On Morita Equivalence of Categories

Bertalan Pécsi

october 2007.

Heteromorphisms

Def. 1.1. Category $\mathbb{H}$ is a *bridge* $\mathbb{A} \rightleftharpoons \mathbb{B}$, if

· $\mathbb{A}$, $\mathbb{B}$ are disjoint, full subcategories of $\mathbb{H}$,

· $\mathtt{Ob}\mathbb{A} \cup \mathtt{Ob}\mathbb{B} = \mathtt{Ob}\mathbb{H}$.

$\mathbb{H}$ is *directed bridge* $\mathbb{A} \Rightarrow \mathbb{B}$, if moreover

· $(b \mid a)_{\mathbb{H}} = \emptyset$ for all $a \in \mathtt{Ob}\mathbb{A}, b \in \mathtt{Ob}\mathbb{B}$.

Let $\mathbb{H}, \mathbb{K}$ be bridges $\mathbb{A} \rightleftharpoons \mathbb{B}$.
A functor $T : \mathbb{H} \to \mathbb{K}$ is a *bridge morphism*, if

· $T\restriction_{\mathbb{A}} = \mathrm{id}_{\mathbb{A}}$ and $T\restriction_{\mathbb{B}} = \mathrm{id}_{\mathbb{B}}$.

Prop. 1.2. Directed bridges $\mathbb{A} \Rightarrow \mathbb{B}$ are just the profunctors (i.e. functors $\mathbb{A}^{op} \times \mathbb{B} \to \mathbb{S}\mathrm{et}$), corresponding (bridge morphisms $\leftrightsquigarrow$ nat. transformations).

Examples:

Let $\mathbb{A}$ be arbitrary, and $\mathbb{B} \leqslant \mathbb{A}$ be a full subcat.

bridge	heteromorphisms
$\mathbb{S}\mathsf{et} \Rightarrow \mathbb{G}\mathsf{rp}$	functions $S \to G$
$\mathbb{A}\mathsf{b} \times \mathbb{A}\mathsf{b} \Rightarrow \mathbb{A}\mathsf{b}$	bilinear morphisms $A \times B \to C$
$\mathbb{A} \Rightarrow \mathbb{A} \times \mathbb{A}$	cones $a \begin{array}{l} \nearrow b \\ \searrow c \end{array}$
$\mathbb{S}\mathsf{et} \Rightarrow \mathbb{S}\mathsf{et}^{op}$	relations between A and B
$\mathbb{B} \rightleftharpoons \mathbb{A}$	copies of arrows $b \to a$ & $a \to b$

A bridge between monoids is just a category with 2 objects.

Profunctors

Theorem 1.3. Let $\mathbb{L} : \mathbb{A} \Rightarrow \mathbb{B}$ be a profunctor.

· $\mathbb{L}$ is induced by a functor $\mathbb{A} \to \mathbb{B}$
iff $\mathbb{B} \leqslant \mathbb{L}$ is reflective.

· $\mathbb{L}$ is induced by a functor $\mathbb{B} \to \mathbb{A}$
iff $\mathbb{A} \leqslant \mathbb{L}$ is coreflective.

· $\mathbb{L}$ is induced by an adjunction $\mathbb{A} \dashv \mathbb{B}$
iff $\mathbb{B} \leqslant \mathbb{L}$ is reflective and $\mathbb{A} \leqslant \mathbb{L}$ is coreflective.

Def. 1.4. *Composition* of $\underset{\mathbb{A}\Rightarrow\mathbb{B}}{\mathbb{F}}$ and $\underset{\mathbb{B}\Rightarrow\mathbb{C}}{\mathbb{G}}$:

$$(a \shortmid c)_{\mathbb{F}\cdot\mathbb{G}} := \{\langle f, g\rangle \mid a \xrightarrow{f} b \xrightarrow{g} c, \ \ {\scriptstyle b\in \mathtt{Ob}\mathbb{B}}\}\big/\!\sim$$

where $\langle f\beta, g\rangle \sim \langle f, \beta g\rangle$ for $\beta \in \mathtt{Mor}\mathbb{B}$.

$\rightsquigarrow$ the bicategory $\mathbb{P}$rof
(of categories, profunctors, bridge morphisms).

Note. A bridge $\mathbb{H} : \mathbb{A} \rightleftharpoons \mathbb{B}$ is determined by its *parts* $\mathbb{H}^{>} : \mathbb{A} \Rightarrow \mathbb{B}$, $\mathbb{H}^{<} : \mathbb{B} \Rightarrow \mathbb{A}$,
and compositions $\mathbb{H}^{>}\cdot\mathbb{H}^{<} \to \mathbb{A}$, $\mathbb{H}^{<}\cdot\mathbb{H}^{>} \to \mathbb{B}$.

Equivalences

Def. 2.1. $\mathbb{H} : \mathbb{A} \rightleftharpoons \mathbb{B}$ is an *equivalence bridge*, if
$\forall a \in \mathsf{Ob}\mathbb{A}\ \exists b \in \mathsf{Ob}\mathbb{B} : a \overset{\simeq}{=} b$ in $\mathbb{H}$, and
$\forall b \in \mathsf{Ob}\mathbb{B}\ \exists a \in \mathsf{Ob}\mathbb{A} : a \overset{\simeq}{=} b$ in $\mathbb{H}$.

