DANIELL'S VERSION OF LEBESGUE INTEGRAL

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In 1918 Percy John Daniell published an article

> *A general form of integral*, Annals of Mathematics **19** (1918) 279–294,

for which I would gladly use epithets like "celebrated" or "pioneering". Only, in spite of its quality, it has not been in my opinion celebrated enough. And one uses the adjective "pioneering" usually for works that found recognition and followers relatively soon (while here it took several decades until the ideas started to appear in monographs). Perhaps one can call it "revolutionary", also in the original sense of the word, that is, "turning over": unlike the more classical approach the roles of integral and measure are reversed: integral comes first, and measure second – almost as an afterthought.

The main idea seems to be to head, from the very start, for an extension of Riemann integral that would be friendlier to limits (no integral can preserve all limits, see the easy example in 6.8 below, but Riemann's integration often fails to behave for the lack of generality only).

Besides the surprising simplicity there are two main features of the Daniell's approach to be emphasized.

First, as already mentioned, there is no measure theory preceding the construction of the integral and proofs of the most important facts about it. Lebesgue measure is obtained practically for free, ex post, as the integral of the characteristic function; its fundamental properties (σ-additivity, measurability of Borel sets) are easy corollaries of (relatively easily proved) facts about the integral.

Secondly, the constructions are not in fact concerned with metric convergence. The limits one uses most of the time are monotone ones, hence suprema or infima (and hence always existing) – one can say that everything is happening, rather, in terms of the natural partial order of real functions; when one has to consider in proving Lebesgue Theorem, at last, a general limit, one works in fact with the common value of limes superior (infimum of suprema) and limes inferior (supremum of infima).

It has been naturally asked whether Daniell proved that his integral coincided with the (15 years older) Lebesgue one. In the paper this fact is just mentioned in passing, without providing details. It does not seem to be hard, though, and one can surmise that the author did not find a more explicit treatment of the equivalence interesting enough.

1. To start with

1.1. Dramatis personae. We will consider a Euclidean space $\mathbb{E}_m$ of a fixed dimension. It does not matter which, but it is fixed.

A *function* is a mapping $f : \mathbb{E}_m \to \mathbb{R} \cup \{-\infty, +\infty\}$. The constant zero function will be denoted by $\mathbf{0}$.

A sequence $(f_n)_n$ of functions is said to be *increasing* if for all x

$$f_1(x) \le f_2(x) \le \cdots \le f_n(x) \le \cdots$$

(usually this is referred to as *non-decreasing*, but "increasing" is shorter and there will not be an occasion for confusion). Similarly we speak of a *decreasing* sequence.

Since we allow the infinite values, an increasing (resp. decreasing) sequence $(f_n)_n$ always has a limit, namely the supremum resp.infimum. We write

$$f_n \nearrow f \quad \text{resp.} \quad f_n \searrow f$$

and if there is a danger of confusion (e.g. in double indexing) we emphasize the index in question as in

$$f_{nk} \overset{k}{\nearrow} f_n, \quad f_{nk} \overset{k}{\searrow} f_n.$$

The notation $a_n \searrow a$, $a_n \nearrow a$ may be used also for monotone sequences of numbers.

Note again that for the increasing and decreasing sequences, whether of functions or of numbers, the limits are *suprema* resp. *infima*; the limit structure is based on the order.

1.2. The class Z. This is the class of

continuous functions with compact carrier,

that is, with values zero on $\mathbb{E}_m \smallsetminus \langle a_1, b_1\rangle \times \cdots \times \langle a_m, b_m\rangle$ where a_i, b_i are finite; the continuity entails finite values $f(x)$.

Note that if $f_n \searrow \mathbf{0}$ and $f_1 \in \mathbf{Z}$ then all f_n are in $\mathbf{Z}$: we can take the carrier of f_1 for all of them.

For the $f \in \mathbf{Z}$ we have the standard Riemann integral which we will denote, so far, by

$$\mathfrak{I}f.$$

1.3. The operations. We will consider the standard sum $f+g$ and the real multiples αf (so far we avoid the case with opposite infinite values in the sum; but see 3.6 below). Further we have the maxima and minima

$$f \vee g \quad \text{and} \quad f \wedge g$$

and the positive and negative parts

$$f^+ = f \vee \mathbf{0}, \quad f^- = -(f \wedge \mathbf{0}),$$

and finally the absolute value

$$|f| = f^+ + f^- \quad \text{(note that} \quad f = f^+ - f^-).$$

1.3.1. Observations. (a) $\mathbf{Z}$ *is closed under all the operations above.*

(b) $f \vee g = \frac{1}{2}(f+g+|f-g|)$ *and* $f \wedge g = \frac{1}{2}(f+g-|f-g|)$. *Thus, if a class of functions is closed under linear combinations and absolute values, it is closed under maxima and minima as well.*

1.4. Theorem. (Dini) *Let* f_n *be continuous real functions on a compact metric space* X *and let* $f_n \searrow \mathbf{0}$. *Then* f_n *converges to* $\mathbf{0}$ *uniformly.*

Proof. It suffices to prove that $m_n = \max_x f_n(x)$ converges to zero, because then $|f_n(x) - 0| < \varepsilon$ for sufficiently large n independently on the choice of $x \in X$.

Suppose it does not. Reducing, possibly, f_n to a smaller system we obtain an example with

$$f_n \searrow 0 \quad \text{and} \quad \forall n,\ m_n > \varepsilon_0$$

for a fixed $\varepsilon_0 > 0$.

Since X is compact there are x_n such that $f_n(x_n) = m_n$; since it is compact metric we can choose a subsequence of x_n converging to some $x \in X$. After a new reduction of the sequence f_n we have

$$f_n \searrow 0, \quad \forall n\ f_n(x_n) > \varepsilon_0 \quad \text{and} \quad \lim_n x_n = x.$$

Now for $k \geq n$,

$$f_n(x_k) \geq f_k(x_k) > \varepsilon_0$$

and hence

$$f_n(x) = \lim_k f_n(x_k) \geq \varepsilon_0 \ \text{ for all } \ n.$$

This is a contradiction since $\lim_n f_n(x) = 0$. $\square$

1.4.1. Corollary. *If* $f_n \in \mathbf{Z}$ *and* $f_n \searrow \mathbf{0}$ *then* $\lim_n \Im f_n$ $(= \inf \Im f_n) = 0$.

1.5. Summary. We have a class $\mathbf{Z}$ of functions such that

(Z1) for all $\alpha, \beta \in \mathbb{R}$ and $f, g \in \mathbf{Z}$, $\alpha f + \beta g \in \mathbf{Z}$,
(Z2) if $f \in \mathbf{Z}$ then $|f| \in \mathbf{Z}$,

and a mapping $\Im : \mathbf{Z} \to \mathbb{R}$ such that

(I1) if $f \geq \mathbf{0}$ then $\Im f \geq 0$,
(I2) $\Im$ is a linear map, and
(I3) if $f_n \searrow \mathbf{0}$ then $\Im f_n \searrow 0$.

