Neighborhoods with respect to a closure operator

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1 Preliminaries

Let $\mathcal{X}$ be a finitely complete category a with a proper $(\mathcal{E},\mathcal{M})$-factorization structure for morphisms. For simplicity, we assume that $\mathcal{E}$ is stable under pullbacks along $\mathcal{M}$-morphisms and that multiple pullbacks of arbitrary large families of $\mathcal{M}$ morphisms with a common codomain exist. Given an $\mathcal{X}$-object X, we denote by sub$X$ the (complete) subobject lattice of X and by o_X the least element of sub$X$. We assume that $f^{-1}(o_Y) = o_X$ whenever $f : X \to Y$ is an $\mathcal{X}$-morphism. For any $m \in$ sub$X$, $\overline{m}$ denotes the pseudocomplement of m - provided it exists.

We suppose there is given a concrete category $\mathcal{K}$ over $\mathcal{X}$ with the corresponding underlying functor $|\ \ | : \mathcal{K} \to \mathcal{X}$. We write f instead of $|f|$ whenever f is a $\mathcal{K}$-morphism and we also write briefly sub$K$ and o_K instead of sub$|K|$ and $o_{|K|}$, respectively, whenever K is a $\mathcal{K}$-object. The category $\mathcal{K}$ is assumed to have finite concrete products and by a (not necessarily finite) product in $\mathcal{K}$ we always mean a concrete one.

We assume there is given a *closure operator* c on $\mathcal{K}$ (with respect to $(\mathcal{E},\mathcal{M})$), i.e., a family of maps $c = (c_K : \text{sub}K \to \text{sub}K)_{K\in\mathcal{K}}$ with the following properties that hold for each $\mathcal{K}$-object K and each $m, p \in$ sub$K$:

(1) $m \le c_K(m)$,

(2) $m \le p \Rightarrow c_K(m) \le c_K(p)$,

(3) $f(c_K(m)) \leq c_L(f(m))$ for each $\mathcal{K}$-morphism $f : K \to L$.

The closure operator c is called

(a) *grounded* if $c_K(o_K) = o_K$ for each $K \in \mathcal{K}$,

(b) *idempotent* if $c_K(c_K(m)) = c_K(m)$ for each $K \in \mathcal{K}$ and each $m \in \mathrm{sub}K$,

(c) *additive* if $c_K(m \vee p) = c_K(m) \vee c_K(p)$ for each $K \in \mathcal{K}$ and each $m, p \in \mathrm{sub}K$,

(d) *hereditary* if, whenever $m : M \to K$ is an embedding in $\mathcal{K}$, $c_M(p) = m^{-1}(c_K(m \circ p))$ for each $p \in \mathrm{sub}M$.

Given a $\mathcal{K}$-object K, a subobject $m \in \mathrm{sub}K$ is said to be *c-closed* (respectively, *c-dense*) provided that $c_K(m) = m$ (respectively, $c_K(m) = \mathrm{id}_K$). A $\mathcal{K}$-morphism $f : K \to L$ is called *c-preserving* if $f(c_K(m)) = c_L(f(m))$ whenever $m \in \mathrm{sub}K$. Thus, if f is c-preserving, then it maps c-closed subobjects to c-closed subobjects, and vice versa provided that c is idempotent.

2 Neighborhoods

Definition 2.1 Let K be an $\mathcal{K}$-object. A subobject $n \in \mathrm{sub}K$ is called a *c-neighborhood* of a given subobject $m \in \mathrm{sub}K$ if n is pseudocomplementable (in $\mathrm{sub}K$) and $m \wedge c_K(\overline{n}) = o_K$. We denote by $\mathcal{N}(m)$ the class of all c-neighborhoods of m. A subclass $\mathcal{B} \subseteq \mathcal{N}(m)$ is called a *base* of c-neighborhoods of m if, for every $n \in \mathcal{N}(m)$, there exists $p \in \mathcal{B}$ such that $p \leq n$.

