimes \hom(\psi(\mathfrak{v}), \psi(\overset{\bullet}{k}\,)) && \text{(by (5))} \\ &\leq \psi(\overset{\bullet}{k}\,). \end{aligned}$$

(2nd) $\forall \mathfrak{v} \in TV\ \ \hom_\xi(\mathfrak{v}, \psi(\overset{\bullet}{k}\,)) = \displaystyle\bigvee_{\mathfrak{u} \in TV} \hom(\xi(\mathfrak{v}), \xi(\mathfrak{u})) \otimes \psi(\mathfrak{u})$:

To check "$\geq$" we just observe that

$$\xi(\mathfrak{v}) \otimes (\hom(\xi(\mathfrak{v}), \xi(\mathfrak{u})) \otimes \psi(\mathfrak{u})) \leq \xi(\mathfrak{u}) \otimes \psi(\mathfrak{u}) \leq \psi(\overset{\bullet}{k}\,).$$

For "$\leq$", first note that

$$\psi(\overset{\bullet}{k}\,) \otimes \varphi(\xi(\mathfrak{u})) \leq \hom(\xi(\overset{\bullet}{k}\,), \xi(\mathfrak{u})) = \hom(k, \xi(\mathfrak{u})) = \xi(\mathfrak{u})$$

from which follows

$$(7) \qquad \xi(T\varphi(\mathfrak{u})) \leq \varphi(\xi(\mathfrak{u})) \leq \hom(\psi(\overset{\bullet}{k}\,), \xi(\mathfrak{u})).$$

From that we conclude that

$$\begin{aligned} \hom(\xi(\mathfrak{v}), \psi(\overset{\bullet}{k}\,)) &\leq \hom(\xi(\mathfrak{v}), \psi(\overset{\bullet}{k}\,)) \otimes \bigvee_{\mathfrak{u} \in TV} \psi(\mathfrak{u}) \otimes \xi(T\varphi(\mathfrak{u})) && \text{(by (6b))} \\ &= \bigvee_{\mathfrak{u} \in TV} \hom(\xi(\mathfrak{v}), \psi(\overset{\bullet}{k}\,)) \otimes \psi(\mathfrak{u}) \otimes \xi(T\varphi(\mathfrak{u})) \\ &\leq \bigvee_{\mathfrak{u} \in TV} \hom(\xi(\mathfrak{v}), \psi(\overset{\bullet}{k}\,)) \otimes \hom(\psi(\overset{\bullet}{k}\,), \xi(\mathfrak{u})) \otimes \psi(\mathfrak{u}) && \text{(by (7))} \\ &\leq \bigvee_{\mathfrak{u} \in TV} \hom(\xi(\mathfrak{v}), \xi(\mathfrak{u})) \otimes \psi(\mathfrak{u}). \end{aligned}$$

(3rd) $\forall \mathfrak{v} \in T\mathsf{V}\ \psi(\mathfrak{v}) = \displaystyle\bigvee_{\mathfrak{u} \in T\mathsf{V}} a(\mathfrak{v}, \mathfrak{u}) \otimes \psi(\mathfrak{u})$:

For "$\leq$" take $\mathfrak{u} = \mathfrak{v}$; for "$\geq$" we use (5): $a^\circ(\mathfrak{u}, \mathfrak{v}) \otimes \psi(\mathfrak{u}) \leq \hom(\psi(\mathfrak{u}), \psi(\mathfrak{v})) \otimes \psi(\mathfrak{u}) \leq \psi(\mathfrak{v})$. $\square$

Lemma. *Assume that $T1 = 1$. Then $T(\hom_\xi) \cdot m_{\mathsf{V}}^\circ = \xi^\circ \cdot \hom \cdot \xi$ provided that*

$$(8) \qquad \xi \cdot T_{\mathrm{o}} \hom(u, \_) \le \hom(u, \_) \cdot \xi,$$

$$\begin{tikzcd} T\mathsf{V} \arrow[r, "T_{\mathrm{o}}(\hom(u,\_))"] \arrow[d, "\xi"'] \arrow[dr, phantom, "\le"] & T\mathsf{V} \arrow[d, "\xi"] \\ \mathsf{V} \arrow[r, "\hom(u,\_)"'] & \mathsf{V} \end{tikzcd}$$

for each $u \in \mathsf{V}$. The inequality (8) is surely true if $\hom(u, \_)$ preserves non-empty suprema.

Proof. First observe that

$$\begin{aligned} T(\hom_\xi) \cdot m_V^\circ &= T\hom \cdot T\xi \cdot m_{\mathsf{V}}^\circ \\ &\le \xi^\circ \cdot \hom \cdot \xi \cdot T\xi \cdot m_{\mathsf{V}}^\circ \qquad \text{(because } \xi \text{ is a } \mathsf{V}\text{-functor, by Proposition 3.2)} \\ &= \xi^\circ \cdot \hom \cdot \xi \cdot m_{\mathsf{V}} \cdot m_V^\circ \\ &\le \xi^\circ \cdot \hom \cdot \xi. \end{aligned}$$

On the other hand, for $\mathfrak{u}, \mathfrak{v} \in T\mathsf{V}$, we have

$$\begin{aligned} a(\mathfrak{u}, \mathfrak{v}) &\ge T\hom_\xi(\overset{\bullet}{\mathfrak{u}}, \mathfrak{v}) \\ &= T\hom(T_{\mathrm{o}}\xi(\overset{\bullet}{\mathfrak{u}}), \mathfrak{v}) \\ &= T\hom(\overset{\bullet}{\xi(\mathfrak{u})}, \mathfrak{v}) \\ &= \xi \cdot T_{\mathrm{o}} \hom \cdot T_{\mathrm{o}}\langle \xi(\mathfrak{u}), 1_{\mathsf{V}}\rangle(\mathfrak{v}) \qquad\qquad (*) \\ &\ge \hom(\xi(\mathfrak{u}), \_) \cdot \xi(\mathfrak{v}) = \hom(\xi(\mathfrak{u}), \xi(\mathfrak{v})). \end{aligned}$$