In the sequel we will consistently use only he properties (Zj) and (Ij) and their consequences. In particular we easily infer that

$$f \leq g \quad \Rightarrow \quad \Im f \leq \Im g, \text{ and}$$
$$f, g \in \mathbf{Z} \quad \Rightarrow \quad f \vee g, f \wedge g, f^+, f^- \in \mathbf{Z}.$$

2. A modest extension

2.1. Define

$$\mathbf{Z}^{\mathsf{up}} = \{f \mid \exists f_n \in \mathbf{Z},\ f_n \nearrow f\},$$
$$\mathbf{Z}^{\mathsf{dn}} = \{f \mid \exists f_n \in \mathbf{Z},\ f_n \searrow f\},$$
$$\mathbf{Z}^* = \mathbf{Z}^{\mathsf{up}} \cup \mathbf{Z}^{\mathsf{dn}}.$$

Note. The functions in $\mathbf{Z}^*$ are not necessarily continuous, they do not have to have a compact carrier, and can reach infinite values. Also note that $\mathbf{Z} \subseteq \mathbf{Z}^{\mathsf{up}} \cap \mathbf{Z}^{\mathsf{dn}}$ and this inclusion is not an equality.

2.2. Proposition. *Let* $f, g \in \mathbf{Z}^*$*, witnessed by sequences* f_n *and* g_n*. Let* $f \leq g$*. Then*

$$\lim \Im f_n \leq \lim \Im g_n.$$

Proof. (a) If $f_n \nearrow f$ and $g_n \searrow g$ then $f_n \leq f \leq g \leq g_n$.

(b) Let $f_n \nearrow f$ and $g_n \nearrow g$. For a fixed k set

$$h_n = g_n \wedge f_k.$$

Then h_n increases and we have

$$\lim h_n = g \wedge f_k = f_k,$$

and hence

$$h_n \nearrow_n f_k, \quad \text{that is,} \quad (f_k - h_n) \searrow^n \mathbf{0}$$

and we obtain, by (I3), that $\lim_n \mathfrak{I}h_n = \mathfrak{I}f_k$. Now $g_n \geq h_n$, hence $\mathfrak{I}g_n \geq \mathfrak{I}h_n$, and hence

$$\lim_n \mathfrak{I}g_n \geq \mathfrak{I}f_k$$

for each k so that finally $\lim_n \mathfrak{I}f_n \leq \lim_k \mathfrak{I}g_k$.

(c) If $f_n \searrow f$ and $g_n \searrow g$ use (b) for $-f, -g$.

(d) Let $f_n \searrow f$ and $g_n \nearrow g$. Then $f_n - g_n \leq h_n = (f_n - g_n)^+$; since $h_n \searrow \mathbf{0}$ we have $\lim \mathfrak{I}h_n = 0$ and finally

$$\lim \mathfrak{I}f_n - \lim \mathfrak{I}g_n = \lim \mathfrak{I}(f_n - g_n) \leq 0. \quad \square$$

2.3. Corollary and definition. For $f \in \mathbf{Z}^*$ we can define

$$\mathfrak{I}f = \lim_n \mathfrak{I}f_n$$

where f_n is an arbitrary monotone sequence converging to f.

2.4. A few immediate facts. (Note that most of them hold specifically for functions from $\mathbf{Z}^{\mathsf{up}}$ resp. $\mathbf{Z}^{\mathsf{dn}}$ while one holds for the whole of $\mathbf{Z}^*$.)

(a) $f \in \mathbf{Z}^{\mathsf{up}}$ *iff* $-f \in \mathbf{Z}^{\mathsf{dn}}$.

(b) *If* $f, g \in \mathbf{Z}^{\mathsf{up}}$ *resp.* $\mathbf{Z}^{\mathsf{dn}}$ *then* $f + g \in \mathbf{Z}^{\mathsf{up}}$ *resp.* $\mathbf{Z}^{\mathsf{dn}}$ *and we have* $\mathfrak{I}(f+g) = \mathfrak{I}f + \mathfrak{I}g$.

(c) *If* $f \in \mathbf{Z}^{\mathsf{up}}$ *and* $\alpha \geq 0$ *resp.* $\alpha \leq 0$ *then* $\alpha f \in \mathbf{Z}^{\mathsf{up}}$ *resp.* $\mathbf{Z}^{\mathsf{dn}}$ *and we have* $\mathfrak{I}(\alpha f) = \alpha \mathfrak{I}f$.

(d) *If* $f, g \in \mathbf{Z}^*$ *and* $f \leq g$ *then* $\mathfrak{I}f \leq \mathfrak{I}g$.

(e) *If* $f, g \in \mathbf{Z}^{\mathsf{up}}$ *then* $f \vee g, f \wedge g \in \mathbf{Z}^{\mathsf{up}}$.

2.5. Proposition. *Let* $f_n \in \mathbf{Z}^{\mathsf{up}}$ *and* $f_n \nearrow f$. *Then* $f \in \mathbf{Z}^{\mathsf{up}}$ *and* $\mathfrak{I}f_n \nearrow \mathfrak{I}f$.

Similarly for $f_n \in \mathbf{Z}^{\mathsf{dn}}$ *and* $f_n \searrow f$.

Proof. Choose $f_{nk} \in \mathbf{Z}$ such that $f_{nk} \nearrow_k f_n$ and set

$$g_n = \bigvee\{f_{ij} \mid 1 \leq i, j \leq n\}.$$

Then $g_n \nearrow g$ for some g. Since

$$g_n(x) = f_{ij}(x) \leq f_i(x) \quad \text{for some} \quad i, j \leq n$$

we have

$$g_n \leq f_n \leq f. \tag{1}$$

On the other hand, for $k \geq n$ we have $g_k \geq f_{nk}$ and hence

$$g \geq f_n. \tag{2}$$

By (1) and (2), $g_n \nearrow f$.

Now for the value of $\mathfrak{I}f$: by (2), $\mathfrak{I}f = \mathfrak{I}g \geq \mathfrak{I}f_n$ and hence $\mathfrak{I}f \geq \lim \mathfrak{I}f_n$; on the other hand, by (1), $\mathfrak{I}f = \lim \mathfrak{I}g_n \leq \lim \mathfrak{I}f_n$. □

3. Lebesgue integral; some basic facts

3.1. For an arbitrary function f set

$$\underline{\int} f = \sup\{\mathfrak{I}g \mid g \leq f,\ g \in \mathbf{Z}^{\mathsf{dn}}\} \text{ and } \overline{\int} f = \inf\{\mathfrak{I}g \mid g \geq f,\ g \in \mathbf{Z}^{\mathsf{up}}\}.$$

$\underline{\int} f$ is called the lower (Lebesgue) integral of f, and $\overline{\int} f$ is the upper one.

3.2. Proposition. (1) $\underline{\int} f = \sup\{\mathfrak{I}g \mid g \leq f,\ g \in \mathbf{Z}^*\}$ *and* $\overline{\int} f = \inf\{\mathfrak{I}g \mid g \geq f,\ g \in \mathbf{Z}^*\}$.

(2) $\underline{\int} f \leq \overline{\int} f$.

(3) *If* $f \leq g$ *then* $\underline{\int} f \leq \underline{\int} g$ *and* $\overline{\int} f \leq \overline{\int} g$.

