Sequential Approach Spaces

Gonçalo Gutierres – CMUC/Universidade de Coimbra

Joint work with D. Hofmann and C. Van Olmen

Approach spaces were introduced by R. Lowen as a natural generalisation of both topological and metric spaces. Therefore many concepts in approach theory are motivated by well-established notions in topology such as compactness or soberness.

With no doubt, sequential spaces form an important classe of topological spaces, however, so far appears no study of sequential approach spaces in the literature. It is the goal of this talk to fill this gap. We propose a definition of sequential approach spaces and some of its properties are studied.

Approach space

$$\delta : X \times PX \to [0, \infty]$$

Approach space

$$\delta : X \times PX \to [0, \infty]$$

$$\delta(x, \{x\}) = 0$$

$$\delta(x, \emptyset) = \infty$$

$$\delta(x, A \cup B) = \min \delta(x, A), \delta(x, B)$$

Approach space

$$\delta : X \times PX \to [0, \infty]$$

$$\delta(x, \{x\}) = 0$$

$$\delta(x, \emptyset) = \infty$$

$$\delta(x, A \cup B) = \min \delta(x, A), \delta(x, B)$$

$$\delta(x, A) \leq \delta(x, A^{(\varepsilon)}) + \varepsilon)$$

$$A^{(\varepsilon)} = \{x \mid \delta(x, A) \leq \varepsilon\}$$

Fréchet space

$$\{\delta \in [0,+\infty]^{X \times PX}\} \underset{\mathcal{S}}{\overset{\mathcal{C}}{\rightleftarrows}} \{a \in [0,+\infty]^{SX \times X}\}$$

Fréchet space

$$\{\delta \in [0,+\infty]^{X\times PX}\} \underset{\mathcal{S}}{\overset{\mathcal{C}}{\rightleftarrows}} \{a \in [0,+\infty]^{SX\times X}\}$$

$$\mathcal{C}(\delta)((x_n),x) := \sup_{(y_n)\preccurlyeq(x_n)} \delta(x,\{y_n \mid n\in\mathbb{N}\})$$

$$\mathcal{S}(a)(x,A) = \inf_{(x_n)\in SA} a((x_n),x)$$

Fréchet space

$$\{\delta \in [0,+\infty]^{X\times PX}\} \underset{\mathcal{S}}{\overset{\mathcal{C}}{\rightleftarrows}} \{a \in [0,+\infty]^{SX\times X}\}$$

$$\mathcal{C}(\delta)((x_n),x) := \sup_{(y_n)\preccurlyeq(x_n)} \delta(x,\{y_n \mid n\in\mathbb{N}\})$$

$$\mathcal{S}(a)(x,A) = \inf_{(x_n)\in SA} a((x_n),x)$$

δ is a Fréchet Approach space if $\delta = \mathcal{S}(\mathcal{C}(\delta))$

Reflection

$$\mathcal{S}(x, A) = \inf_{(x_n)\in SA} a((x_n), x)$$

Reflection

$$\mathcal{S}(x, A) = \inf_{(x_n) \in SA} a((x_n), x)$$

$$\mathcal{S}_{\alpha+1}(x, A) = \inf_{\varepsilon \geq 0} (\mathcal{S}(x, A^{(\varepsilon)}) + \varepsilon)$$

$$A^{(\varepsilon)} = \{x \,|\, \mathcal{S}_\alpha(x, A) \leq \varepsilon\}$$

Reflection

$$\mathcal{S}(x, A) = \inf_{(x_n) \in SA} a((x_n), x)$$

$$\mathcal{S}_{\alpha+1}(x, A) = \inf_{\varepsilon \geq 0} (\mathcal{S}(x, A^{(\varepsilon)}) + \varepsilon)$$

