WEAKLY MAL'TSEV AND DISTRIBUTIVE LATTICES

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Abstract. In this short note we prove that a variety of lattices is distributive if and only if it is weakly Mal'tsev.

1. Introduction

A category is said to be weakly Mal'tsev if it has all pullbacks of split epimorphisms along split epimorphisms, and for every such pullback the two canonical morphisms induced by the sections into the pullback are jointly epimorphic [7]. This is to say that for every diagram as

$$A \underset{r}{\overset{f}{\rightleftarrows}} B \underset{s}{\overset{g}{\leftrightarrows}} C$$

where $fr = 1_B = gs$, the pullback of f along g exists always and the canonical induced morphisms e_1 and e_2, as in

$$\begin{tikzcd} A \times_B C \arrow[r, shift left, "\pi_2"] \arrow[d, shift right, "\pi_1"'] & C \arrow[l, shift left, "e_2"] \arrow[d, shift right, "g"'] \\ A \arrow[u, shift right, "e_1"'] \arrow[r, shift left, "f"] & B \arrow[l, shift left, "r"] \arrow[u, shift right, "s"'] \end{tikzcd} , \tag{1}$$

into the pullback, are jointly epimorphic. In other words, for every diagram as

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[dr, "\alpha"'] & B \arrow[l, shift left, "r"] \arrow[r, shift right, "s"'] \arrow[d, "\beta"] & C \arrow[l, shift right, "g"'] \arrow[dl, "\gamma"] \\ & D & \end{tikzcd} \tag{2}$$

with $fr = 1_B = gs$ and $\alpha r = \beta = \gamma s$ there is at most one morphism

$$\varphi \colon A \times_B C \to D$$

satisfying the two conditions

$$\varphi e_1 = \alpha \ , \ \varphi e_2 = \gamma.$$

The name weakly Mal'tsev is motivated by the fact that if, in the definition of a weakly Mal'tsev category, we require that e_1 and e_2 are jointly strongly epimorphic, then the result is precisely a Mal'tsev category (usually assumed with finite limits, see for instance [4], p. 3836, but also [3, 5] and [2]).

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A general class of examples, containing in particular all the Mal'tsev varieties, may be described as follows.

Let $\mathcal{V}$ denote a variety of algebras containing at least one ternary operation, $p(x, y, z)$, satisfying the following identity

$$p(x, y, y) = p(y, y, x). \tag{3}$$

If the quasi-identity

$$p(x, y, y) = p(x', y, y) \Rightarrow x = x' \tag{4}$$

also holds in $\mathcal{V}$ then $\mathcal{V}$ is weakly Mal'tsev.

A particular case of condition (4) is

$$p(x, y, y) = x$$

which, combined with (3), gives

$$p(x, x, y) = y,$$

the usual Mal'tsev identities.

To see that any variety of universal algebras, containing at least one ternary operation p, and satisfying identity (3) and quasi-identity (4), is a weakly Mal'tsev category: consider a diagram like (2) and the existence of a morphism φ satisfying the required conditions $\varphi e_1 = \alpha$ and $\varphi e_2 = \gamma$; use the identity (3) in order to get

$$p(\varphi(a, c), \beta(f(a)), \beta(f(a))) = p(\alpha(a), \beta(f(a)), \gamma(c));$$

conclude from (4) that any other morphism φ' with the same property as φ must be equal to φ.

Even though this (see also [8]) shows evidence of a general class of varieties with the weakly Mal'tsev property (other than Mal'tsev varieties themselves), there were no known examples of weakly Mal'tsev varieties (besides Mal'tsev ones) defined only in terms of identities, i.e., not involving a quasi-identity as (4) above.

It was a surprise to discover that (a) the variety of distributive lattices is weakly Mal'tsev, and (b) any algebra in a weakly Mal'cev variety of lattices is distributive.

2. A variety of lattices is distributive if and only if it is weakly Mal'tsev

In the following we assume the reader is familiar with basic lattice theory as explained for example in the first two chapters of [6].

A lattice $L = (L, \wedge, \vee)$ is distributive if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all $x, y, z \in L$. We will say that a variety of lattices is distributive if every algebra in it is in particular a distributive lattice.