To see (*), just observe that $T_{\mathrm{o}}\langle \xi(\mathfrak{u}), 1_{\mathsf{V}}\rangle(\mathfrak{v})$ is the only element of $T(\mathsf{V} \times \mathsf{V})$ which projects to both $\overset{\bullet}{\xi(\mathfrak{u})}$ and $\mathfrak{v}$. Assume now that $\hom(u, \_)$ preserves non-empty suprema and let $u \in \mathsf{V}$ and $\mathfrak{u} \in T\mathsf{V}$. Then

$$\begin{aligned} \hom(u, \xi(\mathfrak{u})) &= \hom(u, \bigvee\{v \in \mathsf{V} \mid \mathfrak{u} \in T(\uparrow v)\}) \\ &= \bigvee\{\hom(u, v) \mid v \in \mathsf{V}, \mathfrak{u} \in T(\uparrow v)\}) \\ &\le \bigvee\{v \in \mathsf{V} \mid T_{\mathrm{o}} \hom(u, \_)(\mathfrak{u}) \in T(\uparrow v)\}). \end{aligned}$$

□

4 A Yoneda Lemma for $(\mathbb{T}, \mathsf{V})$-categories

4.1 Function spaces. In this section we wish to obtain the analogue result to Corollary 1.5 in the setting of $(\mathbb{T}, \mathsf{V})$-categories. This in turn requires an understanding of the right adjoint to $X \otimes \_ : (\mathbb{T}, \mathsf{V})\text{-}\mathsf{Cat} \to (\mathbb{T}, \mathsf{V})\text{-}\mathsf{Cat}$, a problem studied in [16]. From there we import the following result.

Proposition. *Let $X = (X, a)$ be a $(\mathbb{T}, \mathsf{V})$-category. Then $X \otimes \_ : (\mathbb{T}, \mathsf{V})\text{-}\mathsf{Cat} \to (\mathbb{T}, \mathsf{V})\text{-}\mathsf{Cat}$ has a right adjoint $\_^X$ provided that $a \cdot Ta = a \cdot m_X$.*

Certainly, each (Eilenberg-Moore) $\mathbb{T}$-algebra, considered as a $(\mathbb{T}, \mathsf{V})$-category, satisfies the condition above. Moreover, the $(\mathbb{T}, \mathsf{V})$-categorical structure (X, a) induced by any V-category $X = (X, r)$ (see 2.3) satisfies this condition if $Te_X \cdot e_X = m_X^\circ \cdot e_X$.

Let $X = (X, a)$ and (Y, b) be $(\mathbb{T}, \mathsf{V})$-categories, and assume that $a \cdot Ta = a \cdot m_X$. Then Y^X has as underlying set

$$\{h : (X, a) \otimes (1, p) \to (Y, b) \mid h \text{ is a } (\mathbb{T}, \mathsf{V})\text{-functor}\},$$

thanks to the bijection (with $P = (1, p)$)

$$\frac{X \otimes P \to Y}{P \to Y^X}.$$

The structure $[\![a, b]\!]$ on Y^X is the largest structure making the evaluation map

$$\mathrm{ev} : X \otimes Y^X \to Y,\ (x, h) \mapsto h(x)$$

a $(\mathbb{T}, \mathsf{V})$-functor: for $\mathfrak{p} \in T(Y^X)$ and $h \in Y^X$ we have

$$[\![a, b]\!](\mathfrak{p}, h) = \bigvee \left\{ v \in \mathsf{V} \;\middle|\; \forall \mathfrak{q} \in T\pi_{Y^X}^{-1}(\mathfrak{p}), x \in X \;\; a(T\pi_X(\mathfrak{q}), x) \otimes v \le b(T\mathrm{ev}(\mathfrak{q}), h(x)) \right\}.$$

4.2 The Yoneda Embedding. By Theorem 3.3, the bimodule $a : X \multimap X$ gives rise to $(\mathbb{T}, \mathsf{V})$-functors $a : |X| \otimes X \to \mathsf{V}$ and $a : X^{\mathrm{op}} \otimes X \to \mathsf{V}$. According to the considerations above, we obtain a $(\mathbb{T}, \mathsf{V})$-functor $y = \ulcorner a \urcorner : X \to \mathsf{V}^{|X|}$. Our next result should be compared with Corollary 1.5.

Theorem (Yoneda)**.** *Let $X = (X, a)$ be a $(\mathbb{T}, \mathsf{V})$-category. Then the following assertions hold.*

(a) For all $\mathfrak{x} \in TX$ and $\varphi \in \mathsf{V}^{|X|}$, $[\![m_X, \hom_\xi]\!](Ty(\mathfrak{x}), \varphi) \le \varphi(\mathfrak{x})$.

(b) Let $\varphi \in \mathsf{V}^{|X|}$. Then

$$\forall \mathfrak{x} \in TX \;\; \varphi(\mathfrak{x}) \le [\![m_X, \hom_\xi]\!](Ty(\mathfrak{x}), \varphi) \qquad \Longleftrightarrow \qquad \varphi : X^{\mathrm{op}} \to \mathsf{V} \text{ is a } (\mathbb{T}, \mathsf{V})\text{-functor.}$$

Proof. Note that the diagrams

$$\begin{tikzcd} & \mathsf{V} \\ TX \times X \arrow[ur, "a"] \arrow[r, "1_{TX} \times y"'] & TX \times \mathsf{V}^{|X|} \arrow[u, "\mathrm{ev}"'] \end{tikzcd} \qquad \begin{tikzcd} TX \times X \arrow[r, "1_{TX} \times y"] \arrow[d, "\pi_2"'] & TX \times \mathsf{V}^{|X|} \arrow[d, "\pi_2"] \\ X \arrow[r, "y"'] & \mathsf{V}^{|X|} \end{tikzcd}$$

commute, where the right-hand side diagram is even a pullback. Let $\mathfrak{x} \in TX$ and $\varphi \in \mathsf{V}^{|X|}$.