$$A^{(\varepsilon)} = \{x \,|\, \mathcal{S}_\alpha(x, A) \leq \varepsilon\}$$

$$\mathcal{S}_\lambda(x, A) = \inf_{\alpha < \lambda} \mathcal{S}_\alpha(x, A)$$

$$\mathcal{S}^\star(x, A) = \inf_{\alpha} \mathcal{S}_\alpha(x, A)$$

Reflection

$$\mathcal{S}(x, A) = \inf_{(x_n) \in SA} a((x_n), x)$$

$$\mathcal{S}_{\alpha+1}(x, A) = \inf_{\varepsilon \geq 0} (\mathcal{S}(x, A^{(\varepsilon)}) + \varepsilon)$$

$$A^{(\varepsilon)} = \{x \,|\, \mathcal{S}_\alpha(x, A) \leq \varepsilon\}$$

$$\mathcal{S}_\lambda(x, A) = \inf_{\alpha < \lambda} \mathcal{S}_\alpha(x, A)$$

$$\mathcal{S}^\star(x, A) = \inf_{\alpha} \mathcal{S}_\alpha(x, A)$$

$\mathcal{S}^*$ is a Fréchet Approach distance.

In general this construction gives a reflection.

Galois connection

$$\hat{S}\varphi : SX \to [0, +\infty], \qquad \hat{S}\varphi((x_n)_n) = \sup_{n\in\mathbb{N}} \inf_{k\geq n} \varphi(x_k)$$

Galois connection

$$\hat{S}\varphi : SX \to [0,+\infty], \qquad \hat{S}\varphi((x_n)_n) = \sup_{n\in\mathbb{N}} \inf_{k\geq n} \varphi(x_k)$$

$$\{\rho \subseteq [0,+\infty]^X\} \underset{\mathcal{R}}{\overset{\mathcal{C}}{\rightleftarrows}} \{a : SX \times X \to [0,+\infty]\}$$

Galois connection

$$\hat{S}\varphi : SX \to [0,+\infty], \qquad \hat{S}\varphi((x_n)_n) = \sup_{n\in\mathbb{N}} \inf_{k\geq n} \varphi(x_k)$$

$$\{\rho \subseteq [0,+\infty]^X\} \underset{\mathcal{R}}{\overset{\mathcal{C}}{\rightleftarrows}} \{a : SX \times X \to [0,+\infty]\}$$

$$\varphi \in \mathcal{R}(a) :\Leftrightarrow \forall \mathfrak{x} \in SX\, \forall x\; \varphi(x) \leq a(\mathfrak{x}, x) + \hat{S}\varphi(\mathfrak{x})$$

$$\mathcal{R}(\rho)(\mathfrak{x}, x) := \sup_{\varphi\in\rho}(\varphi(x) - \hat{S}\varphi(\mathfrak{x}))$$

Galois connection

$$\hat{S}\varphi : SX \to [0,+\infty], \qquad \hat{S}\varphi((x_n)_n) = \sup_{n\in\mathbb{N}} \inf_{k\geq n} \varphi(x_k)$$

$$\{\rho \subseteq [0,+\infty]^X\} \underset{\mathcal{R}}{\overset{\mathcal{C}}{\rightleftarrows}} \{a : SX \times X \to [0,+\infty]\}$$

$$\varphi \in \mathcal{R}(a) :\Leftrightarrow \forall \mathfrak{x} \in SX\, \forall x\; \varphi(x) \leq a(\mathfrak{x}, x) + \hat{S}\varphi(\mathfrak{x})$$

$$\mathcal{R}(\rho)(\mathfrak{x}, x) := \sup_{\varphi\in\rho}(\varphi(x) - \hat{S}\varphi(\mathfrak{x}))$$

$a \geq \mathcal{C}\mathcal{R}(a)$ and $\rho \subseteq \mathcal{R}\mathcal{C}(\rho)$.

Galois connection

$$\hat{S}\varphi : SX \to [0,+\infty], \qquad \hat{S}\varphi((x_n)_n) = \sup_{n\in\mathbb{N}} \inf_{k\geq n} \varphi(x_k)$$

$$\{\rho \subseteq [0,+\infty]^X\} \underset{\mathcal{R}}{\overset{\mathcal{C}}{\rightleftarrows}} \{a : SX \times X \to [0,+\infty]\}$$

$$\varphi \in \mathcal{R}(a) :\Leftrightarrow \forall \mathfrak{x} \in SX \, \forall x \; \varphi(x) \leq a(\mathfrak{x}, x) + \hat{S}\varphi(\mathfrak{x})$$

$$\mathcal{R}(\rho)(\mathfrak{x}, x) := \sup_{\varphi\in\rho}(\varphi(x) - \hat{S}\varphi(\mathfrak{x}))$$

$a \geq \mathcal{C}\mathcal{R}(a)$ and $\rho \subseteq \mathcal{R}\mathcal{C}(\rho)$.