Proposition 2.2 *Let K be a $\mathcal{K}$-object and $m, p \in \mathrm{sub}K$. Then*

(1) $id_K \in \mathcal{N}(m)$ if c is grounded,

(2) $\mathcal{N}(o_K) = \{n \in \mathrm{sub}K;\ n$ is pseudocomplementable$\}$,

(3) if $m > o_K$, then $n > o_K$ for each $n \in \mathcal{N}(m)$,

(4) $n \in \mathcal{N}(m)$ implies $m \leq n$ provided that (a) m is an atom or (b) $\overline{n}$ is pseudocomplementable with $\overline{\overline{n}} = n$ and either (i) $c_K(\overline{n})$ is pseudocomplementable or (ii) both m and $\overline{m}$ are pseudocomplementable,

(5) if $n \in \mathcal{N}(m)$ and $p \in \mathrm{sub}K$ is pseudocomplementable with $p \geq n$, then $p \in \mathcal{N}(m)$,

(6) $p \leq m \Rightarrow \mathcal{N}(m) \subseteq \mathcal{N}(p)$,

(7) if $m > o_K$ and $n_1, n_2, ..., n_k \in \mathcal{N}(m)$ $(k \in \mathbb{N})$, then $m \wedge n_1 \wedge n_2 \wedge ... \wedge n_k > o_K$,

(8) if $m > o_K$ and $n_1, n_2, ..., n_k \in \mathcal{N}(m)$ $(k \in \mathbb{N})$, then $n_1 \wedge n_2 \wedge ... \wedge n_k > o_K$,

(9) if $n_1, n_2 \in \mathcal{N}(m)$, then $n_1 \wedge n_2 \in \mathcal{N}(m)$ provided that c is additive and $\mathrm{sub}K$ is a Boolean algebra.

Proposition 2.3 *Let $f : K \to L$ be a $\mathcal{K}$-morphism, $m \in \mathrm{sub}K$ and $n \in \mathcal{N}(f(m))$. Then $f^{-1}(n) \in \mathcal{N}(m)$.*

Proposition 2.4 *Let K be a $\mathcal{K}$-object and $m, p \in \mathrm{sub}K$, $m > o_K$, and let $\mathcal{B} \subseteq \mathcal{N}(m)$ be a base of c-neighborhoods of m. If $m \leq c_K(p)$, then $n \wedge p > o_K$ for each $n \in \mathcal{B}$, and vice versa provided that m is an atom of $\mathrm{sub}K$ and p, $\overline{p}$, $c_K(p)$ are pseudocomplementable with $\overline{\overline{p}} = p$.*

Definition 2.5 Let K be a $\mathcal{K}$-object and $m \in \mathrm{sub}K$. Then m is said to be *open* (w.r.t. c) if $m \in \mathcal{N}(m)$.

If c is grounded, then the openness is weaker than the c-openness (recall that a subobject $m \in \mathrm{sub}K$ is said to be c-open if $m \wedge c_K(p) \leq c_K(m \wedge p)$ for every $p \in \mathrm{sub}K$). If c is grounded and additive and $\mathrm{sub}K$ is a Boolean algebra, then the openness and c-openness coincide.

Proposition 2.6 *Let K be a $\mathcal{K}$-object and $m \in \mathrm{sub}K$. If m is open, then $\overline{m}$ is c-closed, and vice versa provided that m is pseudocomplementable.*

Corollary 2.7 *Let $f : K \to L$ be a $\mathcal{K}$-morphism, $n \in \mathrm{sub}L$. If n is open, then $f^{-1}(n)$ is open too.*

3 Separation and compactness

Definition 3.1 A $\mathcal{K}$-object K is said to be

(a) *separated* (with respect to c) provided that, whenever $m, p \in \mathrm{sub}K$ are different atoms, there are $n \in \mathcal{N}(m)$ and $q \in \mathcal{N}(p)$ with $n \wedge q = o_K$,

(b) *compact* (with respect to c) if $\bigwedge \mathcal{T} > o_K$ for every centered class $\mathcal{T} \subseteq \mathrm{sub}K$ of c-closed subobjects of K.