Hence

$$
\begin{aligned}
&[\![m_X, \hom_\xi]\!](T_{\mathrm{o}}y(\mathfrak{x}), \varphi)\\
&\qquad= \bigvee\{v \in \mathsf{V} \mid \forall \mathfrak{y} \in TX, \mathfrak{Y} \in m_X^{-1}(\mathfrak{y}), \mathfrak{W} \in T(TX \times \mathsf{V}^{|X|})\\
&\qquad\qquad (T_{\mathrm{o}}\pi_1(\mathfrak{W}) = \mathfrak{Y} \,\&\, T_{\mathrm{o}}\pi_2(\mathfrak{W}) = T_{\mathrm{o}}y(\mathfrak{x})) \Rightarrow v \leq \hom(\xi \cdot T_{\mathrm{o}}\,\mathrm{ev}(\mathfrak{W}), \varphi(\mathfrak{y}))\}\\
&\qquad= \bigvee\{v \in \mathsf{V} \mid \forall \mathfrak{y} \in TX, \mathfrak{Y} \in m_X^{-1}(\mathfrak{y}), \mathfrak{W} \in T(TX \times X)\\
&\qquad\qquad (T_{\mathrm{o}}\pi_1(\mathfrak{W}) = \mathfrak{Y} \,\&\, T_{\mathrm{o}}\pi_2(\mathfrak{W}) = \mathfrak{x}) \Rightarrow v \leq \hom(\xi \cdot T_{\mathrm{o}}a(\mathfrak{W}), \varphi(\mathfrak{y}))\}\\
&\qquad= \bigvee\{v \in \mathsf{V} \mid \forall \mathfrak{y} \in TX, \mathfrak{Y} \in m_X^{-1}(\mathfrak{y})\;\; v \leq \bigwedge_{\substack{\mathfrak{W}\in T(TX\times X)\\ T_{\mathrm{o}}\pi_1(\mathfrak{W})=\mathfrak{Y}\\ T_{\mathrm{o}}\pi_2(\mathfrak{W})=\mathfrak{x}}} \hom(\xi \cdot T_{\mathrm{o}}a(\mathfrak{W}), \varphi(\mathfrak{y}))\}\\
&\qquad= \bigvee\{v \in \mathsf{V} \mid \forall \mathfrak{y} \in TX, \mathfrak{Y} \in m_X^{-1}(\mathfrak{y})\;\; v \leq \hom(\bigvee_{\substack{\mathfrak{W}\in T(TX\times X)\\ T_{\mathrm{o}}\pi_1(\mathfrak{W})=\mathfrak{Y}\\ T_{\mathrm{o}}\pi_2(\mathfrak{W})=\mathfrak{x}}} \xi \cdot T_{\mathrm{o}}a(\mathfrak{W}), \varphi(\mathfrak{y}))\}\\
&\qquad= \bigvee\{v \in \mathsf{V} \mid \forall \mathfrak{y} \in TX, \mathfrak{Y} \in m_X^{-1}(\mathfrak{y})\;\; v \leq \hom(Ta(\mathfrak{Y}, \mathfrak{x}), \varphi(\mathfrak{y}))\}\\
&\qquad= \bigvee\{v \in \mathsf{V} \mid \forall \mathfrak{y} \in TX\;\; Ta \cdot m_X^\circ(\mathfrak{y}, \mathfrak{x}) \otimes v \leq \varphi(\mathfrak{y})\}.
\end{aligned}
$$

In particular we have

$$v = k \otimes v \leq Ta \cdot m_X^\circ(\mathfrak{x}, \mathfrak{x}) \otimes v \leq \varphi(\mathfrak{x}),$$

which proves (a). On the other hand, $\varphi : (TX, Ta \cdot m_X^\circ) \to (\mathsf{V}, \hom)$ is a V-functor if and only if

$$Ta \cdot m_X^\circ(\mathfrak{y}, \mathfrak{x}) \otimes \varphi(\mathfrak{x}) \leq \varphi(\mathfrak{y})$$

for all $\mathfrak{y}, \mathfrak{x} \in TX$, from which follows (b). $\square$

We put $\hat{X} = (\hat{X}, \hat{a})$ where $\hat{X} := \{\varphi \in \mathsf{V}^{|X|} \mid \varphi : X^{\mathrm{op}} \to \mathsf{V}$ is a $(\mathbb{T}, \mathsf{V})$-functor$\}$ considered as a subcategory of $\mathsf{V}^{|X|}$. Recall that $a : X^{\mathrm{op}} \otimes X \to \mathsf{V}$ is a $(\mathbb{T}, \mathsf{V})$-functor, and therefore $a(\_, x) : X^{\mathrm{op}} \otimes P \to \mathsf{V}$ is a $(\mathbb{T}, \mathsf{V})$-functor for each $x \in X$. If $T1 = 1$, then $P = (1, p) = (1, k)$ is the neutral element for $\otimes$ and we can restrict the Yoneda functor y to $\hat{X}$.

Corollary. *Assume $T1 = 1$. Then the Yoneda functor $y : X \to \hat{X}$ is full and faithful.*

If $Te_X \cdot e_X = m_X^\circ \cdot e_X$, we might also consider the transpose $y_0 = \ulcorner a \urcorner : X \to \mathsf{V}^{X^{\mathrm{op}}}$ of $a : X^{\mathrm{op}} \otimes X \to \mathsf{V}$ as below. However, unlike the situation for V-categories, in general we do not have $\hat{X} \cong \mathsf{V}^{X^{\mathrm{op}}}$ (see example below).

Proposition (Yoneda II)**.** *Assume that $Te_X \cdot e_X = m_X^\circ \cdot e_X$ and let $X = (X, a)$ be a $(\mathbb{T}, \mathsf{V})$-category. Then the following assertions hold.*

(a) For all $\mathfrak{x} \in TX$ and $\varphi \in \mathsf{V}^{X^{\mathrm{op}}}$, $[\![a^{\mathrm{op}}, \hom_\xi]\!](Ty_0(\mathfrak{x}), \varphi) \geq \varphi(\mathfrak{x})$.

(b) Let $\mathfrak{x} \in TX$ such that $Ta \cdot e_{TX}(\mathfrak{x}, \mathfrak{x}) \geq k$. Then $[\![a^{\mathrm{op}}, \hom_\xi]\!](Ty_0(\mathfrak{x}), \varphi) \leq \varphi(\mathfrak{x})$.