Proof. (a) Let, say, the second equality not hold. Then there is a $g \geq f$, $g \in \mathbf{Z}^{\mathsf{dn}}$ such that $\mathfrak{I}g < \overline{\int} f$. Let $g_n \searrow g$ with $g_n \in \mathbf{Z}$. Then there has to be a k such that $\mathfrak{I}g_k < \overline{\int} f$. This is a contradiction since $g_n \in \mathbf{Z} \subseteq \mathbf{Z}^{\mathsf{up}}$.

(2) and (3) are trivial. □

3.3. From 3.2(1) we immediately obtain

Corollary. *For* $f \in \mathbf{Z}^*$ *we have* $\underline{\int} f = \overline{\int} f = \mathfrak{I}f$.

3.4. Denote by

$$\mathfrak{L}$$

the set of all functions f such that $\underline{\int} f = \overline{\int} f$ and such that the common value is finite. This common finite value is called the Lebesgue integral of f and denoted by

$$\int f.$$

Note. The assumption of finiteness of the common value is essential. Functions with infinite $\underline{\int} f = \overline{\int} f$ can in general misbehave. We will have functions with infinite Lebesgue integral later, but their class will have to be restricted – see 3.9 below.

3.5. Proposition. *$f \in \mathfrak{L}$ iff for every $\varepsilon > 0$ there exist $g_1 \in \mathbf{Z}^{\mathsf{dn}}$ and $g_2 \in \mathbf{Z}^{\mathsf{up}}$, $g_1 \le f \le g_2$, such that $\mathfrak{I}g_i$ are finite and $\mathfrak{I}g_2 - \mathfrak{I}g_1 < \varepsilon$.*

Proof. The implication $\Rightarrow$ is obvious.

$\Leftarrow$: If g_i have the properties then

$$\mathfrak{I}g_1 \le \underline{\int} f \le \overline{\int} f \le \mathfrak{I}g_2 \le \mathfrak{I}g_1 + \varepsilon$$

so that $\overline{\int} f - \underline{\int} f$ is smaller than any $\varepsilon > 0$. □

3.6. Convention. Functions from $\mathfrak{L}$ can have infinite values. Let us agree that in case of $f(x) = +\infty$ and $g(x) = -\infty$ the value $f(x) + g(x)$ will be chosen arbitrarily. We will see that for our purposes such arbitrariness in the definition of $f + g$ does not matter.

3.7. Proposition. (1) *If $f, g \in \mathfrak{L}$ then $f + g \in \mathfrak{L}$ and one has*

$$\int (f+g) = \int f + \int g.$$

(2) *If $f \in \mathfrak{L}$ then any $\alpha f \in \mathfrak{L}$ and one has*

$$\int \alpha f = \alpha \int f.$$

(3) *If $f, g \in \mathfrak{L}$ then $f \vee g \in \mathfrak{L}$ and $f \wedge g \in \mathfrak{L}$.*
(4) *If $f, g \in \mathfrak{L}$ and $f \le g$ then $\int f \le \int g$.*
(5) *If $f \in \mathfrak{L}$ then $f^+, f^- \in \mathfrak{L}$.*
(6) *If $f \in \mathfrak{L}$ then $|f| \in \mathfrak{L}$ and $|\int f| \le \int |f|$*

Proof. (1) By 3.5. Choose $f_1, g_1 \in \mathbf{Z}^{\mathsf{up}}$ and $f_2, g_2 \in \mathbf{Z}^{\mathsf{dn}}$ such that $f_1 \le f \le f_2$, $g_1 \le g \le g_2$ and $\mathfrak{I}f_1 - \mathfrak{I}f_2 < \varepsilon$, $\mathfrak{I}g_1 - \mathfrak{I}g_2 < \varepsilon$. Then

$$(*) \qquad f_1 + g_1 \le f + g \le f_2 + g_2$$

and the statement follows (realize that the inequalities hold also in the equivocal points from the convention 3.6: if, say, $f(x) = +\infty$ and $g(x) = -\infty$ then $f_2(x) = +\infty$ and $g_1(x) = -\infty$; $f_1(x)$ has to be finite, as a limit of a decreasing sequence of finite numbers, and similarly for $g_2(x)$ so that the inequalities $(*)$ are satisfied trivially).

(2) follows immediately from 3.5.

(3) Take the f_i, g_i as in (1) to obtain

$$f_1 \vee g_1 \le f \vee g \le f_2 \vee g_2 \quad \text{and} \quad f_1 \wedge g_1 \le f \wedge g \le f_2 \wedge g_2$$

and realize that

$$f_2 \vee g_2 - f_1 \vee g_1 \le (f_2 - f_1) + (g_2 - g_1)$$

and similarly for the minimum.

(4) is obvious and (5) follows from (3).

(6) $|\int f| = |\int(f^+ - f^-)| = |\int f^+ - \int f^-| \le \int f^+ + \int f^- = \int |f|$. $\square$

3.8. Lemma. *If $f_n \in \mathfrak{L}$ and if $f_n \nearrow f$ then*

$$\lim \int f_n = \overline{\int} f.$$

Notes before the proof. 1. This lemma is very important and will play a crucial role in the sequel.

2. As $\int f_n \le \underline{\int} f$ we have trivially $\lim \int f_n \le \underline{\int} f$. Hence, under the assumptions it follows from the Lemma that $\lim_n \int f_n = \underline{\int} f = \overline{\int} f$.

Proof. We obviously have $\lim \int f_n \le \overline{\int} f$, and if $\lim \int f_n = +\infty$ the equality is trivial.

Thus, we can assume that the limit is finite. By the definition of $\int f_n$ choose $g_n \in \mathbf{Z}^{\mathsf{up}}$, $g_n \ge f_n$ such that

$$\int f_n + \frac{\varepsilon}{2^{n+1}} > \mathfrak{I} g_n.$$

Set $h_n = \bigvee_{i=1}^n g_i$. Then $h_n \in \mathbf{Z}^{\mathsf{up}}$ and the sequence h_n increases so that by 2.5, $h = \lim h_n \in \mathbf{Z}^{\mathsf{up}}$. Now $h_n \ge g_n \ge f_n$ and hence $h \ge f$, and $\mathfrak{I}h \ge \overline{\int} f$.

Here is an important

Claim.

$$h_n - f_n \le (g_1 - f_1) + (g_2 - f_2) + \cdots + (g_n - f_n).$$

(Indeed, in each point x, some of the summands is $h_n(x) - f_j(x)$ for a $j \le n$. The summands are ≥ 0 and hence the inequality holds for $j = n$; otherwise the sum is $\ge h_n(x) - f_j(x) + g_n(x) - f_n(x) = h_n(x) - f_n(x) + g_n(x) - f_j(x) \ge h_n(x) - f_n(x) + g_n(x) - f_n(x) \ge h_n(x) - f_n(x)$.)