Theorem 3.2 *Let K be a $\mathcal{K}$-object such that, for each atom $r \in \mathrm{sub}K$ and each $n \in \mathcal{N}(r)$, $\overline{n}$ and $c_K(n)$ are pseudocomplementable with $\overline{\overline{n}} = n$. If $r = \bigwedge\{c_K(n);\ n \in \mathcal{N}(r)\}$ for each atom $r \in \mathrm{sub}K$, then K is separated, and vice versa provided that $\mathrm{sub}K$ is atomistic.*

Theorem 3.3 *Let K be a $\mathcal{K}$-object such that $\mathrm{sub}K$ is a complete Boolean algebra. Then K is compact if and only if, for every subclass $\mathcal{S} \subseteq \mathrm{sub}X$ consisting of c-open subobjects, from $\bigvee \mathcal{S} = \mathrm{id}_K$ it follows that there is a finite subset $\mathcal{T} \subseteq \mathcal{S}$ with $\bigvee \mathcal{T} = \mathrm{id}_K$.*

Theorem 3.4 *Let c be hereditary and let $m : M \to K$ be a c-closed embedding in $\mathcal{K}$. If K is compact, then M is compact too.*

Theorem 3.5 *Let $f : K \to L$ be a $\mathcal{K}$-morphism. If K is compact and $f \in \mathcal{E}$, then L is compact too.*

Theorem 3.6 (Absolute closedness) *Let c be grounded, additive, idempotent and hereditary. Let $m : M \to K$ be an embedding in $\mathcal{K}$ where M is compact, K is separated and* sub$K$ *is an atomistic Boolean algebra. Then m is c-closed.*

Corollary 3.7 *Let $\mathcal{K}$ have embeddings and $(\mathcal{E}, Emb_{\mathcal{M}})$-factorization structure and let c be grounded, additive, idempotent and hereditary. Let $f : K \to L$ be a $\mathcal{K}$-morphism where K is compact and L is separated with the property that* sub$L$ *is an atomistic Boolean algebra. Then f is c-preserving.*

Remark 3.8 (Maximality) Let the assumptions of Corollary 3.7 be satisfied and let $\mathcal{K}$ have the property that each $\mathcal{K}$-morphism which is a c-preserving $\mathcal{X}$-isomorphism is a $\mathcal{K}$-isomorphism. Then f is a $\mathcal{K}$-isomorphism whenever it is an $\mathcal{X}$-isomorphism. Moreover, let $|K| = |L|$ and suppose that $c_K \leq c_L$ implies that $\mathrm{id}_{|K|}$ is a $\mathcal{K}$-morphism $\mathrm{id}_{|K|} : K \to L$. Then $c_K \leq c_L$ (i.e., $c_K(m) \leq c_L(m)$ for each $m \in \mathrm{sub}K = \mathrm{sub}L$) implies $c_K = c_L$ by Corollary 3.7 (putting $f = \mathrm{id}_{|K|}$). Thus, given an $\mathcal{X}$-object X, in the class of all c_K with K a separated $\mathcal{K}$-object such that $|K| = X$, c_K with K compact are maximal (provided that the class is nonempty).

Corollary 3.9 *Let $\mathcal{E}$ be stable under pullbacks and c be idempotent. Let $K = \prod_{i \in I} K_i$ be a product in K such that all projections $pr_i : K \to K_i$, $i \in I$, belong to $\mathcal{E}$. If K is compact, then K_i is compact for each $i \in I$.*

Theorem 3.10 (Tychonoff's Theorem) *Let c be idempotent and let $K = \prod_{i \in I} K_i$ be a product in $\mathcal{K}$ such that* subK_i *is atomic for each $i \in I$ and all atoms of* subK_i*, $i \in I$, have the same domain (up to isomorphisms). Let every centered class $\mathcal{T} \subseteq \mathrm{sub}K$ have the property that t, $\bar{t}$ and $c_K(t)$ are pseudocomplementable with $\bar{\bar{t}} = t$ for each $t \in \mathcal{T}$.*

Finally, whenever $x_i \in \mathrm{sub}K_i$ *is an atom for each* $i \in I$, *let the atom* $x = \langle x_i;\ i \in I\rangle$ *fulfill the following condition:*

> *There exists a neighborhood base* $\mathcal{B} \subseteq \mathcal{N}(x)$ *such that, for each* $p \in \mathcal{B}$, *there is a finite subset* $I' \subseteq I$ *with* $p = \bigcap_{i\in I'} \mathrm{pr}_i^{-1} n_i$ *where* $n_i \in \mathcal{N}(x_i)$ *for each* $i \in I'$.

