Extensions in the theory of lax algebras

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Barr's Theorem

Barr's Theorem (1970)

A relation $r \subset \beta X \times X$ *is the convergence relation of some topology on* X *if and only if* $1_X \subset r \cdot e_X$ *and* $r \cdot \overline{\beta} r \subset r \cdot m_X$*. In diagrams*

$$
\begin{tikzcd}
X \arrow[r, "e_X"] \arrow[dr, equal] \arrow[dr, phantom, "\subset", near end, shift left=3] & \beta X \arrow[d, "r"] \\
 & X
\end{tikzcd}
\qquad
\begin{tikzcd}
\beta\beta X \arrow[r, "\overline{\beta} r"] \arrow[d, "m_X"'] \arrow[dr, phantom, "\supset"] & \beta X \arrow[d, "r"] \\
\beta X \arrow[r, "r"] & X
\end{tikzcd}
$$

Equivalently, $e_X^\circ \subset r$ and $r \cdot \overline{\beta} r \cdot m_X^\circ \subset r$.

Here $s \cdot r = \{\, (x, z) \mid \exists y : (x, y) \in r, (y, z) \in s \,\}$ and $\overline{\beta}$ is some extension of β to relations.

Barr-extension

Fact: Every relation $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ factors as $r = r_1 \cdot r_0^\circ$ with

$$\begin{tikzcd} & R \arrow[dl, "r_0"'] \arrow[dr, "r_1"] & \\ X & & Y \end{tikzcd}$$

Given a functor $T : \mathbf{Set} \longrightarrow \mathbf{Set}$, we define $\overline{T} : \mathbf{Rel} \longrightarrow \mathbf{Rel}$ via

$$\overline{T}(r) = Tr_1 \cdot (Tr_0)^\circ.$$

$\overline{T}$ preserves identities and commutes with $-^\circ$. Moreover:

$$\overline{T} \text{ is lax} \iff T \text{ nearly preserves pullbacks,}$$
$$\overline{T} \text{ is oplax} \iff T \text{ preserves regular epimorphisms.}$$

"Lax algebras" for other monads (using Barr extension)

- identity monad $\rightsquigarrow$ ordered sets;
- powerset monad $\rightsquigarrow$ "closure spaces" without monotonicity;
- filter monad $\rightsquigarrow$ "topological closure spaces" without monotonicity.

Other extension needed?

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Other extension needed?

Life w/o extensions: Kleisli algebras (t.a.f.k.a. monadic topologies)

Let $\mathbb{F} = (F, e, m)$ denote the filter monad.

Proposition

$a : X \longrightarrow FX$ arises as the "neighborhood function" of a topology on X if and only if

$$e_X \leq a \qquad\qquad a * a \leq a$$

with $a * a = m_X \cdot Fa \cdot a$ the Kleisli-composition.

Replacing $\mathbb{F}$ with an arbitrary (ordered) monad $\mathbb{T}$ yields *Kleisli-algebras*.

Proof:

$$\frac{\dfrac{X \longrightarrow FX \longrightarrow 2^{2^X}}{2^X \times X \longrightarrow 2}}{2^X \longrightarrow 2^X}$$

and $X \xrightarrow{a} 2^{2^X}$ factors through FX if and only if the corresponding function $2^X \xrightarrow{\hat{a}} 2^X$ preserves finite intersections.
Moreover,

$$e_X \leq a \iff \hat{a} \subset 1_{2^X},$$
$$a * a \leq a \iff \hat{a} \subseteq \hat{a} \cdot \hat{a}.$$

Sup-enriched monads

A *sup-enriched monad* is a monad $\mathbb{T}$ with a monad morphism $\mathbb{P} \xrightarrow{\tau} \mathbb{T}$; $\mathbb{P}$ the powerset monad.

Thus, every TX is a complete lattice, each m_X and each Tf preserve suprema.

A sup-enriched monad is called *coherently sup-enriched* if the associated extension operations $-^{\mathbb{T}}$, given by

$$(X \xrightarrow{a} TY) \longmapsto m_Y \cdot Ta,$$

is monotone.

Kleisli-extensions

For a coherently sup-enriched monad $(\mathbb{T}, \tau)$, we define the Kleisli-extension $T^{\tau} : \mathbf{Rel} \longrightarrow \mathbf{Rel}$ via

$$(\mathfrak{x}, \mathfrak{y}) \in T^{\tau} r \iff \mathfrak{x} \leq r^{\tau}(\mathfrak{y})$$

with $r^{\tau} = (\tau_X \cdot r^{\flat})^{\mathbb{T}} = m_X \cdot T(\tau_X \cdot r^{\flat})$ and $r^{\flat} : Y \longrightarrow PX$ given by pre-image under r.

$$\begin{tikzcd} TY \arrow[r, "r^{\tau}"] \arrow[d, "Tr^{\flat}"'] & TX \\ TPX \arrow[r, "T\tau_X"'] & TTX \arrow[u, "m_X"'] \end{tikzcd}$$

Properties of the Kleisli-extension

Let $(\mathbb{T}, \tau)$ be a coherently sup-enriched monad. Then:

- $e : 1 \longrightarrow T^\tau$ and $m : T^\tau T^\tau \longrightarrow T^\tau$ are oplax transformations.
- $T^\tau 1_X$ is the order-relation on TX induced by τ.
- T^τ preserves compositions.
- The categories of Kleisli-algebras for $(\mathbb{T}, \tau)$ and of lax algebras with respect to T^τ are isomorphic.