Thus we have

$$\mathfrak{I}h_n - \int f_n \le \sum_{i=1}^n \frac{\varepsilon}{2^{i+1}} < \varepsilon$$

so that $\mathfrak{I}h_n \le \int f_n + \varepsilon$ and finally $\overline{\int} f \le \mathfrak{I}h_n \le \lim \int f_n + \varepsilon$. $\square$

3.9. Some more notation. Set

$\mathfrak{L}^{\mathsf{up}} = \{f \mid \exists f_n \in \mathfrak{L},\ f_n \nearrow f\}$, $\mathfrak{L}^{\mathsf{dn}} = \{f \mid \exists f_n \in \mathfrak{L},\ f_n \searrow f\}$, and $\mathfrak{L}^* = \mathfrak{L}^{\mathsf{up}} \cup \mathfrak{L}^{\mathsf{dn}}$.

Now we obtain from 3.8

3.9.1. Corollary. *For each* $f \in \mathfrak{L}^*$ *we have* $\underline{\int} f = \overline{\int} f$. *Consequently,*

$$\mathfrak{L}^{\mathsf{up}} \cap \mathfrak{L}^{\mathsf{dn}} = \mathfrak{L}.$$

3.9.2. Convention. For $f \in \mathfrak{L}^*$ we will use the symbol $\int f$ for the common value of $\underline{\int} f$ and $\overline{\int} f$, even if it is infinite.

3.9.3. Proposition. *If* $f \in \mathfrak{L}^*$ *and if* $\int f$ *from 3.9.2 is finite then* $f \in \mathfrak{L}$ *and the integral coincides with the standard integral in* $\mathfrak{L}$.

Proof. Let, say, $f \in \mathfrak{L}^{\mathsf{up}}$, let $f_n \nearrow f$ with $f_n \in \mathfrak{L}$. Then by 3.8 and Note 2, $\int f = \lim \int f_n = \overline{\int} f = \underline{\int} f$. $\square$

4. Null sets

4.1. The characteristic function of a subset $M \subseteq \mathbb{E}_m$ will be denoted by

$$c_M.$$

Thus we have

$$M \subseteq N \quad \text{iff} \quad c_M \leq c_N,$$
$$c_{M \cup N} = c_M \vee c_N \quad \text{and} \quad c_{M \cap N} = c_M \wedge c_N,$$

and if $M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq \cdots$ then we have for $M = \bigcup_{n=1}^{\infty} M_n$

$$c_{M_n} \nearrow c_M.$$

M is a *null set* if $\overline{\int} c_M = 0$ (then, since $c_m \geq \mathbf{0}$, we also have $\underline{\int} c_M = 0$ and hence $c_M \in \mathfrak{L}$).

4.2. Proposition. (1) *If* M *is a null set and* $N \subseteq M$ *then* N *is a null set.*

(2) *If* M_n *are null sets then also* $\bigcup_{n=1}^{\infty} M_n$ *is a null set.*

Proof. (1) is trivial. For (2) consider $N_n = M_1 \cup \cdots \cup M_n$. Then $c_{N_n} \leq c_{M_1} + \cdots c_{M_n}$ and hence N_n is a null set by 3.7. Now $c_{N_n} \nearrow c_M$ and hence $\overline{\int} c_M = 0$ by 3.8. $\square$

4.3. Let $V(x)$ be a proposition concerning points of our $\mathbb{E}_m$. We say that

$$V \textit{ holds almost everywhere} \text{ (briefly, } \textit{a.e.})$$

if the set

$$\{x \mid \text{not} V(x)\}$$

is a null one.

The statement that $f(x) = g(x)$ almost everywhere will be indicated by

$$f \sim g.$$

4.4. Proposition. (1) *If $f \in \mathfrak{L}$ then $f(x)$ is almost everywhere finite.*

(2) *If $f \in \mathfrak{L}^{\mathsf{up}}$ (resp. $\mathfrak{L}^{\mathsf{dn}}$) then $f(x) > -\infty$ (resp. $< +\infty$) almost everywhere.*

Proof. (1) Recall the convention on sums on 3.6 and Proposition 3.2(1). For $f + (-f)$ we can choose equally well $\mathbf{0}$ and c_M where $M = \{x \mid f(x) = \pm\infty\}$ and hence $\int c_M = \int \mathbf{0} = 0$.

(2) For $f \in \mathfrak{L}^{\mathsf{up}}$ take $f_n \in \mathfrak{L}$ with $f_n \nearrow f$. Then $\{x \mid f(x) = -\infty\} \subseteq \{x \mid f_1(x) = \pm\infty\}$ and the latter set is a null one by (1). □

4.5. Proposition. *If $f \sim g$ then $\underline{\int} f = \underline{\int} g$ and $\overline{\int} f = \overline{\int} g$.*

Proof will be done for $\overline{\int}$. If we do not have $\overline{\int} f = \overline{\int} g = +\infty$ we can assume that $\overline{\int} f < +\infty$. Set $M = \{x \mid f(x) \neq g(x)\}$ and $r_n = n \cdot c_M$. By 3.8 we have $\overline{\int} r = 0$ for $r = \lim r_n$.

Choose $h_1, h_2 \in \mathbf{Z}^{\mathsf{up}}$ such that $h_1 \geq f$, $h_2 \geq r$, $\mathfrak{I}h_1 < \overline{\int} f + \varepsilon$ and $\mathfrak{I}h_2 < \varepsilon$. Then we have $h_1 + h_2 \in \mathbf{Z}^{\mathsf{up}}$, $h_1 + h_2 \geq g$, and hence $\overline{\int} g \leq \mathfrak{I}h_1 + \mathfrak{I}h_2 < \overline{\int} f + 2\varepsilon$. Thus, $\overline{\int} g \leq \overline{\int} f$, in particular $\overline{\int} g < +\infty$, and we can repeat the procedure with f, g interchanged. □

4.6. Corollary. (1) *If $f \in \mathfrak{L}$ and $f \sim g$ then $g \in \mathfrak{L}$.*

(2) *If $f \in \mathfrak{L}^{\mathsf{up}}$ resp. $\mathfrak{L}^{\mathsf{dn}}$ and $f \sim g$ then $g \in \mathfrak{L}^{\mathsf{up}}$ resp. $\mathfrak{L}^{\mathsf{dn}}$.*

4.7. Proposition. *If $f \geq 0$ and $\overline{\int} f = 0$ then $f \sim \mathbf{0}$.*

Proof. Set $M_n = \{x \mid f(x) \geq \frac{1}{n}\}$. Since $\mathbf{0} \leq c_{M_n} \leq nf$ we have $\overline{\int} c_{M_n} = 0$, hence M_n is a null set, and consequently $\{x \mid f(x) \neq 0\} = \bigcup_{n=1}^{\infty} M_n$ is a null set. □

5. Levi and Lebesgue theorems

5.1. Theorem. (Levi) *Let $f_n \in \mathfrak{L}^{\mathsf{up}}$ and let $f_n \nearrow f$ a.e.. Then $f \in \mathfrak{L}^{\mathsf{up}}$ and $\int f = \lim \int f_n$. Similarly for $f_n \in \mathfrak{L}^{\mathsf{dn}}$ and $f_n \searrow f$.*

Proof. We can assume that $f_n \nearrow f$. Choose $f_{nk} \in \mathfrak{L}$ such that $f_{nk} \nearrow_k f_n$ and set

$$g_n = \bigvee \{f_{ij} \mid i, j \leq n\}.$$

Now $g_n \nearrow g$ with $g_n \in \mathfrak{L}$. Since $g_n \le f$ we have $g \le f$. On the other hand, however, $g_p \ge f_{mp}$ for $p \ge n$ and hence $g \ge f_n$, and finally $g \ge f$ Thus, $f = g \in \mathfrak{L}^{\mathsf{up}}$.