$\mathcal{R}(a)$ is a regular function frame on X.

Approach space

$\rho \subseteq [0, \infty]$ is a regular function frame if:

Approach space

$\rho \subseteq [0,\infty]$ is a regular function frame if:

is closed under arbitrary joins and finite meets,

$\varphi \in \rho \Rightarrow \varphi + \varepsilon,\ \varphi - \varepsilon \in \rho$

Approach space

$\rho \subseteq [0,\infty]$is a regular function frame if:

is closed under arbitrary joins and finite meets,

$\varphi \in \rho \Rightarrow \varphi + \varepsilon,\ \varphi - \varepsilon \in \rho$

(X, ρ) is a sequential approach space if $\rho = \mathcal{RC}(\rho)$

(X, δ) is a sequential approach space if $\delta = \mathcal{S}^\star$

Examples

Topological sequential spaces

Examples

Topological sequential spaces

Metric spaces

Examples

Topological sequential spaces

Metric spaces

$$\mathbb{P} = ([0, \infty], \delta)$$

$$\delta(x, A) := x - \sup A$$

$$\delta(x, \emptyset) := \infty$$

Sequential spaces

(1) $\dot{x} \longrightarrow x$

Sequential spaces

(1) $\dot{x} \longrightarrow x$

(2) if $(x_n) \longrightarrow x$ and $(y_n) \preccurlyeq (x_n)$, then $(y_n) \longrightarrow x$

Sequential spaces

(1) $\dot{x} \longrightarrow x$

(2) if $(x_n) \longrightarrow x$ and $(y_n) \preccurlyeq (x_n)$, then $(y_n) \longrightarrow x$

(3) $[\forall (y_n) \preccurlyeq (x_n)\, \exists (z_n) \preccurlyeq (y_n)\ (z_n) \longrightarrow x] \;\Rightarrow (x_n)_n \longrightarrow x$

Sequential spaces

(1) $\dot{x} \longrightarrow x$

(2) if $(x_n) \longrightarrow x$ and $(y_n) \preccurlyeq (x_n)$, then $(y_n) \longrightarrow x$

(3') $[\forall (y_n) \preccurlyeq (x_n)\ \exists \mathfrak{z} \preccurlyeq (y_n)\ \ \mathfrak{z} \longrightarrow x] \Rightarrow (x_n) \longrightarrow x$

Sequential spaces

(1) $\dot{x} \longrightarrow x$

(2) if $(x_n) \longrightarrow x$ and $(y_n) \preccurlyeq (x_n)$, then $(y_n) \longrightarrow x$

(3') $[\forall (y_n) \preccurlyeq (x_n)\ \exists \mathfrak{z} \preccurlyeq (y_n)\ \ \mathfrak{z} \longrightarrow x] \Rightarrow (x_n) \longrightarrow x$

(4) $(\dot{x}_n)_n \longrightarrow (y_n)_n \longrightarrow x \ \Rightarrow\ (x_n)_n \longrightarrow x$

Higher order subsequences

$$\preccurlyeq^{\alpha} \colon S^{\alpha} X \relbar\joinrel\mapstochar\joinrel\rightarrow SX$$

Higher order subsequences

$\preccurlyeq^{\alpha}: S^{\alpha}X \longrightarrow SX$

- If $\mathfrak{z} \in SX$, $\mathfrak{z} \preccurlyeq (y_n)$ means that $\mathfrak{z}$ is a subsequence of (y_n).

