# Are limits of presheaves really pointwise?

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Given a small category $\mathcal{C}$, the corresponding presheaf category $\mathcal{V} = [\mathcal{C}^{\mathsf{op}}, \mathsf{S}et]$ is a complete, cocomplete cartesian closed category. We are interested here in the consideration of weighted $\mathcal{V}$-limits and $\mathcal{V}$-colimits.

Given a small enriched $\mathcal{V}$-category $\mathcal{D}$ and two $\mathcal{V}$-functors

$$F\colon \mathcal{D} \to \mathcal{V}, \quad G\colon \mathcal{D} \to \mathcal{V}$$

we can compose these with the evalution functor at some object $C \in \mathcal{C}$

$$\mathsf{e}v_C\colon \mathcal{V} = [\mathcal{C}^{\mathsf{op}}, \mathsf{Set}] \to \mathsf{Set}, \quad H \mapsto H(C)$$

and ask the question: is it the case that

$$(\lim_G F)(C) = \lim_{(\mathsf{e}v_C \circ G)}(\mathsf{e}v_C \circ F)$$

and analogously for weighted colimits (with then $G$ contravariant). The answer is negative in general for weighted limits, but turns out to be positive for weighted colimits. Let us remember that an analogous observation holds for internal limits and colimits in the topos $\mathcal{V} = [\mathcal{C}^{\mathsf{op}}, \mathsf{Set}]$.

The covariant weight $G$ above is $\mathcal{V}$-*finite* when $\mathcal{D}$ has finitely many objects, while each $\mathcal{D}(D, D')$ and each $G(D)$ are finitely presentable in $\mathcal{V}$. A weighted $\mathcal{V}$-colimit is *filtered* when it commutes with all finite weighted $\mathcal{V}$-limits.

Under a mild condition on $\mathcal{C}$ — the existence of finite weak multi-limits (which certainly holds when $\mathcal{C}$ is finitely cocomplete) — we prove that the filtered $\mathcal{V}$-weights are precisely those which are pointwise filtered over $\mathsf{Set}$. This applies in particular when $\mathcal{V}$ is the category of simplicial sets.

References


[1] G.M. Kelly, *Basic concepts of Enriched Category Theory*, London Math. Soc. Lect. Note Series **64**, Cambridge Univ. Press (1982)

[2] J. Adàmek and J. Rosický *Locally presentable and accessible categories*, Cambridge Univ. Press (1994)


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