Now about the value of $\int f$. If $\lim \int f_n = +\infty$ the equality is trivial; hence we can assume that $\lim \int f_n$ is finite. Then $f_n \in \mathfrak{L}$ and we can use 3.8 to obtain $\lim \int f_n = \overline{\int} f = \int f$. □

5.2. Theorem. (Lebesgue) *Let $f_n \in \mathfrak{L}$, let $\lim f_n(x) = f(x)$ a.e., and let there exist a $g \in \mathfrak{L}$ such that $|f_n(x)| \le g(x)$ a.e.. Then $f \in \mathfrak{L}$ and $\int f = \lim \int f_n$.*

Note. An attentive reader may worry about the sloppy formulation: does one mean "almost everywhere one has for all n that $|f_n(x)| \le g(x)$" or "for each n one has almost everywhere that $|f_n(x)| \le g(x)$"? But it is an easy exercise to show it is the same.

Proof. Again, we can forget about the provisa of almost everywhere. Set

$$h_n = \bigvee\{f_k \mid k \ge n\}, \quad g_n = \bigwedge\{f_k \mid k \ge n\}.$$

Since $\bigvee_{j=0}^{p} f_{n+j} \nearrow_p h_n$ we have $h_n \in \mathfrak{L}^{\mathsf{up}}$, and similarly $g_n \in \mathfrak{L}^{\mathsf{dn}}$. But we have, moreover,

$$-g \le g_n \le f_n \le h_n \le g$$

and hence $\int g_n$ and $\int h_n$ are finite and we have in fact $g_n, h_n \in \mathfrak{L}$, and consequently $g_n \in \mathfrak{L}^{\mathsf{up}}$ and $h_n \in \mathfrak{L}^{\mathsf{dn}}$ and we can use Levi theorem. Now obviously $g_n \nearrow f$ and $h_n \searrow f$, by Levi theorem we have $\lim \int g_n = \lim \int h_n = \int f$, and finally since $g_n \le f_n \le h_n$ we conclude that $\int f = \lim \int f_n$. □

5.3. Proposition. *Let $g \in \mathfrak{L}$, let $f_n \in \mathfrak{L}^*$, let $f_n \ge g$ a.e. and let $\lim_n f_n(x) = f(x)$ a.e.. Then $f \in \mathfrak{L}^{\mathsf{up}}$. Similarly for $f_n \le f$ we obtain $f \in \mathfrak{L}^{\mathsf{dn}}$.*

Proof. Since $-\infty < \int g \le \int f_n$, $f_n \in \mathfrak{L}^{\mathsf{up}}$ (if $f_n \in \mathfrak{L}^{\mathsf{dn}}$ it has, hence, a finite integral so that, by 3.9.3, $f_n \in \mathfrak{L} \subseteq \mathfrak{L}^{\mathsf{up}}$ as well). Set $\varphi = \sup_n f_n$. We have $\bigvee_{k \le n} f_k \nearrow_n \varphi$ and hence $\varphi \in \mathfrak{L}^{\mathsf{up}}$ by 5.1, and there exist $\varphi_n \in \mathfrak{L}$ such that $\varphi_n \nearrow \varphi$. Obviously $\varphi \ge f \ge g$ and we can assume that $\varphi_n \ge g$ (else replace φ_n by $\varphi_n \vee g$). Set

$$g_{kn} = \varphi_k \wedge f_n.$$

We have $g \le g_{kn} \le \varphi_k$ and hence $g_{kn} \in \mathfrak{L}$ and, moreover, we can use Lebesgue theorem for $\lim_n g_{kn}$ and obtain

$$\varphi_k \wedge f = \lim_n g_{kn} \in \mathfrak{L}.$$

Now we conclude that $\varphi_k \wedge f \nearrow_k f$ and hence $f \in \mathfrak{L}^{\mathsf{up}}$. □

6. The class $\mathbf{\Lambda}$ (measurable functions)

6.1. Stating that $\lim_n f_n = f$ will be abbreviated by writing $f_n \to f$. Set

$$\mathbf{\Lambda} = \{f \mid \exists f_n \in \mathfrak{L},\ f_n \to f\}$$

(unlike in the definition of $\mathfrak{L}^{\mathsf{up}}$ and $\mathfrak{L}^{\mathsf{dn}}$ there is no assumption on the nature of the convergence).

6.2. Proposition. *If $f \sim g$ and $f \in \mathbf{\Lambda}$ then $g \in \mathbf{\Lambda}$.*

Proof. Let $f_n \in \mathfrak{L}$ and $f_n \to f$. Define $M = \{x \mid f(x) \neq g(x)\}$ and set

$$g_n(x) = g(x) \text{ for } x \in M, \quad g_n(x) = f_n(x) \text{ otherwise.}$$

Then by 4.6, $g_n \in \mathfrak{L}$. □

6.3. From 5.3 we immediately infer

Corollary. *If $f \in \mathbf{\Lambda}$ and $f \geq 0$ then $f \in \mathfrak{L}^{\mathsf{up}}$.*

6.4. The following is trivial.

Proposition. (a) *If $f, g \in \mathbf{\Lambda}$ and if $f + g$ makes sense a.e. then $f + g \in \mathbf{\Lambda}$.*

(b) *If $f \in \mathbf{\Lambda}$ and $\alpha \in \mathbb{R}$ then $\alpha f \in \mathbf{\Lambda}$.*

(c) *If $f, g \in \mathbf{\Lambda}$ then $f \vee g, f \wedge g \in \mathbf{\Lambda}$.*

(d) *If $f \in \mathbf{\Lambda}$ then $|f| \in \mathbf{\Lambda}$.*

6.5. Proposition. *$f \in \mathbf{\Lambda}$ if and only if both f^+ and f^- are in $\mathfrak{L}^{\mathsf{up}}$.*

Proof. If f_n are in $\mathfrak{L}$ and $f_n \to f$ then obviously $f_n^+ \to f^+$ and $f_n^- \to f^-$. Use 6.3. The other implication is trivial. □

6.5.1. Corollary. *Let $f \in \mathbf{\Lambda}$ and let there be a $g \in \mathfrak{L}$ such that $|f| \leq g$. Then $f \in \mathfrak{L}$.*

6.6. Proposition. *If $f_n \in \mathbf{\Lambda}$ and if $f_n \to f$ a.e. then $f \in \mathbf{\Lambda}$.*

Proof. We have $f_n^+, f_n^- \in \mathfrak{L}^{\mathsf{up}}$ and $f_n^+ \to f^+$, $f_n^- \to f^-$. Thus, by 5.3, both f^+ and f^- are in $\mathfrak{L}^{\mathsf{up}}$. □

6.7. Proposition. *$f \in \mathfrak{L}^*$ iff f^+ and f^- are in $\mathfrak{L}^{\mathsf{up}}$ and if the difference $\int f^+ - \int f^-$ makes sense.*

Consequently, $f \in \mathbf{\Lambda} \smallsetminus \mathfrak{L}^*$ iff f^+ and f^- are in $\mathfrak{L}^{\mathsf{up}}$ and $\int f^+ = \int f^- = +\infty$.