If K_i *is compact for each* $i \in I$, *then* K *is compact too.*

Recall that a $\mathcal{K}$-object K is called

(a) *c-separated* if the diagonal morphism $\delta_K : |K| \to |K| \times |K|$ is c-closed,

(b) *c-compact* if the projection $pr_L : K \times L \to L$ is c-preserving for every $\mathcal{K}$-object L.

Theorem 3.11 *Let* K *be a* $\mathcal{K}$*-object such that* $\mathrm{sub}(K \times K)$ *is atomistic and both* δ_K *and* $c_K(\delta_K)$ *are pseudocomplementable. Let, for every atom* $m \in sub(K \times K)$, *both the projections* $pr_i : |K| \times |K| \to |K|$, $i = 1, 2$, *fulfill* $pr_i \circ m \in \mathcal{M}$ *and let from* $p \in \mathcal{N}(pr_1 \circ m)$ *and* $q \in \mathcal{N}(pr_2 \circ m)$ *it follows that* $p \times q \in \mathcal{N}(m)$. *If* K *is separated, then it is c-separated.*

Theorem 3.12 *Let* K *be a* $\mathcal{K}$*-object such that all atoms of* $\mathrm{sub}K$ *have the same domain (up to isomorphisms) and* δ_K, $\overline{\delta_K}$, $c_{K\times K}(\delta_K)$ *are pseudocomplementable with* $\overline{\overline{\delta_K}} = \delta_K$. *Let, for any pair of atoms* $p, q \in \mathrm{sub}K$, *from* $n \in \mathcal{N}(\langle p, q\rangle)$ *it follows that* $pr_1 \circ n \in \mathcal{N}(p)$ *and* $pr_2 \circ n \in \mathcal{N}(q)$. *If* K *is c-separated, then it is separated.*

Theorem 3.13 *Let* c *be additive and* $\mathrm{sub}L$ *be an atomic Boolean algebra for each* $\mathcal{K}$*-object* L. *Let* K *be a* $\mathcal{K}$*-object satisfying the following condition:*

> *Given a* $\mathcal{K}$ *object* L, *an atom* $y \in \mathrm{sub}L$ *and a subobject* $m \in \mathrm{sub}(K \times L)$ *with* $pr_L(c_{K\times L}(m)) \wedge y = o_L$, *for each*

atom $x \in \mathrm{sub}K$ *there are subobjects* $u_x \in \mathrm{sub}K$ *and* $v_x \in \mathrm{sub}L$*,* u_x *c-closed, such that* $u_x \wedge x = o_K$*,* $c_L(v_x) \wedge y = o_L$*, and* $c_{K\times L}(m) \leq pr_K^{-1}(u_x) \vee pr_L^{-1}(v_x)$*.*

If K *is compact, then it is c-compact.*

Theorem 3.14 *Let* c *be idempotent and* K *be a* $\mathcal{K}$*-object with the properties that* $\mathrm{sub}K$ *is a Boolean algebra and for any centered subclass* $\mathcal{F} \subseteq \mathrm{sub}K$ *of c-closed subobjects of* K *there exist a* $\mathcal{K}$*-object* L *and a c-dense subobject* $m : |K| \to |L|$ *of* L *such that the following conditions are satisfied:*

(1) $\mathrm{sub}(K \times L)$ *is atomic.*

(2) For any atom $z \in \mathrm{sub}(K \times L)$*, from* $p \in \mathcal{N}(pr_K(z))$ *and* $q \in \mathcal{N}(pr_L(z))$ *it follows that* $p \times q \in \mathcal{N}(z)$*.*