Higher order subsequences

$\preccurlyeq^\alpha : S^\alpha X \longrightarrow\!\!\!\!\!\!\!\!\!\!+\quad SX$

- If $\mathfrak{z} \in SX$, $\mathfrak{z} \preccurlyeq (y_n)$ means that $\mathfrak{z}$ is a subsequence of (y_n).

- If $\mathfrak{z} = (\mathfrak{z}_n)_n \in S^{\alpha+1}X$,
$\mathfrak{z} \preccurlyeq (y_n)_n$ if for all n, $\mathfrak{z}_n \preccurlyeq^\alpha (y_n)_n$.

Higher order subsequences

$\preccurlyeq^\alpha \colon S^\alpha X \longrightarrow\!\!\!\!\!\!\!\!+\;\;\; SX$

- If $\mathfrak{z} \in SX$, $\mathfrak{z} \preccurlyeq (y_n)$ means that $\mathfrak{z}$ is a subsequence of (y_n).
- If $\mathfrak{z} = (\mathfrak{z}_n)_n \in S^{\alpha+1}X$, $\mathfrak{z} \preccurlyeq (y_n)_n$ if for all n, $\mathfrak{z}_n \preccurlyeq^\alpha (y_n)_n$.
- $\preccurlyeq^\lambda \colon S^\lambda X \longrightarrow\!\!\!\!\!\!\!\!+\;\;\; SX$ is the colimit.

Convergence

$$a : SX \times X \longrightarrow [0, \infty]$$

Convergence

$$a : SX \times X \longrightarrow [0, \infty]$$

$$Sa : S^2X \times SX \longrightarrow [0, \infty]$$

$$Sa((x_m^n), x_n) = \inf_n \sup_{k \geq n} a((x_m^k)_m, x_k)$$

Convergence

$$a : SX \times X \to [0, \infty]$$

$$Sa : S^2X \times SX \to [0, \infty]$$

$$Sa((x_m^n), x_n) = \inf_n \sup_{k \geq n} a((x_m^k)_m, x_k)$$

$$a^{\alpha+1} : S^{\alpha+1}X \times X \to [0, \infty]$$

$$a^{\alpha+1} = a \cdot Sa^{\alpha} \text{ i.e } a^{\alpha+1}(\mathfrak{z}, x) = \inf_{(y_n)} Sa^{\alpha}(\mathfrak{z}, (y_n)) + a((y_n), x)$$

Convergence

$$a : SX \times X \longrightarrow [0, \infty]$$

$$Sa : S^2X \times SX \longrightarrow [0, \infty]$$

$$Sa((x^n_m), x_n) = \inf_n \sup_{k \geq n} a((x^k_m)_m, x_k)$$

$$a^{\alpha+1} : S^{\alpha+1}X \times X \longrightarrow [0, \infty]$$

$$a^{\alpha+1} = a \cdot Sa^{\alpha} \quad \text{i.e} \quad a^{\alpha+1}(\mathfrak{z}, x) = \inf_{(y_n)} Sa^{\alpha}(\mathfrak{z}, (y_n)) + a((y_n), x)$$

$$a^{\lambda} = \inf_{\alpha < \lambda} a^{\alpha}$$

Convergence

$$a : SX \times X \to [0, \infty]$$

$$Sa : S^2X \times SX \to [0, \infty]$$

$$Sa((x^n_m), x_n) = \inf_n \sup_{k \geq n} a((x^k_m)_m, x_k)$$

$$a^{\alpha+1} : S^{\alpha+1}X \times X \to [0, \infty]$$

$$a^{\alpha+1} = a \cdot Sa^{\alpha} \ \text{ i.e } \ a^{\alpha+1}(\mathfrak{z}, x) = \inf_{(y_n)} Sa^{\alpha}(\mathfrak{z}, (y_n)) + a((y_n), x)$$

$$a^{\lambda} = \inf_{\alpha < \lambda} a^{\alpha}$$

Lemma $\mathcal{S}^{\alpha}(x, A) = \inf_{\mathfrak{z} \in S^{\alpha}A} a(\mathfrak{z}, x)$

