Elgot Monads

Stefan Milius*

It was the idea of Calvin Elgot [4] to use Lawvere theories for the study of the semantics of recursion. He introduced iterative theories as those Lawvere theories in which every ideal morphism $e : n \to n+p$ (representing a system of recursive equations in n variables and p parameters) has a unique solution, i. e., a unique morphism $e^\dagger$ such that the equation $e^\dagger = [e^\dagger, \mathsf{id}_p] \cdot e$ holds. Elgot proved that for a finitary signature Σ the theory R_Σ of rational trees over Σ is a free iterative theory on Σ.

Later Stephen Bloom and Zoltan Ésik introduced iteration theories where there is an operation $(-)^\dagger$ assigning to every morphism e as above a solution $e^\dagger$ subject to additional axioms, see [3]. They then prove a completeness theorem stating that iteration theories axiomatize all identities valid for the least fixed point operation in domains. We recently gave an category theoretic explanation of this completeness theorem, see [2]. In fact, Bloom and Ésik had proved that the rational tree theory $R_{\Sigma_\perp}$, where $\Sigma_\perp$ is the finitary signature Σ extended by a new constant symbol $\perp$, is the free iteration theory on Σ. More precisely, the forgetful functor U from the category ITh of iteration theories to the category $\mathsf{Sgn} = \mathsf{Set}/\mathbb{N}$ has a left adjoint given by $\Sigma \mapsto R_{\Sigma_\perp}$. We proved that, moreover, U is monadic, and from this result one can obtain the completeness theorem mentioned before.

In recent years, we have studied a category-theoretic generalization and extension of the classical work of Elgot, see [1]. We proved that for every finitary endofunctor H of a locally finitely presentable category there is a free iterative monad R_H on H, and we gave a coalgebraic construction of R_H.

In this talk we present a generalized version of (our monadicity result for) iteration theories. We work with a base category $\mathcal{A}$ that is locally finitely presentable and hyper extensive, i. e., countable coproducts are disjoint and universal and, in addition, the copairing of a countable family of disjoint coproduct components is itself a coproduct component. In this setting we define the notion of an Elgot monad, viz. a finitary monad M together with an operation $(-)^\dagger$ that assigns to any equation morphism $e : X \to M(X + Y)$, X finitely presentable, a morphism $e^\dagger : X \to MY$ subject to certain axioms. These axioms resemble very much the axioms of iteration theories. One example of an Elgot monad is the finite power set functor. More generally, any finitary monad M is an Elgot monad on $\mathcal{A}$ if and only if the full subcategory of the Kleisli category $\mathcal{A}_M$ given by the finitely presentable objects of $\mathcal{A}$ is traced cocartesian and the trace is uniform for all morphisms $\eta_B \cdot f : A \to MB$, where $f : A \to B$ is

*Joint work with Jiří Adámek and Jiří Velebil.

a morphism of $\mathcal{A}$ and η is the unit of the monad M. So, for example, all iteration theories satisfying the functorial dagger implication for base morphisms give rise to Elgot monads on Set.

Let C_1 denote the constant functor on the terminal object of $\mathcal{A}$.

Theorem 1. *For every finitary functor H of $\mathcal{A}$ the rational monad R_{H+C_1} is a free Elgot monad on H.*

In other words the assignment $H \mapsto R_{H+C_1}$ gives rise to a left adjoint to the forgetful functor V from the category $\mathsf{Elgot}(\mathcal{A})$ of Elgot monads on $\mathcal{A}$ to the category $\mathsf{Fin}(\mathcal{A})$ of finitary endofunctors of $\mathcal{A}$. In addition, we prove

Theorem 2. *The functor $V : \mathsf{Elgot}(\mathcal{A}) \to \mathsf{Fin}(\mathcal{A})$ is monadic.*

This theorem states that Elgot monads can be understood as Eilenberg-Moore algebras for the monad $\mathbb{R}\mathrm{at}$ given by the adjunction from Theorem 1.

References

[1] J. Adámek, S. Milius, J. Velebil, *Iterative Algebras at Work*, Math. Structures Comput. Sci. 16 (2006), no. 6, 1085-1131.

[2] J. Adámek, S. Milius, J. Velebil, *What are Iteration Theories?*, in Proceedings of MFCS 2007, Lecture Notes Comput. Sci. 4708, 240-252.

[3] S. L. Bloom and Z. Ésik, *Iteration Theories: The Equational Logic of Iterative Processes*, EATCS Monograph Series on Theoretical Computer Science, Springer-Verlag, 1993.

[4] C. C. Elgot, *Monadic Computation and Iterative Algebraic Theories*, in: Logic Colloquium '73 (eds: H. E. Rose and J. C. Shepherdson), North-Holland Publishers, Amsterdam, 1975.