(3) There exists a subobject $y \in \mathrm{sub}L$ *with* $y > o_L$*,* $y \wedge m = o_L$*, and* $y \vee m(s) \in \mathcal{N}(y)$ *for each* $s \in \mathcal{F}$*.*

If K *is c-compact, then it is compact.*

4 Convergence

For each $\mathcal{X}$-object X we denote by $\mathbf{R}_X$ the conglomerate of all centered subclasses of $\mathrm{sub}X$. Given a $\mathcal{K}$-object K, we write briefly $\mathbf{R}_K$ instead of $\mathbf{R}_{|K|}$.

Definition 4.1 Let K be a $\mathcal{K}$-object, $m \in \mathrm{sub}K$ and $\mathcal{R} \in \mathbf{R}_K$.

(a) We say that $\mathcal{R}$ *converges* to m, in symbols $\mathcal{R} \to m$, if, for each $p \in \mathrm{sub}K$ with $o_K < p \leq m$ and each $n \in \mathcal{N}(p)$ there exists $r \in \mathcal{R}$ such that $r \leq n$.

(b) m is called a *clustering* of $\mathcal{R}$ provided that $m \leq c_K(r)$ for each $r \in \mathcal{R}$ (i.e., provided that $m \leq \bigwedge_{r\in\mathcal{R}} c_K(r)$).

Proposition 4.2 *Let* K *be a* $\mathcal{K}$*-object. Then*

(1) $\mathcal{R} \to o_K$ *for each* $\mathcal{R} \in \mathbf{R}_K$.

(2) $\mathcal{N}(m) \to m$ *whenever* m *is an atom of* $\mathrm{sub}K$.

(3) For any $\mathcal{R} \in \mathbf{R}_K$ *and any* $m \in \mathrm{sub}K$, *from* $\mathcal{R} \to m$ *it follows that* $\mathcal{R} \to p$ *for each* $p \in \mathrm{sub}K$, $p \leq m$.

(4) Let the lattice $\mathrm{sub}K$ *be atomic, let* $\mathcal{R} \in \mathbf{R}_K$ *and let* $m \in \mathrm{sub}K$. *If* $\mathcal{R} \to a$ *for each atom* $a \in \mathrm{sub}K$ *with* $a \leq m$, *then* $\mathcal{R} \to m$.

(5) For any $\mathcal{R} \in \mathbf{R}_K$ *and any* $m \in \mathrm{sub}K$, *from* $\mathcal{R} \to m$ *it follows that* $\mathcal{S} \to m$ *whenever* $\mathcal{S} \in \mathbf{R}_K$ *is finer than* $\mathcal{R}$.

(6) If $\mathcal{R} \in \mathbf{R}_K$ *is a stack on* $\mathrm{sub}K$ *and* $m \in \mathrm{sub}K$, *then* $\mathcal{R} \to m$ *if and only if* $\mathcal{N}(p) \subseteq \mathcal{R}$ *for each* $p \in \mathrm{sub}K$ *with* $o_K < p \leq m$.

(7) o_K *is a clustering of every* $\mathcal{R} \in \mathbf{R}_K$.

(8) Let $\mathcal{R} \in \mathbf{R}_K$ *and* $m, n \in \mathrm{sub}K$. *If* m *is a clustering of* $\mathcal{R}$ *and* $n \leq m$, *then* n *is a clustering of* $\mathcal{R}$ *too.*

Proposition 4.3 *Let* K *be a* $\mathcal{K}$*-object,* $m, p \in \mathrm{sub}K$, *and let* m *be an atom of* $\mathrm{sub}K$. *If* $m \leq c_K(p)$, *then there exists* $\mathcal{R} \in \mathbf{R}_K$ *such that* $\mathcal{R} \to m$ *and* $n \wedge p > o_K$ *for each* $n \in \mathcal{R}$, *and vice versa provided that* $\mathrm{sub}K$ *is a Boolean algebra.*