Proof. $\Rightarrow$: Let, say, $f \in \mathfrak{L}^{\mathsf{up}}$ and let $f_n \nearrow f$ and $f_n \in \mathfrak{L}$. As $f_1 = f_1^+ - f_1^- \leq f = f^+ - f^-$ we have $f^- \leq f_1^- \in \mathfrak{L}$ and hence the value of $\int f^-$ is finite.

$\Leftarrow$: If $\int f^+ - \int f^-$ makes sense then at least one of the integrals is finite and either f^+ or f^- is in $\mathfrak{L}$. Thus, $f^+ - f^-$ is either in $\mathfrak{L}^{\mathsf{up}}$ or in $\mathfrak{L}^{\mathsf{dn}}$. $\square$

6.8. Remark. Some of the statements in this section may be somewhat surprising. It turns out that for integrability of a limit of integrable functions, the nature of the limiting process is not very important: all what one needs is that the positive and negative parts of the limit are not both infinite.

For the value of the integral of the limit, however, the nature of the convergence is of the essence.

Example. Define functions f_n, g_n on $\mathbb{E}_1$ by setting

$$f_n(x) = \begin{cases} 0 & \text{for} \quad x \neq 0, \\ n & \text{for} \quad x = 0, \end{cases} \qquad g_n(x) = \begin{cases} 0 & \text{for} \quad x \leq -\frac{1}{n} \quad \text{and} \quad x \geq \frac{1}{n}, \\ n + n^2 x & \text{for} \quad -\frac{1}{n} \leq x \leq 0, \\ n - n^2 x & \text{for} \quad 0 \leq x \leq \frac{1}{n}. \end{cases}$$

Then $f_n \nearrow f$ and $g_n \to f$ where $f(x) = 0$ for $x \neq 0$ and $f(0) = +\infty$, $\int f_n = \int f = 0$ and $\int g_n = 1$. The functions g_n converge to f the wrong way.

7. Lebesgue measure

7.1. A set $A \subseteq \mathbb{E}_m$ is said to be *(Lebesgue) measurable* if the characteristic function c_A is in $\mathbf{\Lambda}$ (then, of course, it is in $\mathfrak{L}^{\mathsf{up}}$, by 6.3). We set

$$\mu(A) = \int c_A$$

and call $\mu(A)$ the *(Lebesgue) measure* of A.

Note that the null sets from 3.4 are precisely the measurable sets with measure zero.

7.2. General facts. For measurable A, B we have $A \cup B$ measurable and

$$\mu(A \cup B) \leq \mu(A) + \mu(B)$$

(by 6.4 we have $c_{A\cup B} = c_A \vee c_B \quad (\leq c_A + c_B)$ in $\mathbf{\Lambda}$) and if A, B are disjoint then

$$\mu(A \cup B) = \mu(A) + \mu(B) \tag{7.2.1}$$

as then $c_{A\cup B} = c_A + c_B$.

But we have much more: the measure is countably additive (σ-additive, as this fact is usually expressed). Here are some facts on measurability.

Proposition. (1) *Let A_n, $n = 1, 2, \dots,$ be measurable. Then $\bigcup_{n=1}^{\infty} A_n$ is measurable. If for any two n, k the intersection $A_n \cap A_k$ is a null set then*

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n).$$

(2) *The intersection of a countable system of measurable sets is measurable.*

(3) *If A, B are measurable then the difference $A \smallsetminus B$ is measurable.*

(4) *$\mu(\emptyset) = 0$ and for measurable $A \subseteq B$, $\mu(A) \leq \mu(B)$.*

Proof. (1) We have $c_{A_1\cup\dots\cup A_n} \nearrow_n c_{\bigcup_{n=1}^{\infty} A_n}$ and hence $c_{\bigcup_{n=1}^{\infty} A_n} \in \mathfrak{L}^{\mathsf{up}}$. In the almost disjoint case we obtain the value from the finite additivity (7.2.1) and from Levi Theorem.

(2) We have $c_{A_1\cap\dots\cap A_n} \searrow^n c_{\bigcap_{n=1}^{\infty} A_n}$.

(3) $c_{A\smallsetminus B} = (c_A - c_B) \vee \mathbf{0}$.

(4) is trivial. □

7.3. Special sets.

Proposition. (1) *Each open set in $\mathbb{E}_m$ is measurable.*

(2) *Each closed set in $\mathbb{E}_m$ is measurable.*

(3) *For the interval $J = \langle a_1, b_1\rangle \times \dots \times \langle a_m, b_m\rangle$ one has*

$$\mu(J) = (b_1 - a_1)(b_2 - a_2)\cdots(b_n - a_n).$$

(4) *Each countable set is measurable, with measure 0.*

Proof. The distance in $\mathbb{E}_m$ will be denoted by $\rho(x, y)$.

(1) It suffices to show that bounded open sets are measurable: for a general open U consider the open balls $\mathbb{B}_n = \{x \mid \rho(x, (0, \dots, 0)) < n\}$ and use Proposition 7.2(1) for $U = \bigcup_n U \cap \mathbb{B}_n$.

Thus, let U be a bounded open set. Set

$$A_n = \{x \mid \rho(x, \mathbb{E}_m \smallsetminus U) \geq \frac{1}{n}\}$$

and define $f_n : \mathbb{E}_m \to \mathbb{R}$ by

$$f_n(x) = \frac{\rho(x, \mathbb{E}_m \smallsetminus U)}{\rho(x, \mathbb{E}_m \smallsetminus U) + \rho(x, A_n)}.$$

Since A_n and $\mathbb{E}_m \smallsetminus U$ are disjoint closed sets, f_n is a continuous map. Since $f_n(x) = 0$ for $x \notin U$, we have $F_n \in \mathbf{Z} \subseteq \mathbf{\Lambda}$. Now if $x \in U$ then

$\rho(x, \mathbb{E}_m \smallsetminus U) \geq \frac{1}{n_0}$ for some n_0 and hence $x \in A_n$ – and $f_n(x) = 1$ – for all $n \geq n_0$. Thus,

$$f_n \to c_U$$

and $c_U \in \mathbf{\Lambda}$.

(2) Use (1) and 7.2(3).

(3) Note that for a bounded closed set C we can use a similar procedure like in (1): this time set

$$A_n = \{x \mid \rho(x, C) \geq \frac{1}{n}\}$$

and define $f_n : \mathbb{E}_m \to \mathbb{R}$ by

$$f_n(x) = \frac{\rho(x, A_n)}{\rho(x, A_n) + \rho(x, C)}.$$

Now obviously $f_n(x) = 1$ for $x \in C$ and $f_n(x) = 0$ for $\rho(x, C) \geq \frac{1}{n}$ if n. Furthermore, if $k \geq n$ then $\rho(x, A_k) \leq \rho(x, A_n)$, and $f_k(x) \leq f_n(x)$. Thus,

$$f_n \searrow c_C.$$

In particular this holds for the interval J. Moreover, $f_n(x) = 0$ outside

$$\langle a_1 - \frac{1}{n}, b_1 + \frac{1}{n}\rangle \times \cdots \times \langle a_m - \frac{1}{n}, b_m + \frac{1}{n}\rangle$$

and $0 \leq f_n(x) \leq 1$ so that by the standard estimate of Riemann integrals

$$(b_1 - a_1) \cdots (b_n - a_n) \ \leq \ \int f_n \ \leq \ (b_1 - a_1 + \frac{2}{n}) \cdots (b_n - a_n + \frac{2}{n})$$

and $\int c_J = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n)$ by Levi Theorem (actually already by Dini Theorem).