Theorem 2.1. *Let $\mathcal{L}$ be a variety of lattices. The following conditions are equivalent.*

(*i*) *$\mathcal{L}$ is distributive;*

(*ii*) *for every lattice L, given any element $a \in L$, the following implication*

$$\begin{cases} x \wedge a = x' \wedge a \\ x \vee a = x' \vee a \end{cases} \Rightarrow x = x'$$

holds;

(*iii*) *$\mathcal{L}$ is a weakly Mal'tsev category.*

Proof. $(i) \Rightarrow (ii)$ Using the absorption, commutative and distributive laws, together with the hypotheses in (ii), (see for example [1], p. 340) we have

$$x = x \wedge (x \vee a) = x \wedge (x' \vee a) = (x \wedge x') \vee (x \wedge a) =$$
$$(x' \wedge x) \vee (x' \wedge a) = x' \wedge (x \vee a) = x' \wedge (x' \vee a) = x'.$$

$(ii) \Rightarrow (iii)$ Consider a diagram as displayed in (2) and suppose that two morphisms $\varphi, \varphi' \colon A \times_B C \to D$ exist such that $\varphi e_1 = \varphi' e_1 = \alpha$ and $\varphi e_2 = \varphi' e_2 = \gamma$.

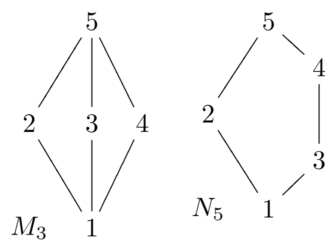
For every $a \in A$ and $c \in C$, with $f(a) = b = g(c)$, a consequence of the lattice axioms (and the fact that $e_1(a) = (a, s(b))$ and $e_2(c) = (r(b), c)$), is that

$$\begin{cases} \varphi(a,c) \wedge \beta(b) &= \alpha(a) \wedge \gamma(c) &= \varphi'(a,c) \wedge \beta(b) \\ \varphi(a,c) \vee \beta(b) &= \alpha(a) \vee \gamma(c) &= \varphi'(a,c) \vee \beta(b) \end{cases}.$$

The implication of condition (ii) is now used to conclude that the two morphisms φ and φ' are the same, and hence the category $\mathcal{L}$ is weakly Mal'tsev.

$(iii) \Rightarrow (i)$ We will show that if $\mathcal{L}$ is not distributive then it is not weakly Mal'tsev.

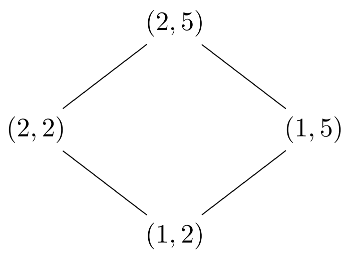
A useful result in Lattice Theory states that a lattice L is distributive if and only if L does not contain a sublattice isomorphic to M_3 or N_5, the two lattices displayed in the following picture.



Assume $\mathcal{L}$ is not distributive, and suppose D is a lattice containing M_3 as a sublattice. If considering $A = \{1, 2\}$, $B = \{2\}$ and $C = \{2, 5\}$ as sublattices of M_3, and hence of D, together with the obvious morphisms

$$A \underset{r}{\overset{f}{\rightleftarrows}} B \underset{s}{\overset{g}{\leftrightarrows}} C \quad , \quad fr = 1 = gs,$$

the resulting $A \times_B C$ is the lattice



and then there are two different lattice homomorphisms

$$\varphi, \varphi' \colon A \times_B C \to D$$

with the property that $\varphi e_1 = \varphi' e_1$ and $\varphi e_2 = \varphi' e_1$, namely:

	φ	φ'
$(1,2)$	1	1
$(2,2)$	2	2
$(1,5)$	3	4
$(2,5)$	5	5

Similarly, if some lattice D has a sublattice isomorphic to N_5 we may consider $A = \{1,2\}$, $B = \{2\}$ and $C = \{2,5\}$ in the same way (now considered as sublattices of N_5 and hence D) and conclude again the existence of two different morphisms $\varphi, \varphi' \colon A \times_B C \to D$, such that $\varphi e_1 = \varphi' e_1$ and $\varphi e_2 = \varphi' e_1$, namely the ones defined by the table above. This shows that the category $\mathcal{L}$ is not weakly Mal'tsev, and concludes the proof. □

3. Commutative semigroups

A lattice is build up from two commutative and idempotent semigroup structures with two compatibility conditions. A semilattice is just a commutative and idempotent semigroup. We could ask what is the role of weakly Mal'cev in a variety of semilattices. The following result provides a characterization for the weakly Mal'tsev property in the more general context of commutative semigroups.

