Cocomplete $\mathcal{T}$-categories, injectivity, and Kan-extensions

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V-categories

Quantale

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The category V-Rel of V-relations

has V-relations $r : X \times Y \longrightarrow \mathsf{V}$ as morphisms $r : X \relbar\!\!\mapsto Y$, and composition is given by (with $s : Y \relbar\!\!\mapsto Z$)

$$s \cdot r(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z).$$

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The category V-Cat of V-categories

A V-category $X = (X, a)$ consists of a set X and a V-relation $a : X {\longrightarrow\hspace{-1em}\mapsto} X$ such that $\quad 1_X \le a,\ \ a \cdot a \le a.$

$f : (X, a) \longrightarrow (Y, b)$ is a V-functor if $a(x, x') \le b(f(x), f(x'))$.

Examples

Metric spaces, $(\mathsf{P}_{\!+} = [0,\infty]^{\mathrm{op}}, +, 0)$

X with $d : X \times X \longrightarrow \mathsf{P}_{\!+}$ such that

$$0 \geq d(x,x), \qquad d(x,y) + d(y,z) \geq d(x,z).$$

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Ordered sets, $(2 = \{\mathrm{false}, \mathrm{true}\}, \&, \mathrm{true})$

X with $\leq : X \times X \longrightarrow 2$ such that

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Categories, $(\mathsf{Set}, \times, 1)$

X with $\hom : X \times X \longrightarrow \mathsf{Set}$ such that

$$1 \longrightarrow \hom(x,x), \qquad \hom(x,y) \times \hom(y,z) \longrightarrow \hom(x,z)$$

and . . . (commutative diagrams in Set).

V-Cat is monoidal closed

The tensor product on V-Cat

Let $X = (X, a)$, $Y = (Y, b)$ be V-categories. We put

$$X \otimes Y = (X \times Y, c), \text{ where } c((x, y), (x', y')) = a(x, x') \otimes b(y, y').$$

The V-category $E = (1, k)$ is a neutral element for $\otimes$.

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The function space structure

Let $X = (X, a)$, $Y = (Y, b)$ be V-categories. We put

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Presheaf categories

$\hat{X}$ denotes $\mathsf{V}^{X^{\mathrm{op}}}$, and $[\_, \_]$ its structure. Here $\mathsf{V} = (\mathsf{V}, \hom)$.

Modules

Definition

A V-module $\varphi : (X, a) \multimap (Y, b)$ is a V-relation $\varphi : X \relbar\joinrel\mapsto Y$ such that

$$b \cdot \varphi \leq \varphi \qquad \text{and} \qquad \varphi \cdot a \leq \varphi.$$

V-Mod denotes the category of V-categories and V-modules.

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Functors induce modules

Each map $f : X \longrightarrow Y$ between V-categories (X, a), (Y, b) induces V-relations

$$f_* : X \relbar\!\!\!\mapsto Y;\ f_*(x, y) = b(f(x), y),$$

$$f^* : Y \relbar\!\!\!\mapsto X;\ f^*(y, x) = b(y, f(x)).$$

Lemma

f is a V-functor. $\iff$ *f_* is a V-module.* $\iff$ *f^* is a V-module.*

Modules as functors

Theorem

For V-categories (X,a) and (Y,b), and a V-relation $\psi : X {\longrightarrow\!\!\!\!\!\!\mapsto} Y$, the following assertions are equivalent.

1. $\psi : (X,a) {-\!\!\circ\!\!\rightarrow} (Y,b)$ *is a* V*-module.*
2. $\psi : X^{\mathrm{op}} \otimes Y \longrightarrow \mathsf{V}$ *is a* V*-functors.*

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Theorem (Yoneda)

Let $y = \ulcorner a \urcorner : X \longrightarrow \mathsf{V}^{X^{\mathrm{op}}} = \hat{X}$*. Then* $\quad [y(x), \psi] = \psi(x)$*.*
In particular, y *is fully faithful (that is,* $[y(x), y(y)] = a(x, y)$*).*

V-Cat is an ordered category

The order between V-functors

For $f, g : X \longrightarrow Y$ in V-Cat, we define

$$f \leq g \qquad \text{whenever} \qquad f^* \leq g^* \quad (\iff g_* \leq f_*).$$

Hence: $(\_)_* : \text{V-Cat}^{\text{co}} \longrightarrow \text{V-Mod}$ and $(\_)^* : \text{V-Cat}^{\text{op}} \longrightarrow \text{V-Mod}$.

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Lemma

Let $f, g : (X, a) \longrightarrow (Y, b)$ be V-functors. Then

$$f \leq g \iff \forall x \in X \,.\, k \leq b(f(x), g(x)).$$

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Adjoint V-functors

$f : X \longrightarrow Y$ is left adjoint to $g : Y \longrightarrow X$ if $1_X \leq gf$ and $1_Y \geq fg$.