Proof. Let $\mathfrak{x} \in TX$ and $\varphi \in \mathsf{V}^{X^{\mathrm{op}}}$. As above, we obtain

$$\begin{aligned}
&[\![a^{\mathrm{op}}, \hom_\xi]\!](T_{\mathrm{o}}y(\mathfrak{x}), \varphi)\\
&\quad = \bigvee\{v \in \mathsf{V} \mid \forall \mathfrak{y} \in TX, \mathfrak{Y} \in T^2X, \mathfrak{W} \in T(TX \times \mathsf{V}^{X^{\mathrm{op}}})\\
&\qquad\qquad (T_{\mathrm{o}}\pi_1(\mathfrak{W}) = \mathfrak{Y} \,\&\, T_{\mathrm{o}}\pi_2(\mathfrak{W}) = T_{\mathrm{o}}y(\mathfrak{x})) \Rightarrow a^{\mathrm{op}}(\mathfrak{Y}, \mathfrak{x}) \otimes v \leq \hom(\xi \cdot T_{\mathrm{o}}\operatorname{ev}(\mathfrak{W}), \varphi(\mathfrak{y}))\}\\
&\quad = \bigvee\{v \in \mathsf{V} \mid \forall \mathfrak{y} \in TX, \mathfrak{Y} \in T^2Y \,.\, Ta(\mathfrak{Y}, \mathfrak{x}) \otimes a^{\mathrm{op}}(\mathfrak{Y}, \mathfrak{y}) \otimes v \leq \varphi(\mathfrak{y})\}\\
&\quad = \bigvee\{v \in \mathsf{V} \mid \forall \mathfrak{y} \in TX \;\; a^{\mathrm{op}} \cdot Ta^\circ(\mathfrak{x}, \mathfrak{y}) \otimes v \leq \varphi(\mathfrak{y})\}.
\end{aligned}$$

Furthermore, we have

$$a^{\mathrm{op}} \cdot Ta^\circ = e_{TX}^\circ \cdot Tm_X^\circ \cdot TTa^\circ \cdot Ta^\circ = e_{TX}^\circ \cdot Ta^\circ \leq m_X \cdot Ta^\circ.$$

Hence $\varphi(\mathfrak{x}) \leq [\![a^{\mathrm{op}}, \hom_\xi]\!](T_{\mathrm{o}}y_0(\mathfrak{x}), \varphi)$ and, if $k \leq Ta \cdot e_{TX}(\mathfrak{x}, \mathfrak{x}) = a^{\mathrm{op}} \cdot Ta^\circ(\mathfrak{x}, \mathfrak{x})$, we also have $[\![a^{\mathrm{op}}, \hom_\xi]\!](T_{\mathrm{o}}y_0(\mathfrak{x}), \varphi) \leq \varphi(\mathfrak{x})$. □

Example. Unlike y, the functor y_0 does not need to be full and faithful. In fact, consider $X = \mathbb{N}$ as a $(\mathbb{U}, 2)$-category, i.e. a topological space, equipped with the discrete topology $a = e_{\mathbb{N}}^\circ$. Then $\mathbb{N}^{\mathrm{op}}$ is the discrete space $\mathbb{N}^{\mathrm{op}} = (U\mathbb{N}, e_{U\mathbb{N}}^\circ)$. Let $\mathfrak{x}$ be a free ultrafilter on $\mathbb{N}$. Then, for each $\mathfrak{y} \in U\mathbb{N}$, $a^{\mathrm{op}} \cdot Ua^\circ(\mathfrak{x}, \mathfrak{y}) = e_{\mathbb{N}}^\circ \cdot Ue_{\mathbb{N}}(\mathfrak{x}, \mathfrak{y}) = \mathrm{false}$ and therefore $Uy_0(\mathfrak{x}) \to \varphi$ for each $\varphi \in 2^{\mathbb{N}^{\mathrm{op}}}$. On the other hand, for $\varphi = a(\_, x)$ (x any element of $\mathbb{N}$) we have $\varphi(\mathfrak{x}) = \mathrm{false}$. In particular we see that $y_0 : \mathbb{N} \to 2^{\mathbb{N}^{\mathrm{op}}}$ is not full and faithful.

5 Examples

5.1 Ordered sets. Recall that $2\text{-}\mathsf{Cat} = \mathsf{Ord}$. Given an ordered set $X = (X, \leq)$, by Theorem 1.5 we have that a bimodule $\phi : 1 \,{-\!\!\circ\!\!\to}\, X$ is an order-preserving map $\phi : X \to 2$, while a bimodule $\psi : X \,{-\!\!\circ\!\!\to}\, 1$ is an order-preserving map $X^{\mathrm{op}} \to 2$. We can identify φ with the upclosed set $A = \varphi^{-1}(\mathrm{true})$ and ψ with the downclosed set $B = \psi^{-1}(\mathrm{true})$. Under this identification, $\varphi \dashv \psi$ translates to

$$A \cap B \neq \varnothing \qquad\qquad \text{and} \qquad\qquad \forall x \in A \;\; \forall y \in B \;\; y \leq x.$$

Then any $z \in A \cap B$ is simultaneously a smallest element of A and a largest element of B, therefore z represents $\varphi \dashv \psi$. Hence, by Proposition 1.6, each ordered set is Lawvere-complete. Note that the proof of Proposition 1.6 makes use of the Axiom of Choice; in fact, as pointed out in [6], here we have no choice.