Theorem 3.1. *Let $\mathcal{S}$ be a variety of commutative semigroups. The following two conditions are equivalent:*

(i) *Given any three elements a, b, c in a commutative semigroup D (in $\mathcal{S}$), the equation*
$$x \cdot b = a \cdot c$$
has at most one solution $x \in D$;

(ii) *The category $\mathcal{S}$ is weakly Mal'tsev.*

Proof. $(i) \Rightarrow (ii)$ Given any diagram of commutative semigroups of the form

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[dr, "\alpha"'] & B \arrow[l, shift left, "r"] \arrow[r, shift right, "s"'] \arrow[d, "\beta"] & C \arrow[l, shift right, "g"'] \arrow[dl, "\gamma"] \\ & D & \end{tikzcd},$$

such that $fr = 1_B = gs$ and $\alpha r = \beta = \gamma s$, suppose there is a morphism $\varphi \colon A \times_B C \to D$ such that
$$\varphi(a, sf(a)) = \alpha(a)\ ,\ \varphi(rg(c), c) = \gamma(c).$$
This morphism is uniquely determined since it satisfies the equation
$$\varphi(a, c) \cdot \beta(b) = \alpha(a) \cdot \gamma(c)$$
for every $a \in A$ and $c \in C$, with $f(a) = b = g(c)$. Indeed we have
$$\begin{aligned} \varphi(a, c) \cdot \beta(b) &= \varphi(a, c) \cdot \varphi(r(b), s(b)) \\ &= \varphi(a \cdot r(b), c \cdot s(b)) \\ &= \varphi(a \cdot r(b), s(b) \cdot c) \\ &= \varphi(a, s(b)) \cdot \varphi(r(b) \cdot c) \\ &= \alpha(a) \cdot \gamma(c). \end{aligned}$$

$(ii) \Rightarrow (i)$ Given any three elements a, b, c in a commutative semigroup D (in $\mathcal{S}$), we may consider A, B and C the free commutative semigroups generated by the sets $\{a, b\}$, $\{b\}$ and $\{b, c\}$, respectively, together with the obvious maps

$$\begin{tikzcd} \{a,b\} \arrow[r, shift left, "f"] & \{b\} \arrow[l, shift left, "r"] \arrow[r, shift right, "s"'] & \{b,c\} \arrow[l, shift right, "g"'] \end{tikzcd} \quad , \quad fr = 1 = gs.$$

Note that A, B and C are not necessarily in $\mathcal{S}$, nevertheless we observe that the following argument still holds if we trade A, B and C by their respective inclusions in D.

Suppose the equation

$$x \cdot b = a \cdot c \tag{5}$$

has a solution x in D, we will show that if the variety $\mathcal{S}$ is weakly Mal'tsev then this solution is unique.

The existence of a solution $x \in D$ to the equation (5) gives us a morphism

$$\varphi\colon A \times_B C \to D$$

with the property that $\varphi e_1 = \alpha$ and $\varphi e_2 = \gamma$, where e_1, e_2 are the already mentioned canonical morphisms into the pullback whilst α and γ are the inclusions of A and C, respectively, into D, considered as subobjects. This means that the existence of any other solution $x' \in D$ to equation (5) would produce a morphism $\varphi'\colon A \times_B C \to D$ satisfying the two conditions $\varphi' e_1 = \alpha$ and $\varphi' e_2 = \gamma$, however, the weakly Mal'tsev property requires the existence of at most one such morphism. Hence the solution is unique, provided it exist.

The construction of the morphism φ may be obtained as described in the following procedure.