(4) By (3), $\mu(\{x\}) = 0$. Use 7.2(1). □

7.4. Borel sets. The smallest class of subsets of $\mathbb{E}_m$ containing all open subsets and closed under

- complements,
- countable unions, and
- countable intersections

(of course, the last follows from the first two) is called the class of *Borel sets*.

Thus one has here e.g. all the open sets, all the closed ones, the G_δ, F_σ, $G_{\sigma\delta}$ etc. – the sets of the Baire classification.

From 7.2 and 7.3 we immediately obtain

7.4.1. Corollary. *Each Borel set is measurable.*

7.5. Let us finish this section with a trivial remark. From 7.2(1) and 7.3(1) we immediately obtain the often used somewhat paradoxical observation that for every $\varepsilon > 0$ there is a dense open set U of the unit interval I such that $\mu(U) < \varepsilon$: order all the rationals in I in a sequence $r_1, r_2, \dots, r_n, \dots$ and set

$$U = \bigcup_{n=1}^{\infty}(r_n - \frac{1}{2^{n+2}}, r_n + \frac{1}{2^{n+2}})$$

(where (a, b) designate open intervals).

8. The integral over a set

8.1. Unlike the additivity of the classes $\mathfrak{L}$ etc. we do not have anything like multiplicativity. Nevertheless, multiplying by measurable c_M gives satisfactory results.

Proposition. *Let M be a measurable set and let f be in $\mathfrak{L}$. Then $c_M \cdot f$ is in $\mathfrak{L}$.*

Proof. Set $\varphi_n = nc_M \wedge (f \vee (-n \cdot c_M))$. Then $\varphi_n \in \mathbf{\Lambda}$ and since $|\varphi_n| \leq |f|$ we have $c_M f = \lim \varphi_n$ in $\mathfrak{L}$ by 6.5.1. $\square$

8.2. By 8.1. we can define for measurable M and $f \in \mathfrak{L}$,

$$\int_M f \quad \equiv_{\mathrm{df}} \quad \int c_M f,$$

the *integral of f over M*.

8.3. Sometimes we have a real function defined on a measurable set M only. Then, before determining about an "integral of f over M" one has to decide whether f can be extended to a function $\overline{f}$ on the whole of $\mathbb{E}_m$ belonging to $\mathfrak{L}$ (by 3.1 this amounts to deciding whether $\overline{f}$ defined by

$$\overline{f}(x) = \begin{cases} f(x) & \text{for } x \in M, \\ 0 & \text{otherwise} \end{cases} \tag{3.3.1}$$

is in $\mathfrak{L}$. This is often obvious (and often, of course, we have another obvious extension). Here is an expedient

Proposition. *Let M be compact. Consider the class of all the functions f obtainable from continuous maps $g : M \to \mathbb{R}$ by subsequent taking limits. Then all the associated $\overline{f}$ are in $\mathbf{\Lambda}$.*

Thus, $\overline{f}^+$ and $\overline{f}^-$ are in $\mathfrak{L}^{\mathsf{up}}$, and if at least one of the $\int \overline{f}^+$ and $\int \overline{f}^-$ is finite then $\int_M f$ makes sense.

Proof. By Tietze theorem we can extend a continuous $f : M \to \mathbb{R}$ to a continuous $f : \mathbb{E}_n \to \mathbb{R}$ such that

$$f_n(x) = 0 \quad \text{for} \quad \rho(x, M) \geq \frac{1}{n}, \quad \text{and}$$
$$a \leq f_n(x) \leq b \quad \text{where} \quad a = \min_M f(x) \quad \text{and} \quad b = \max_M f(x).$$

Then f_n converge to the $\overline{f}$ from (3.3.1) and by Lebesgue theorem, $\overline{f} \in \mathfrak{L}$. Now replace the limits $g_n \to g$ in the subsequent constructions by $\overline{g}_n \to \overline{g}$.

8.4. Proposition. *Let M_n, $n = 1, 2, \dots$ be measurable.*

(a) *Let for $n \neq k$, M_n, M_k be almost disjoint, $f \in \mathbf{\Lambda}$, and let for $M = \bigcup M_n$ $\int_M f$ make sense. Then*

$$\int_M f = \sum_{n=1}^{\infty} \int_{M_n} f.$$

(b) *Let $M_1 \subseteq M_2 \subseteq \cdots$, $M = \bigcup M_n$ and let $\int_M f$ make sense. Then*

$$\int_M f = \lim_n \int_{M_n} f.$$

(c) *Let $M_1 \supseteq M_2 \supseteq \cdots$, $M = \bigcap M_n$ and let $\int_M f$ make sense. Then*

$$\int_M f = \lim_n \int_{M_n} f.$$

Proof. For $f \geq \mathbf{0}$ the statement immediately follows from Levi theorem and the fact that the sum formula obviously holds for finitely many M_n. Thus, we have the equality for f^+ and f^-. Now if $\int_M f$ makes sense then by 6.7 one of $\int_m f^+$, $\int_m f^-$ is finite, and hence at least one of the series $\sum_{n=1}^{\infty} \int_{M_n} f^+$, $\sum_{n=1}^{\infty} \int_{M_n} f^-$ converges, and since the summands are non-negative, it converges absolutely. Thus,

$$\int_M f = \int_M f^+ - \int_M f^- = \sum_{n=1}^{\infty} \int_{M_n} f^+ - \sum_{n=1}^{\infty} \int_{M_n} f^- = \sum_{n=1}^{\infty} \int_{M_n} (f^+ - f^-),$$

the last reshuffling being made possible by the absolute convergence of at least one of the series (and the other being a sum of non-negative numbers).

(b) Apply (a) for $M_1, M_2 \smallsetminus M_1, M_3 \smallsetminus M_2, \dots$.

(c) Set $N_n = M_1 \smallsetminus M_n$. Then $M = M_1 \smallsetminus \bigcup N_n$. Use (b). □

8.4.1. Note. For the general statement the assumption that $\int_M f$ make sense is essential. The point is that we could have both $\int_M f^+$

and $\int_M f^-$ infinite. If this is excluded by some other information, we do not have to make the assumption explicitly.

9. Parameters

This section is included for completeness sake only. Unlike in the others, there is nothing specific Daniel here: we just use the already proved Lebesgue theorem.