Theorem. *The following assertions are equivalent.*

(i) *Each ordered set is Lawvere-complete.*

(ii) *The Axiom of Choice.*

Proof. To see (ii)⇒(i), let $f : X \to Y$ be a surjective map. We equip Y with the discrete order Δ_Y and X with the kernel relation of f; then we have not only $f_* \dashv f^*$ but also $f^* \dashv f_*$. Hence there exists some $g : Y \to X$ which represents $f^* \dashv f_*$, and such g necessarily satisfies $f \cdot g = 1_Y$. □

5.2 Metric spaces. For $\mathsf{V} = \mathsf{P}_+$ we have $\mathsf{P}_+\text{-}\mathsf{Cat} \cong \mathsf{Met}$. Let $X = (X, d)$ be a metric space. A pair of adjoint bimodules $\varphi \dashv \psi$ corresponds to a pair of non-expansive maps $\varphi : X \to \mathsf{P}_+$ and $\psi : X^{\mathrm{op}} \to \mathsf{P}_+$ which satisfy

$$\inf_{x \in X} \varphi(x) + \psi(x) = 0 \qquad \text{and} \qquad \forall x, y \in X \;\; \psi(y) + \varphi(x) \geq d(y, x).$$

As observed in [18], pairs of adjoint bimodules on X correspond exactly to equivalence classes of Cauchy sequences. To see this, recall first that a sequence $s = (x_n)_{n \in \mathbb{N}}$ is called *Cauchy* if

$$\inf_{k \in \mathbb{N}} \sup_{n, n' \geq k} d(x_n, x_{n'}) = 0.$$

Given a Cauchy sequence $s = (x_n)_{n \in \mathbb{N}}$, we have

$$\inf_{m \in \mathbb{N}} \sup_{n \geq m} d(x_n, x) = \sup_{m \in \mathbb{N}} \inf_{n \geq m} d(x_n, x)$$

as well as

$$\inf_{m \in \mathbb{N}} \sup_{n \geq m} d(x, x_n) = \sup_{m \in \mathbb{N}} \inf_{n \geq m} d(x, x_n),$$

and s gives rise to non-expansive maps

$$\begin{aligned} \varphi_s : X &\to \mathsf{P}_+ \\ x &\mapsto \sup_{m \in \mathbb{N}} \inf_{n \geq m} d(x_n, x) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \psi_s : X^{\mathrm{op}} &\to \mathsf{P}_+. \\ x &\mapsto \sup_{m \in \mathbb{N}} \inf_{n \geq m} d(x, x_n) \end{aligned}$$

One sees easily that $\varphi_s \dashv \psi_s$; moreover, two equivalent Cauchy sequences induce the same maps.

On the other hand, given an adjunction $\varphi \dashv \psi$, we may define $s = (x_n)_{n \in \mathbb{N}}$ such that $\varphi(x_n) + \psi(x_n) \leq \frac{1}{n}$, hence $d(x_n, x_m) \leq \frac{1}{n} + \frac{1}{m}$, and therefore s is a Cauchy sequence. Any two such sequences are equivalent. Furthermore, $\varphi \leq \varphi_s$ as well as $\psi \leq \psi_s$, therefore, since $\varphi \dashv \psi$ and $\varphi_s \dashv \psi_s$, we have even equality. Starting with a Cauchy sequence $s = (x_n)_{n \in \mathbb{N}}$, then for any sequence $t = (y_n)_{n \in \mathbb{N}}$ chosen for $\varphi \dashv \psi$ as above we have

$$\inf_{m \in \mathbb{N}} \inf_{k \in \mathbb{N}} \sup_{n \geq k} d(x_n, y_m) = 0 \qquad \text{and} \qquad \inf_{m \in \mathbb{N}} \inf_{k \in \mathbb{N}} \sup_{n \geq k} d(y_m, x_n) = 0,$$

hence s and t are equivalent. Finally, $s = (x_n)_{n \in \mathbb{N}}$ converges to x (i.e. s is equivalent to $(x)_{n \in \mathbb{N}}$) if and only if $\varphi_s \dashv \psi_s$ is represented by x.

The same argumentation applies also to the case $\mathsf{V} = \mathsf{P}_{\max}$: pairs of adjoint bimodules $\varphi \dashv \psi : 1 \to X$ with X an ultrametric space correspond precisely to Cauchy sequences in X, and convergence to representability.

Remark. A notion of non-symmetric Cauchy-sequence was introduced and studied in [3].

5.3 Topological spaces. We consider now $\mathbb{T} = \mathbb{U} = (U, e, m)$ the *ultrafilter monad* and $\mathsf{V} = \mathsf{2}$. As already stated, Proposition 3.1 describes our extension U in terms of $U_\mathrm{o} : \mathsf{Set} \to \mathsf{Set}$ (for a direct calculation of U, see [7, Example 6.4]). Then $(\mathbb{U}, \mathsf{2})\text{-}\mathsf{Cat} = \mathsf{Top}$, as it was shown by Barr [1]. By Theorem 3.3, a bimodule $\varphi : U1 \multimap X$ from the one-element space 1 to a topological space X is essentially a continuous map $\varphi : X \to \mathsf{2}$ from X into the Sierpinski space 2, hence we can identify it with a closed subset $A \subseteq X$. A bimodule $\psi : UX \multimap 1$ is basically a map $\psi : UX \to \mathsf{2}$ such that $\mathcal{A} = \psi^{-1}(\text{true})$ is closed in $|X|$ as well as in X^{op}. The topology on $|X|$

is given by the Zariski closure, that is, $\mathfrak{x} \in UX$ is in the closure of $\mathcal{M} \subseteq UX$ if $\bigcap \mathcal{M} \subseteq \mathfrak{x}$. To understand the structure of X^{op}, observe first that the order on $M^\circ X$ is given by

$$\begin{aligned}\mathfrak{x} \le \mathfrak{y} &\iff \exists \mathfrak{X} \in U^2X\; m_X(\mathfrak{X}) = \mathfrak{x} \text{ and } \mathfrak{X} \to \mathfrak{y} \\ &\iff \forall A \in \mathfrak{x}, B \in \mathfrak{y}\;\; \exists \mathfrak{a} \in UA, y \in B\; \mathfrak{a} \to y \\ &\iff \forall A \in \mathfrak{x}, B \in \mathfrak{y}\;\; \overline{A} \cap B \neq \varnothing.\end{aligned}$$

Denoting the filter base $\{\overline{A} \mid A \in \mathfrak{x}\}$ by $\overline{\mathfrak{x}}$, we have

$$\mathfrak{x} \le \mathfrak{y} \iff \overline{\mathfrak{x}} \subseteq \mathfrak{y}.$$