Considering that a generic element of A (resp. C) is of the form $a^n b^m$ (resp. $c^n b^m$) where n and m are two nonnegative integers that cannot be zero at the same time, we have

$$f(a^n b^m) = b^{n+m} = g(c^n b^m);$$

a generic element of B is then of the form b^n with n a positive integer, and

$$r(b^n) = a^0 b^n \quad , \quad s(b^n) = c^0 b^n.$$

In this case, a generic element in the pullback $A \times_B C$ is of the form

$$(a^n b^m, c^{n'} b^{m'})$$

where n, m, n', m' are nonnegative integers (n and m are not both zero at the same time, nor n' and m') such that

$$n + m = n' + m'.$$

The existence of an element $x \in D$, satisfying $x \cdot b = a \cdot c$, induces a morphism $\varphi\colon A \times_B C \to D$ defined in the following way

$$\varphi(a^n b^m, c^{n'} b^{m'}) = \begin{cases} x^n c^{n'-n} b^{m'} & \text{if } n \leq n' \\ x^{n'} a^{n-n'} b^m & \text{if } n > n' \end{cases}.$$

Indeed φ is a morphism of semigroups, i.e, for every $n, m, n', m', u, v, u', v'$ nonnegative integers satisfying $n + m = n' + m' \neq 0 \neq u + v = u' + v'$,

$$\varphi((a^n b^m, c^{n'} b^{m'}) \cdot (a^u b^v, c^{u'} b^{v'})) = \varphi(a^n b^m, c^{n'} b^{m'}) \cdot \varphi(a^u b^v, c^{u'} b^{v'})$$

or even

$$\varphi(a^{n+u}b^{m+v}, c^{n'+u'}b^{m'+v'}) = \varphi(a^n b^m, c^{n'}b^{m'}) \cdot \varphi(a^u b^v, c^{u'}b^{v'}).$$

Several cases have to be checked:

- if $n \leq n'$ and $u \leq u'$, then $n+u \leq n'+u'$ and we have
$$x^{n+u}c^{n'+u'-(n+u)}b^{m'+v'} = (x^n c^{n'-n} b^{m'})(x^u c^{u'-u} b^{v'});$$
- if $n \leq n'$ and $u > u'$ then we have to consider two further cases:
  - if $n+u \leq n'+u'$, then
$$\begin{aligned} & x^{n+u}\, c^{n'+u'-(n+u)}\, b^{m'+v'} = \\ = \quad & x^{n+u'}\, x^{u-u'}\, b^{u-u'}\, c^{n'+u'-n-u}\, b^{m'+v'-u+u'} \\ = \quad & x^{n+u'}\, a^{u-u'}\, c^{n'-n}\, b^{m'+v'-u+u'} \text{ , since } xb = ac \\ = \quad & x^{n+u'}\, a^{u-u'}\, c^{n'-n}\, b^{m'+v} \text{ , since } v+u = v'+u' \\ = \quad & (x^n\, c^{n'-n}\, b^{m'})(x^{n'}\, a^{u-u'}\, b^v); \end{aligned}$$
  - if $n+u > n'+u'$, then
$$\begin{aligned} & x^{n'+u'}\, a^{n+u-(n'+u')}\, b^{m+v} = \\ = \quad & x^{n+u'}\, x^{n'-n}\, b^{n'-n}\, a^{n+u-n'-u'}\, b^{m+v-n'+n} \\ = \quad & x^{n+u'}\, c^{n'-n}\, a^{u-u'}\, b^{m+v-n'+n} \text{, since } xb = ca \\ = \quad & x^{n+u'}\, c^{n'-n}\, a^{u-u'}\, b^{m'+v} \text{, since } m'+n' = m+n \\ = \quad & (x^n\, c^{n'-n}\, b^{m'})(x^{u'}\, a^{u-u'}\, b^v); \end{aligned}$$

and similarly for the remaining cases.

Finally we observe that this morphism is such that

$$\begin{aligned} \varphi(a^n b^m, c^0 b^{m+n}) &= x^0 a^{n-0} b^m = a^n b^m \\ \varphi(a^0 b^{m'+n'}, c^{n'} b^{m'}) &= x^0 c^{n'-0} b^{m'} = c^{n'} b^{m'}. \end{aligned}$$

□

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