Proposition 4.4 *Let* K *be a* $\mathcal{K}$*-object such that* $\mathrm{sub}K$ *is a Boolean algebra, let* $\mathcal{R} \in \mathbf{R}_K$ *be a stack on* $\mathrm{sub}K$ *and let* $m \in \mathrm{sub}K$ *be a join of atoms. If there exists* $\mathcal{S} \in \mathbf{R}_K$ *with* $\mathcal{R} \subseteq \mathcal{S}$ *and* $\mathcal{S} \to m$, *then* m *is a clustering of* $\mathcal{R}$, *and vice versa provided that* $\mathrm{sub}K$ *is atomic,* c *is additive and* $\mathcal{R}$ *is a filter.*

Corollary 4.5 *Let* $K \in \mathcal{K}$ *be an object such that* $\mathrm{sub}K$ *is a Boolean algebra, let* $\mathcal{R} \in \mathbf{R}_K$ *be a stack and let* $m \in \mathrm{sub}K$ *be a join of atoms. If* $\mathcal{R} \to m$, *then* m *is a clustering of* $\mathcal{R}$.

Corollary 4.6 *Let c be additive, K be a $\mathcal{K}$-object such that* $\mathrm{sub}K$ *is an atomic Boolean algebra, and let $\mathcal{R} \in \mathbf{R}_K$ be an ultrafilter. Then $\mathcal{R} \to m$ if and only if m is a clustering of $\mathcal{R}$.*

Theorem 4.7 *Let $f : K \to L$ be a $\mathcal{K}$-morphism, $m \in \mathrm{sub}K$ and $\mathcal{R} \in \mathbf{R}_K$. If $\mathcal{R} \to m$, then $f(\mathcal{R}) \to f(m)$.*

Let $K = \prod_{i\in I} K_i$ be a product in $\mathcal{K}$ and let $\mathcal{R} \in \mathbf{R}_K$. By Theorem 4.7, given $m \in \mathrm{sub}K$, $\mathcal{R} \to m$ implies $\mathrm{pr}_i(\mathcal{R}) \to \mathrm{pr}_i(m)$ for each $i \in I$. If the converse implication is also valid, we say that the centered class $\mathcal{R}$ is *convergence-compatible* with the product K.

Proposition 4.8 *Let in $\mathcal{X}$ the non-trivial objects be stable under products and let all projections in $\mathcal{K}$ belong to $\mathcal{E}$. Let $K = \prod_{i\in I} K_i$ be a product in $\mathcal{K}$ and, for each $i \in I$, let $\mathcal{R}_i \in \mathbf{R}_{K_i}$, $m_i \in \mathrm{sub}K_i$ and $\mathcal{R}_i \to m_i$. If $\prod_{i\in I} \mathcal{R}_i \in \mathbf{R}_K$ is convergence-compatible with K, then $\prod_{i\in I} \mathcal{R}_i \to \prod_{i\in I} m_i$.*

Theorem 4.9 *Let K be a $\mathcal{K}$-object. If K is separated, then from $\mathcal{R} \to m$ and $\mathcal{R} \to p$ it follows that $m = p$ whenever $m, p \in \mathrm{sub}K$ are atoms and $\mathcal{R} \in \mathbf{R}_K$, and vice versa provided that c is additive and* $\mathrm{sub}K$ *is a Boolean algebra.*

Theorem 4.10 *Let K be a $\mathcal{K}$-object. If every $\mathcal{R} \in \mathbf{R}_K$ has a clustering different from o_K, then K is compact, and vice versa provided that c is idempotent.*

Remark 4.11 The introduced concept of convergence may be strengthened by saying that $\mathcal{R} \in \mathbf{R}_K$ converges to $m \in \mathrm{sub}K$ if $\mathcal{N}(p) \subseteq \mathcal{R}$ for each $p \in \mathrm{sub}K$ with $o_K < p \leq m$. Then all statements concerning convergence remain valid (and this is true even if the assumption that $\mathcal{R}$ is a stack is omitted wherever it occurs).