Hence bimodules $\psi : UX {\longrightarrow\!\!\!\!\!\!\!\circ\;\;} 1$ can be identified with subsets $\mathcal{A} \subseteq UX$ which are Zariski closed and down-closed for the order described above. Now $\varphi \dashv \psi$ translates to

$$\exists \mathfrak{x}_0 \in UX\;\; \mathfrak{x}_0 \in \mathcal{A} \;\&\; A \in \mathfrak{x}_0 \qquad\qquad \text{and} \qquad\qquad \forall \mathfrak{x} \in \mathcal{A}, x \in A\;\; \mathfrak{x} \to x.$$

Clearly, each $\mathfrak{x} \in \mathcal{A}$ converges to all points of A. On the other hand, for any $\mathfrak{x} \in UX$ with this property we have $\mathfrak{x} \le \mathfrak{x}_0$ and therefore $\mathfrak{x} \in \mathcal{A}$. We conclude that

$$\mathcal{A} = \{\mathfrak{x} \in UX \mid \forall x \in A\;\; \mathfrak{x} \to x\}.$$

A closed subset $A \subseteq X$ admits an ultrafilter $\mathfrak{x}_0 \in UA$ which converges to all $x \in A$ if and only if $\{V \subseteq X \mid V \text{ open}, V \cap A \neq \varnothing\}$ is a filter base. In the language of closed sets this is expressed by saying that A is not the union of two proper closed subsets, i.e. A is *irreducible*. Finally, ψ (and hence φ) is representable if and only if $\mathfrak{x}_0$ can be chosen principal, that is, if and only if there exists some point $x_0 \in A$ which converges to all $x \in A$. In conclusion, we have

Theorem. *The following assertions are equivalent for a topological space X.*

(i) *X is Lawvere-complete.*

(ii) *Each irreducible closed subset $A \subseteq X$ is of the form $A = \overline{\{x\}}$ for some $x \in A$, i.e. X is weakly sober.*

5.4 Approach spaces. Recall that $\mathsf{App} = (\mathbb{U}, \mathsf{P}_{\!+})\text{-}\mathsf{Cat}$ is the category of approach spaces and non-expansive maps. We fix an approach space $X = (X, a)$. As above, a bimodule $\varphi : U1 {\longrightarrow\!\!\!\!\!\!\!\circ\;\;} X$ is a non-expansive map $\varphi : X \to \mathsf{P}_{\!+}$, by Theorem 3.3. There is a bijective correspondence between maps $\varphi : X \to \mathsf{P}_{\!+}$ and families $(A_v)_{v \in \mathsf{P}_+}$ of subsets $A_v \subseteq X$ satisfying

$$A_v = \bigcap_{u > v} A_u, \tag{9}$$

where $\varphi \mapsto (\varphi^{-1}([0, v]))_{v \in \mathsf{P}_+}$ and a family $(A_v)_{v \in \mathsf{P}_+}$ defines the map $x \mapsto \inf\{v \in \mathsf{P}_{\!+} \mid x \in A_v\}$. Under this bijection, non-expansive maps correspond precisely to those families $(A_v)_{v \in \mathsf{P}_+}$ which satisfy in addition

$$\forall u, v \in \mathsf{P}_{\!+}\; \forall x \in X\; (d(A_u, x) \le v \Rightarrow x \in A_{u+v}), \tag{10}$$

where $d(A, x) = \inf\{a(\mathfrak{x}, x) \mid \mathfrak{x} \in UA\}$.

We may think of the family $A = (A_v)_{v\in \mathsf{P}_+}$ satisfying (9) as a *variable set*[6]; we call A *closed* if it satisfies (10). Now it is not difficult to see that a right adjoint $\psi : X \,{-\!\!\circ\!\!\rightarrow}\, 1$ to $\varphi : 1 \,{-\!\!\circ\!\!\rightarrow}\, X$ is determined by the variable set $\mathcal{A} = (\mathcal{A}_v)_{v\in\mathsf{P}_+}$ given by

$$\mathcal{A}_v = \{\mathfrak{x} \in UX \mid \forall u \in \mathsf{P}_+ \;\; \forall x \in A_u \;\; a(\mathfrak{x}, x) \leq u + v\},$$

for each $v \in \mathsf{P}_+$. Furthermore, given $\varphi : 1 \,{-\!\!\circ\!\!\rightarrow}\, X$, the variable set $\mathcal{A}$ defined as above corresponds to a right adjoint of φ if and only if

$$\forall u \in \mathsf{P}_+ \; (u > 0 \Rightarrow UA_u \cap \mathcal{A}_u \neq \varnothing). \tag{11}$$

In analogy to the situation in Top, we call a variable set A *irreducible* if it satisfies (11). Finally, we remark that the bimodule $\varphi : 1 \,{-\!\!\circ\!\!\rightarrow}\, X$ is represented by $x \in X$ precisely if the corresponding variable set A is of the form

$$A_v = \{y \in X \mid d(x, y) \leq v\},$$

for each $v \in \mathsf{P}_+$. Naturally, we say that such a variable set is *representable* (by x).

Theorem. *The following assertions are equivalent for an approach space X.*

(i) *X is Lawvere-complete.*

(ii) *Each irreducible closed variable set A is representable.*

We point out that this setting satisfies the conditions of Theorem 3.4, therefore it assures that P_+ is Lawvere-complete.

Remark. The notion of approach frame and its connection with approach spaces was recently studied by Christophe Van Olmen in his PhD thesis [23]. In particular, the concept of *sober approach space* as a fixed point of the dual adjunction between App and the category AFrm of approach frames and homomorphisms was introduced. As confirmed by the author of [23], these are precisely the approach spaces where each irreducible closed variable set is uniquely representable.

6 Appendix: Lawvere-complete quasi-uniform spaces

6.1 Cauchy-complete quasi-uniform spaces. We recall that a *quasi-uniformity* U on a set X is a set of binary relations on X such that:

$\forall u \in U \;\; \Delta \subseteq u;$

$\forall u \in U \;\; \exists v \in U \;\; v \cdot v \subseteq u.$

The pair (X, U) is called a *quasi-uniform space*; it is a *uniform space* when, for all $u \in U$, $u^{-1} \in U$. Given quasi-uniform spaces (X, U) and (Y, V), a map $f : X \to Y$ is *uniformly continuous* if

$$\forall v \in V \;\; \exists u \in U \;\; \forall x, y \in X \;\; x\,u\,y \;\Rightarrow\; f(x)\,v\,f(y).$$

[6] In fact, we may consider $A : \mathsf{P}_+ \to \mathsf{Set}$ as a sheaf where, for each $u \in \mathsf{P}_+$, $\{v < u\}$ is a cover of u.