Axioms

(1) $a(\dot{x}, x) = 0$,

Axioms

(1) $a(\dot{x}, x) = 0$,

(2) if $(y_n)_n \preccurlyeq (x_n)_n$, then $a((y_n), x) \leq a((x_n), x)$,

Axioms

(1) $a(\dot{x}, x) = 0$,

(2) if $(y_n)_n \preccurlyeq (x_n)_n$, then $a((y_n), x) \leq a((x_n), x)$,

(3) $a((x_n), x) \leq \sup\limits_{(y_n) \preccurlyeq (x_n)} \inf\limits_{\mathfrak{z} \preccurlyeq (y_n)} a(\mathfrak{z}, x)$,

Axioms

(1) $a(\dot{x}, x) = 0$,

(2) if $(y_n)_n \preccurlyeq (x_n)_n$, then $a((y_n), x) \leq a((x_n), x)$,

(3) $a((x_n), x) \leq \sup_{(y_n) \preccurlyeq (x_n)} \inf_{\mathfrak{z} \preccurlyeq (y_n)} a(\mathfrak{z}, x)$,

(4) $a((x_k), x) \leq Sa((\dot{x}_n), (y_n)) + a((y_n), x)$

$$a((x_k), x) \leq \inf_n \sup_{k \geq n} a(\dot{x}_k, y_k) + a((y_n), x)$$

Some results

- An Approach space is Fréchet iff each subspace is sequential.

Some results

- An Approach space is Fréchet iff each subspace is sequential.
- Sequential approach spaces are coreflective in Approach spaces.

Some results

- An Approach space is Fréchet iff each subspace is sequential.
- Sequential approach spaces are coreflective in Approach spaces.
- Sequential approach spaces is a topological category.

$$\mathbb{V} = (V, \otimes, k), \qquad \hat{S}\varphi((x_n)_n) = \bigwedge_{n \in \mathbb{N}} \bigvee_{k \geq n} \varphi(x_k)$$

$$\mathbb{V} = (V, \otimes, k), \qquad \hat{S}\varphi((x_n)_n) = \bigwedge_{n\in\mathbb{N}} \bigvee_{k\geq n} \varphi(x_k)$$

$$\{\rho \subseteq V^X\} \underset{\mathcal{R}}{\overset{\mathcal{C}}{\rightleftarrows}} \{a : SX \times X \to V\}$$

$$\mathbb{V} = (V, \otimes, k), \qquad \hat{S}\varphi((x_n)_n) = \bigwedge_{n\in\mathbb{N}} \bigvee_{k\geq n} \varphi(x_k)$$

$$\{\rho \subseteq V^X\} \underset{\mathcal{R}}{\overset{\mathcal{C}}{\rightleftarrows}} \{a : SX \times X \to V\}$$

$$\varphi \in \mathcal{R}(a) :\Leftrightarrow \forall \mathfrak{x} \in SX \, \forall x \; \varphi(x) \geq a(\mathfrak{x}, x) + \hat{S}\varphi(\mathfrak{x})$$

$$\mathcal{R}(\rho)(\mathfrak{x}, x) := \bigwedge_{\varphi\in\rho} (\varphi(x) - \hat{S}\varphi(\mathfrak{x}))$$

$$\mathbb{V} = (V, \otimes, k), \qquad \hat{S}\varphi((x_n)_n) = \bigwedge_{n\in\mathbb{N}} \bigvee_{k\geq n} \varphi(x_k)$$

$$\{\rho \subseteq V^X\} \underset{\mathcal{R}}{\overset{\mathcal{C}}{\rightleftarrows}} \{a : SX \times X \to V\}$$

$$\varphi \in \mathcal{R}(a) :\Leftrightarrow \forall \mathfrak{x} \in SX\, \forall x\; \varphi(x) \geq a(\mathfrak{x}, x) + \hat{S}\varphi(\mathfrak{x})$$

$$\mathcal{R}(\rho)(\mathfrak{x}, x) := \bigwedge_{\varphi\in\rho} (\varphi(x) - \hat{S}\varphi(\mathfrak{x}))$$

$a \leq \mathcal{C}\mathcal{R}(a)$ and $\rho \subseteq \mathcal{R}\mathcal{C}(\rho)$.