$\rightsquigarrow$ topological spaces via filter-convergence.

What about ultrafilter convergence?

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What about ultrafilter convergence?

Back to ultrafilters: initial extensions

Let $\mathbb{T}$ be a monad with lax extension $\tilde{T}$ and $\alpha : \mathbb{S} \longrightarrow \mathbb{T}$ be a monad morphism. We obtain a lax extension $\alpha^* \tilde{T}$ of $\mathbb{S}$ via

$$\alpha^* \tilde{T}(r) = \alpha_Y^\circ \cdot \tilde{T}(r) \cdot \alpha_X .$$

We call $\alpha^* \tilde{T}$ the *initial lift* of S.

Comparing lax algebras and Kleisli algebras

Fix a coherently sup-enriched monad $(\mathbb{T}, \tau)$ and a monad morphism $\mathbb{S} \xrightarrow{\alpha} \mathbb{T}$. We define an adjunction

$$\begin{tikzcd} \mathbf{Rel}(SX, Y) \arrow[r, shift left, "\phi"] \arrow[r, phantom, "\perp"] & \mathbf{Set}(Y, TX) \arrow[l, shift left, "\psi"] \end{tikzcd},$$

for sets X, Y via:

$$\phi(r)(y) = \bigvee \alpha_X[r^\flat(y)] \qquad \text{and} \qquad (\mathfrak{x}, y) \in \psi(c) \iff \alpha_X(\mathfrak{x}) \leq c(y).$$

ϕ induces a functor $\mathbf{KleiAlg}(\mathbb{T}) \longrightarrow \mathbf{Alg}(\mathbb{S})$.

ψ induces a functor $\mathbf{Alg}(\mathbb{S}) \longrightarrow \mathbf{KleiAlg}(\mathbb{T})$ provided α is *interpolating*, that is:

$$\mathfrak{x} \leq \bigvee \alpha_X[r^\flat(y)] \implies \exists \mathfrak{X} \in SSX : \mathfrak{x} \leq m_X(\mathfrak{X}) \text{ and } \alpha_{SX}(\mathfrak{X}) \leq r^\tau \cdot e_Y(y)$$

and

$$\ldots$$

holds for all relations $SX \overset{r}{\nrightarrow} Y$, $y \in Y$.

Clearly, we have $\Phi \dashv \Psi$.

Φ, Ψ induce an isomorphism $\mathbf{KleiAlg}(\mathbb{T}) \cong \mathbf{Alg}(\mathbb{S})$ provided α is sup-generating, that is:

$$\forall \mathfrak{f} \in TX \; \exists \mathcal{A} \subset SX : \mathfrak{f} = \bigvee \alpha_X[\mathcal{A}].$$

Note: If α is sup-generating, then it satisfies the second part of the interpolation condition (...).

Theorem

Let $(\mathbb{T}, \tau)$ *be a coherently sup-enriched monad. If* $\alpha : \mathbb{S} \longrightarrow \mathbb{T}$ *is an interpolating and sup-generating monad morphism, then we have an isomorphism*

$$\mathbf{KleiAlg}(\mathbb{T}) \cong \mathbf{Alg}(\mathbb{S}).$$

Corollary

For any coherently sup-enriched monad $(\mathbb{T}, \tau)$*, the categories* $\mathbf{KleiAlg}(\mathbb{T})$ *and* $\mathbf{Alg}(\mathbb{T})$ *are isomorphic.*

Proof.

$1_{\mathbb{T}}$ is interpolating and sup-generating. □

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Proof.

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Examples

- $\beta \hookrightarrow \mathbb{F}$ is sup-generating and interpolating.
  Hence, we can obtain topological spaces via neighborhoods, filter or ultrafilter convergence.
- $\mathbb{P} \longrightarrow \mathbb{S}$ ("stacks") by $A \longmapsto \{ B \subset X \mid A \subset B \}$ is interpolating and sup-generating: leads to interior spaces.
- $\mathbb{P} \longrightarrow \mathbb{S}$ by $A \longmapsto \{ B \subset X \mid \exists x \in A \cap B \}$ is interpolating, but not sup-generating.
- (prime functional ideals) $\hookrightarrow$ (functional ideals) interpolating and sup-generating.
- "fuzzy stuff".