Definition. Let (X, U) be a quasi-uniform space.

1. A pair $(\mathfrak{f}, \mathfrak{g})$ is a *filter* in (X, U) if $\mathfrak{f}$ and $\mathfrak{g}$ are filters in X such that

$$\forall F \in \mathfrak{f}\ \ \forall G \in \mathfrak{g}\ \ F \cap G \neq \varnothing.$$

2. A filter $(\mathfrak{f}, \mathfrak{g})$ in (X, U) is a *Cauchy filter* if

$$\forall u \in U\ \ \exists F \in \mathfrak{f}\ \ \exists G \in \mathfrak{g}\ \ F \times G \subseteq X_u := \{(x, x') \,|\, x\,u\,x'\}.$$

3. A filter $(\mathfrak{f}, \mathfrak{g})$ in (X, U) *converges to* $x_0 \in X$ if

$$\forall u \in U\ \ \exists F \in \mathfrak{f}\ \ \exists G \in \mathfrak{g}\ \ F \times G \subseteq X_{-ux_0} \times X_{x_0u-},$$

where $X_{-ux_0} := \{x \in X \,|\, x\,u\,x_0\}$ and $X_{x_0u-} := \{x \in X \,|\, x_0\,u\,x\}$.

Lemma. *Given a quasi-uniformity U in X and $x_0 \in X$, the neighbourhood filter of x_0*

$$(\{X_{-ux_0} \,|\, u \in U\}, \{X_{x_0u-} \,|\, u \in U\})$$

is a minimal Cauchy filter in (X, U).

Proposition. *For a quasi-uniform space (X, U), the following conditions are equivalent.*

(i) *Every Cauchy filter converges.*

(ii) *Every minimal Cauchy filter is the neighbourhood filter of a point x_0.*

A quasi-uniform space is said to be *Cauchy-complete* if it satisfies any of the equivalent conditions of the Proposition.

For further information see [12] and [13].

6.2 Quasi-uniform spaces as lax algebras. In order to describe quasi-uniform spaces as lax algebras, we turn back to the setting described in [7] and substitute the bicategory V-Mat of 2.1 by the bicategory Y having sets as objects and (possibly improper) filters in $\mathsf{Rel}(X, Y)$ as morphisms, where Rel is the bicategory of relations. The composition of two filters $R : X \rightarrow\!\!\!\!\!+ Y$ and $S : Y \rightarrow\!\!\!\!\!+ Z$ is the filter obtained by pointwise composition of relations $R \cdot S = \{s \cdot r \,|\, s \in S$ and $r \in R\}$, while $R \leq R'$ whenever $R' \subseteq R$ (as sets).

We define a *lax algebra* now exactly like a V-category: it is a Y-morphism $A : X \rightarrow\!\!\!\!\!+ X$ such that

$$1_X \leq A \quad \text{and} \quad A \cdot A \leq A,$$

or, equivalently,

$$\forall x \in X\ \ \forall a \in A\ \ x\,a\,x \quad \text{and} \quad \forall a \in A\ \exists a' \in A\ \ a' \cdot a' \leq a.$$

A *lax morphism* $f : (X, A) \to (Y, B)$ between lax algebras is a map $f : X \to Y$ such that $f \cdot A \leq B \cdot f$, i.e.

$$\forall b \in B\ \ \exists a \in A\ \ f \cdot a \leq b \cdot f.$$

It was shown in [7, Theorem 3.6] that this category of lax algebras and lax morphisms is equivalent to the category of quasi-uniform spaces and uniformly continuous maps.

6.3 Adjoint pairs of bimodules in quasi-uniform spaces. A *bimodule* $\Psi:(X,A){\circ\!\!\!\longrightarrow}(Y,B)$ between lax algebras is a Y-morphism $\Psi:X\relbar\joinrel\mapstochar\joinrel\rightarrow Y$ such that $\Psi\cdot A\le\Psi$ and $B\cdot\Psi\le\Psi$. As in the context of V-categories, A and B act as identities for the composition with bimodules, so that a pair of bimodules $(\Phi:(Y,B){\circ\!\!\!\longrightarrow}(X,A),(\Psi:(X,A){\circ\!\!\!\longrightarrow}(Y,B))$ is an *adjoint pair*, with $\Phi\dashv\Psi$, if $B\le\Psi\cdot\Phi$ and $\Phi\cdot\Psi\le A$. As before, every lax morphism $f:(X,A)\to(Y,B)$ defines a pair of adjoint bimodules $(f_*=B\cdot f:(X,A){\circ\!\!\!\longrightarrow}(Y,B),f^*=f^\circ\cdot A:(Y,B){\circ\!\!\!\longrightarrow}(X,A))$. It is easy to check that Proposition 2.7 is still valid in this context.

Proposition. *For a lax algebra (X,A), the following conditions are equivalent.*

(i) *Each pair of adjoint bimodules $(\Phi:(Y,B){\circ\!\!\!\longrightarrow}(X,A))\dashv(\Psi:(X,A){\circ\!\!\!\longrightarrow}(Y,B))$ is induced by a lax morphism $(Y,B)\to(X,A)$.*

(ii) *Each pair of adjoint bimodules $(\Phi:1{\circ\!\!\!\longrightarrow}(X,A))\dashv(\Psi:(X,A){\circ\!\!\!\longrightarrow}1)$ is induced by a lax morphism $1\to(X,A)$ (or simply a map).*

Theorem. *For Y-morphisms $\Phi:1\relbar\joinrel\mapstochar\joinrel\rightarrow X$ and $\Psi:X\relbar\joinrel\mapstochar\joinrel\rightarrow 1$, the following conditions are equivalent.*

(i) $\Phi\dashv\Psi$.