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That is,

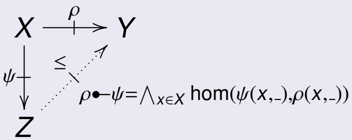
$$f \dashv g \iff g_* \dashv f_* \iff f_* = g^* \iff b(f\_, \_) = a(\_, g\_).$$

Extensions

Let $\psi : X \relbar\!\!\mapsto Z$ be a V-relation, and consider

$$\_ \cdot \psi : \text{V-Rel}(Z, Y) \longrightarrow \text{V-Rel}(X, Y).$$

$\_ \cdot \psi$ has a right adjoint $\_ \bullet\!\!- \psi$:



Note: ρ,ψ are V-modules. $\Rightarrow \rho \bullet\!\!- \psi$ is a V-module.

Liftings and extensions

Extensions

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$$\_ \cdot \psi : \mathsf{V}\text{-}\mathsf{Rel}(Z, Y) \longrightarrow \mathsf{V}\text{-}\mathsf{Rel}(X, Y).$$

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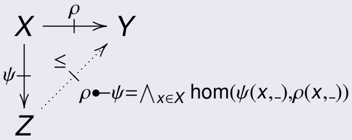


Note: ρ,ψ are V-modules. $\Rightarrow \rho \multimapdotinv \psi$ is a V-module.

Remark

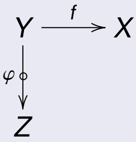
With $\ulcorner\psi\urcorner : Z \longrightarrow \hat{X}$, $\ulcorner\rho\urcorner : Y \longrightarrow \hat{X}$, we have

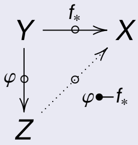
$$\rho \multimapdotinv \psi(z, y) = [\ulcorner\psi\urcorner(z), \ulcorner\rho\urcorner(y)].$$

Colimits

Definition

Let X, Y, Z be V-categories, $f : Y \longrightarrow X$ a V-functor and $\varphi : Y \multimap Z$ a V-module.





$g : Z \longrightarrow X$ is called φ-weighted colimit of f if $g_* = \varphi \bullet\!\!- f_*$.
Notation: $g \cong \mathrm{colim}(\varphi, f)$.

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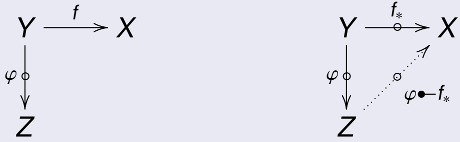

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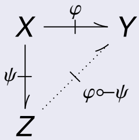
Proposition

1. *Each $\varphi \in \hat{X}$ is a colimit of representables.*
2. *Left adjoints preserve weighted colimits.*

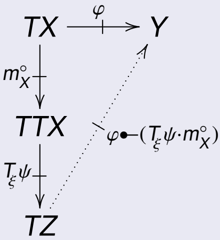
Extensions of $\mathcal{T}$-relations (resp. $\mathcal{T}$-modules)

Extensions in $\mathcal{T}$-Rel

We pass from



to



and define $\varphi \circ\!- \psi = \varphi \bullet\!- (T_\xi \psi \cdot m_X^\circ)$.

Theorem (Yoneda)

For $\psi : X \multimap Z$, $\varphi : X \multimap Y$, $\mathfrak{z} \in TZ$ and $y \in Y$, we have

$$(\varphi \circ\!- \psi)(\mathfrak{z}, y) = [\![T\ulcorner\psi\urcorner(\mathfrak{z}), \ulcorner\varphi\urcorner(y)]\!].$$

What does this all mean in Top?

Let X be a topological space.

1. $M(X) = (UX, \leq)$ where $\mathfrak{x} \leq \mathfrak{y}$ whenever $\overline{\mathfrak{x}} \subseteq \mathfrak{y}$.
2. X^{op} = the Alexandroff space induced by $M(X)^{\mathrm{op}}$.
3. $|X| = (UX, \text{Zariski-closure})$.
4. $2^{|X|} \cong \{\mathcal{A} \subseteq UX \mid \mathcal{A} \text{ is Zariski-closed}\}$ with open sets
$$B(\mathcal{B}, \{0\}) = \{\mathcal{A} \subseteq UX \mid \mathcal{A} \cap \mathcal{B} = \varnothing\}. \qquad (\mathcal{B} \text{ Zariski-closed})$$
5. $\hat{X} \cong \{\mathcal{A} \subseteq UX \mid \mathcal{A} \text{ is Zariski-closed and down-closed}\}$.
6. $y_X : X \longrightarrow \hat{X},\ x \longmapsto \{\mathfrak{x} \in UX \mid \mathfrak{x} \longrightarrow x\}$.
7. $\mu_X : \hat{\hat{X}} \longrightarrow \hat{X},\ \mathscr{A} \longmapsto \{\mathfrak{x} \in UX \mid Uy_X(\mathfrak{x}) \in \mathscr{A}\}$.
8. $(\hat{(\_)}, y, \mu)$ is isomorphic to the filter monad (F, e, m) on Top_0:
$$\mathcal{A} \longmapsto \Big(\bigcap \mathcal{A}\Big) \cap \tau, \qquad \mathfrak{f} \longmapsto \{\mathfrak{x} \in UX \mid \mathfrak{f} \subseteq \mathfrak{x}\}$$