Theorem 2.2. $\mathbb{A} \simeq \mathbb{B}$ iff $\exists \mathbb{A} \rightleftharpoons \mathbb{B}$ equiv. bridge.

Note. Axiom of choice is used in constructing a functor from an equivalence bridge.
(cf. Makkai: "Avoiding the Axiom of Choice...")

Def. 2.3. $\mathbb{M} : \mathbb{A} \rightleftharpoons \mathbb{B}$ is a *Morita bridge*, if every morphism is composition of heteromorphisms.

Def. 2.4 (*Idempotent completion*).

· $\mathtt{Ob}(\mathbb{A}^{id}) := \{e \in \mathtt{Mor}\mathbb{A} \mid e^2 = e\}$,

· $(e \mid f)_{\mathbb{A}^{id}} := \{\alpha \mid e\alpha f = \alpha\}$.

Theorem 2.5. The followings are equivalent:

a) There are profunctors $\underset{\mathbb{A}\Rightarrow\mathbb{B}}{\mathbb{F}}$, $\underset{\mathbb{B}\Rightarrow\mathbb{A}}{\mathbb{G}}$, such that $\mathbb{F}\cdot\mathbb{G} \cong \mathbb{A}$ and $\mathbb{G}\cdot\mathbb{F} \cong \mathbb{B}$.

b) There is a Morita bridge $\mathbb{M} : \mathbb{A} \rightleftharpoons \mathbb{B}$.

c) $\mathbb{A}^{id} \simeq \mathbb{B}^{id}$.

Theorem 2.5. The followings are equivalent:
a) There are profunctors $\underset{\mathbb{A}\Rightarrow\mathbb{B}}{\mathbb{F}}$, $\underset{\mathbb{B}\Rightarrow\mathbb{A}}{\mathbb{G}}$,
such that $\mathbb{F}\cdot\mathbb{G} \cong \mathbb{A}$ and $\mathbb{G}\cdot\mathbb{F} \cong \mathbb{B}$.
b) There is a Morita bridge $\mathbb{M} : \mathbb{A} \rightleftharpoons \mathbb{B}$.
c) $\mathbb{A}^{id} \simeq \mathbb{B}^{id}$.

Proof.
a) $\Rightarrow$ b): $\mathbb{M} := \mathbb{F} \cup \mathbb{G}$ can be made a bridge by

Lemma 2.6. Let $f : A \to B$, $g : B \to A$ be an equivalence in a bicategory, with isomorphisms $\underset{fg \rightharpoonup 1_A}{\varphi}$ and $\underset{gf \rightharpoonup 1_B}{\psi}$.
Then $\exists \underset{gf \rightharpoonup 1_B}{\psi'} : \underset{fgf \rightharpoonup f}{f\cdot\psi' = \varphi\cdot f}$ and $\underset{gfg \rightharpoonup g}{\psi'\cdot g = g\cdot\varphi}$.

b) $\Rightarrow$ c): Consider $\mathbb{M}^{id}$.

c) $\Rightarrow$ b): Let $\mathbb{H} : \mathbb{A}^{id} \rightleftharpoons \mathbb{B}^{id}$ be an equivalence bridge.
Set $\mathbb{M} := \mathbb{H}\restriction_{\mathbb{A}\cup\mathbb{B}}$. It is a Morita bridge.

b) $\Rightarrow$ a): Consider the parts of $\mathbb{M}$:
$\mathbb{F} := \mathbb{M}^{>}$ and $\mathbb{G} := \mathbb{M}^{<}$. They give an equivalence by

Lemma 2.7. Let $\mathbb{K} : \mathbb{A} \rightleftharpoons \mathbb{B}$ be a bridge, with surjective composition $\chi : \mathbb{K}^{>}\cdot\mathbb{K}^{<} \to \mathbb{A}$. Then χ is isomorphism.

$\square$

A next level in abstraction

Def. 3.1. In a bicategory,

$\langle \underset{A\to B}{f}, \underset{B\to A}{g}, \underset{fg\rightharpoonup 1_A}{\varphi}, \underset{gf\rightharpoonup 1_B}{\psi} \rangle$ is a *bridge*, if

· $\underset{fgf\rightharpoonup f}{f\cdot\psi = \varphi\cdot f}$ and $\underset{gfg\rightharpoonup g}{\psi\cdot g = g\cdot\varphi}$.

Examples:

Let $\mathbb{B}$imod be the bicategory of (rings, bimodules).

Then ${}_R\left(R^{1\times n}\right)_{R^{n\times n}}$ with ${}_{R^{n\times n}}\left(R^{n\times 1}\right)_R$

is a bridge in $\mathbb{B}$imod.

By lemma 2.6 every equivalence in a bicategory can be made a bridge.

Note that lemma 2.7 also holds in $\mathbb{B}$imod.

Question.

Search for more examples of bridges.

Bertalan Pécsi
ELTE University, Budapest

aladar@renyi.hu
http://www.renyi.hu/~aladar