(ii) $(\{X_{-\psi\star}\,|\,\psi\in\Psi\},\{X_{\star\varphi-}\,|\,\varphi\in\Phi\})$ *is a minimal Cauchy filter in (X,A).*

Proof. The conditions $1\le\Psi\cdot\Phi$ and $\Phi\cdot\Psi\le A$ read as

$$\forall\psi\in\Psi\ \ \exists\varphi\in\Phi\ \ X_{\star\varphi-}\cap X_{-\psi\star}\neq\varnothing,$$

$$\forall a\in A\ \ \exists\varphi\in\Phi\ \ \exists\psi\in\Psi\ \ X_{-\psi\star}\times X_{\star\varphi-}\subseteq X_a,$$

where the former condition means that $(\{X_{-\psi\star}\,|\,\psi\in\Psi\},\{X_{\star\varphi-}\,|\,\varphi\in\Phi\})$ is a filter, while the latter one means that it is Cauchy.

(i) $\Rightarrow$ (ii): It remains to be shown that this Cauchy filter is minimal. Let $(\mathfrak{f},\mathfrak{g})$ be a filter contained in it. If $\mathfrak{f}\subsetneq\{X_{-\psi\star}\,|\,\psi\in\Psi\}$, i.e. if there exists $\psi\in\Psi$ such that $X_{-\psi\star}\notin\mathfrak{f}$, then there exist $a\in A$ and $\psi'\in\Psi$ with $\psi'\cdot a=\psi$, because ψ is a bimodule, hence a and ψ' are such that $\bigcup\limits_{x'\in X_{-\psi'\star}}X_{-ax'}\notin\mathfrak{f}$. Therefore

$$\forall F\in\mathfrak{f}\ \ \exists x\in F\ \ \forall x'\in X_{-\psi'\star}\ \ (x,x')\notin X_a.$$

Moreover, since

$$\forall G\in\mathfrak{g}\ \ G\in\{X_{\star\varphi-}\,|\,\varphi\in\Phi\}\ \Rightarrow\ \forall G\in\mathfrak{g}\ \ \exists y\in X_{-\psi'\star}\cap G,$$

we obtain

$$\forall F\in\mathfrak{f}\ \ \forall G\in\mathfrak{g}\ \ \exists x\in F\ \ \exists y\in G\ \ (x,y)\notin X_a,$$

that is $(\mathfrak{f},\mathfrak{g})$ is not a Cauchy filter.

(ii) $\Rightarrow$ (i): Let $\Phi:1{\circ\!\!\!\longrightarrow}(X,A)$ and $\Psi:(X,A){\circ\!\!\!\longrightarrow}1$ be a pair of bimodules and consider $(\{X_{-\psi\star}\,|\,\psi\in\Psi\},\{X_{\star\varphi-}\,|\,\varphi\in\Phi\})$. We concluded already that the adjunction conditions are equivalent to this pair being a Cauchy filter. But we did not show yet that Φ and Ψ are bimodules. For any $a \in A$,

$$(\{\bigcup_{x\in X_{-\psi\star}} X_{-ax}\,|\,\psi\in\Psi,\,a\in A\},\,\{\bigcup_{y\in X_{\star\varphi-}} X_{ya-}\,|\,\varphi\in\Phi,a\in A\})$$

is a Cauchy filter contained in the former one, as we show next. First,

$$\bigcup_{x\in X_{-\psi\star}} X_{-ax}\;\cap\;\bigcup_{y\in X_{\star a-}} X_{ya-}\;\supseteq\;X_{-\psi\star}\cap X_{\star\varphi-}\neq\varnothing.$$

To prove the other condition, let $a \in A$, and consider $b \in A$ such that $b \cdot b \cdot b \le a$. There exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that $X_{-\psi\star} \times X_{\star\varphi-} \subseteq X_b$, and this implies that

$$\bigcup_{x\in X_{-\psi\star}} X_{-bx}\times\bigcup_{y\in X_{\star\varphi-}} X_{yb-}\subseteq X_a,$$

since

$$x'\in\bigcup_{x\in X_{-\psi\star}} X_{-bx}\;\Rightarrow\;\exists x\in X_{-\psi\star}\;\;(x',x)\in X_b,$$

$$y'\in\bigcup_{y\in X_{\star\varphi-}} X_{yb-}\;\Rightarrow\;\exists y\in X_{\star\varphi-}\;\;(y,y')\in X_b;$$

hence, since also $(x, y) \in X_b$, we conclude that $(x', y') \in X_a$ as claimed. □

6.4 Lawvere-complete=Cauchy-complete. It is now straightforward to prove that the two notions of completeness coincide.

Theorem. *For a quasi-uniform space (X, A) the following conditions are equivalent.*

(i) (X, A) *is a Lawvere-complete lax algebra.*

(ii) (X, A) *is a Cauchy-complete quasi-uniform space.*

Proof. (i) $\Rightarrow$ (ii): Each minimal Cauchy filter in (X, A) defines an adjoint pair of bimodules $(\Phi : 1 \multimap (X, A)) \dashv (\Psi : (X, A) \multimap 1)$, which, by (i), is induced by a map $f : 1 \to X$, $\star \mapsto x_0$. Hence $\Phi = \{\varphi_b = b \cdot f\,|\,b \in B\}$ and $\Psi = \{\psi_b = f^\circ \cdot b\,|\,b \in B\}$. Moreover, $x \in X_{\star\varphi_b-}$ exactly when $b(x_0, x) = \top$, that is $X_{\star\varphi_b-} = X_{x_0b-}$, and $x \in X_{-\psi_b\star}$ exactly when $b(x, x_0) = \top$, which means $X_{-\psi_b\star} = X_{-bx_0}$.

(ii) $\Rightarrow$ (i): Given an adjoint pair of bimodules $(\Phi : 1 \multimap (X, A)) \dashv (\Psi : (X, A) \multimap 1)$, by (ii) the minimal Cauchy filter it induces is the neighbourhood filter of a point x_0. It is straightforward to check that $\Phi = A \cdot f$ and $\Psi = f^\circ \cdot A$ for $f : 1 \to X$, $\star \mapsto x_0$. □

Final remark. The results of this section can be investigated in the more general setting introduced in [9], i.e., in proalgebras; here, for simplicity, we decided to state them only at the level of quasi-uniform structures, which are proalgebras for the identity monad.

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