9.1. Theorem. *Let T be a metric space, $t_0 \in T$, and let $f : T \times \mathbb{E}_m \to \mathbb{R} \cup \{+\infty, -\infty\}$ be a function such that*

(1) *for almost all x, $f(-, x)$ is continuous in a point t_0,*
(2) *there is a neighbourhood U of t_0 such that the functions $f(t, -)$ belong to $\mathfrak{L}$ for all $t \in U \setminus \{t_0\}$, and*
(3) *there exists a $g \in \mathfrak{L}$ and a neighbourhood U of t_0 such that for almost all x and for all $t \in U \setminus \{t_0\}$ one has $|f(t, x)| \leq g(x)$.*

Then $f(t_0, -)$ is in $\mathfrak{L}$ and we have

$$\int f(t_0, -) = \lim_{t \to t_0} \int f(t, -).$$

Proof. Choose $t_n \in U \setminus \{t_0\}$ such that $\lim_n t_n = t_0$ and use Lebesgue Theorem. □

9.2. Theorem. *Let $f : \mathbb{R} \times \mathbb{E}_m \to \mathbb{R} \cup \{+\infty, -\infty\}$ be such that in a neighbourhood U of t_0*

(1) *there exist partial derivatives $\frac{\partial f(t,x)}{\partial t}$ for almost all x,*
(2) *there is a $g \in \mathfrak{L}$ such that for almost all x and for all $f \in U$ one has*
$$\left| \frac{\partial f(t, x)}{\partial t} \right| \leq g(x),$$
(3) *and for $t \in U$ there exist $\int f(t, -)$.*

Then there exist the integral $\int \frac{\partial f(t_0, -)}{\partial t}$ and one has

$$\int \frac{\partial f(t_0, -)}{\partial t} = \frac{\mathrm{d}}{\mathrm{dt}} \int f(t_0, -).$$

Proof. We have $\frac{\partial f(t_0,x)}{\partial t} = \lim_{h \to 0} \frac{1}{h}(f(t_0 + h, x) - f(t_0, x))$. Set $\varphi(h, x) = \frac{1}{h}(f(t_0 + h, x) - f(t_0, x))$. By Lagrange theorem we have

$$|\varphi(h, x)| = \left| \frac{\partial f(t_0 + \theta h, x)}{\partial t} \right| \leq g(x)$$

and hence we can apply Theorem 9.1. □

10. Fubini Theorem

In this section we will have to indicate the Euclidean space in which we work. In case of $\mathbb{E}_m$ we will specify the $\mathbf{Z}, \mathbf{Z}^{\mathsf{up}}, \mathfrak{L}$ etc. as $\mathbf{Z}_m, \mathbf{Z}^{\mathsf{up}}_m, \mathfrak{L}_m$ etc., and for the integral symbols we will use $\int^{(m)}, \underline{\int}^{(m)}, \overline{\int}^{(m)}$ instead of plain $\int, \underline{\int}, \overline{\int}$.

We will abandon the integral symbol $\mathfrak{I}$ since we already know that for $f \in \mathbf{Z}^*$ we have $\mathfrak{I}f = \int f$.

Finally, to avoid confusion in the case of two variables we will sometimes use the classical

$$\int f(x,y)\mathrm{d}y \ \text{ or } \int f(x,y)\mathrm{d}x \quad \text{for} \quad \int f(x,-) \ \text{ or} \int f(-,y).$$

10.1. Lemma. *For a function f defined on $\mathbb{E}_{m+n}$ define functions $\overline{F}$ and $\underline{F}$ on $\mathbb{E}_m$ by setting*

$$\overline{F}(x) = \overline{\int}^{(m+n)} f(x,y)\mathrm{d}y \quad (\textit{resp.} \quad \underline{F}(x) = \underline{\int}^{(m+n)} f(x,y)\mathrm{d}y\ ).$$

Then one has

$$\overline{\int}^{(m+n)} f \geq \overline{\int}^{(m)} \overline{F} \quad (\textit{resp.} \quad \underline{\int}^{(m+n)} f \leq \underline{\int}^{(m)} \underline{F}\ ).$$

Proof. I. If $f \in \mathbf{Z}_{m+n}$ then we have equalities, by the standard Fubini theorem for continuous maps on compact intervals. Furthermore we have for $F = \overline{F} = \underline{F}$,

$$F \in \mathbf{Z}_m.$$

Indeed, choose a compact interval J carrying the function f. The function F obviously has a compact carrier, namely the projection of J (the values elsewhere are integrals of $\mathbf{0}$). Further, let K be the volume of J. For an $\varepsilon > 0$ there is a $\delta > o$ such that for $\rho(x,x') < \delta$, $|f(x,y) - f(x',y)| < \frac{\varepsilon}{K}$, independently on y. Thus we have

$$\left| \int F(x) - \int F(x') \right| < \frac{\varepsilon}{K} \cdot K = \varepsilon,$$

and F is continuous.

II. Now let $f_k \in \mathbf{Z}_{m+n}$, $f_k \nearrow_k f$. Then

$$F_k(x) = \int f_k(x,y)\mathrm{d}y \nearrow F(x) \quad \text{and also} \quad f_k(x,-) \nearrow f(x,-)$$

for all y. Thus we still have

$$\int^{(m+n)} f(x,y)\mathrm{d}y = \lim_k \int^{(m+n)} f_k = \lim_k \int^{(m)} F_k = \int^{(m)} F.$$

III. Now let f be general and let $g \in \mathbf{Z}^{\mathsf{up}}$ be such that $g \geq f$. Set $G(x) = \int^{(m+n)} g(x,y)\mathrm{d}y$, Then $G \geq F$ and by II we have

$$\int^{(m+n)} g = \int^{(m)} G \geq \overline{\int}^{(m)} \overline{F}$$

and hence

$$\overline{\int} f = \inf\{\int g \mid g \in \mathbf{Z}^{\mathsf{up}},\ g \geq f\} \geq \overline{\int} \overline{F}. \qquad \square$$

10.2. Theorem. (Fubini) *Let $f \in \mathfrak{L}^*_{m+n}$. Then for almost all x there exists the integral $\int^{(m+n)} f(x,y)\mathrm{d}y$. If we denote its value by $F(x)$, and define the values $F(x)$ arbitrarily in the remaining points, we have $F \in \mathfrak{L}^*_m$ and*

$$\int^{(m+n)} f = \int^{(m)} F.$$

Proof. Set $\overline{F}(x) = \overline{\int} f(x,y)\mathrm{d}y$ and $\underline{F}(x) = \underline{\int} f(x,y)\mathrm{d}y$. By Lemma 10.1 we have

$$\int f = \overline{\int} f \geq \overline{\int} \overline{F} \geq \left\{ \begin{array}{c} \overline{\int} \underline{F} \\ \underline{\int} \overline{F} \end{array} \right\} \geq \underline{\int} \underline{F} \geq \underline{\int} f = \int f.$$

Let f be in $\mathfrak{L}_{m+n}$. Then the values are finite and we obtain, first of all, that $\overline{\int} \overline{F} = \underline{\int} \overline{F}$ finite and hence $\overline{F} \in \mathfrak{L}_m$, and similarly $\underline{F} \in \mathfrak{L}_m$. Further, $\int \overline{F} = \int \underline{F}$ and hence $\int (\overline{F} - \underline{F}) = 0$ and hence $\overline{F} - \underline{F} = 0$ almost everywhere, by 4.7. If $f \in \mathfrak{L}^*_{m+n}$ use Levi theorem. $\square$

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