Representability Relative to a Doctrine

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Many classical notions in CT are representability notions

1. Limits $=$ representability of cone functors.
2. Adjunctions $=$ representability of $\mathscr{A}(L-, A)$.
3. Monoidal structures $=$ representability of promonoidal structures.
4. $\ldots$

Weakened representability $\Rightarrow$ weakened notions.

Example: Weak Limits

$F : \mathscr{A}^{op} \longrightarrow \mathrm{Set}$ is weakly representable if there is an epimorphism $\mathscr{A}(-, A) \to F$ for some A.
For a weak limit of $D : \mathscr{D} \to \mathscr{A}$, choose F to be the cone functor

$$\mathrm{Cone}(D) : X \mapsto \text{the set of } D\text{-cones with vertex } X$$

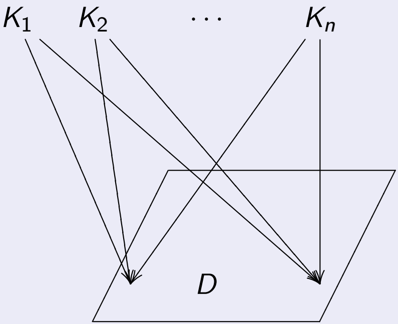
Hence an epimorphism

$$\mathscr{A}(-, A) \to \mathrm{Cone}(D)$$

meaning: there is a distinguished cone for D with vertex A through which any other factors (not necessarily uniquely).

Example: A Finite Plurilimit of a Finite Diagram D (P.Karazeris, J.Rosický,JV, JPAA, 2005)

There exists a finite set of distinguished cones



for D through which any other factors uniquely up to a zig-zag:

$$\mathrm{Cone}(D) \cong \mathrm{colim}_i \mathscr{A}(-, K_i)$$

Plurirepresentability

$F : \mathscr{A}^{op} \longrightarrow \mathrm{Set}$ is plurirepresentable if there is a natural isomorphism

$$F \cong \mathrm{colim}_i \mathscr{A}(-, K_i)$$

for some finite diagram $K : \mathscr{K} \to \mathscr{A}$.

Many Other Such Notions

1. Multirepresentability:

$$F \cong \coprod_i \mathscr{A}(-, K_i)$$

for some finite discrete diagram $K : \mathscr{K} \to \mathscr{A}$.

2. . . .

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Many Other Such Notions

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2. ...

Common Important Features

Given $\mathscr{A}$, form a category $\mathbb{C}(\mathscr{A})$ that

1. contains $\mathscr{A}$
2. is contained in $[\mathscr{A}^{op}, \mathrm{Set}]$

$$\mathscr{A} \dashrightarrow \mathbb{C}(\mathscr{A}) \dashrightarrow [\mathscr{A}^{op}, \mathrm{Set}]$$

to measure the “degree of representability” of (say) cone functors for $D : \mathscr{D} \to \mathscr{A}$.

The Goals of the Talk

1. To give a uniform environment where weak notions can be studied.
2. To show that weak notions abound: in domain theory, in general algebra, ...
3. Weak limits have connections to honest limits in free cocompletions.

For this, it is convenient to work in enriched categories.
In fact, it does not make the reasoning any harder.

Motivation: Representability as a Factorization

Let $\mathscr{I}$ be the one-morphism category.
For $F : \mathscr{A}^{op} \to \mathrm{Set}$, denote by $\ulcorner F \urcorner : \mathscr{I} \to [\mathscr{A}^{op}, \mathrm{Set}]$ the name of F, i.e., $\ulcorner F \urcorner(*) = F$.
F is representable if there is a factorization

$$\begin{tikzcd} \mathscr{I} \arrow[d, dotted] \arrow[dr, "\ulcorner F \urcorner"] & \\ \mathscr{A} \arrow[r, "Y_{\mathscr{A}}"'] & {[\mathscr{A}^{op}, \mathrm{Set}]} \end{tikzcd}$$

to within an isomorphism.

The Weakening Strategy

In the diagram

$$
\begin{tikzcd}
\mathscr{I} \arrow[d, dotted] \arrow[dr, "\ulcorner F \urcorner"] & \\
\mathscr{A} \arrow[r, "Y_{\mathscr{A}}"'] & {[\mathscr{A}^{op}, \mathrm{Set}]}
\end{tikzcd}
$$

replace

1. $\mathscr{A}$ by $\mathbb{C}(\mathscr{A})$ (with a fully faithful $\gamma_{\mathscr{A}} : \mathscr{A} \to \mathbb{C}(\mathscr{A})$).
2. $Y_{\mathscr{A}} : \mathscr{A} \to [\mathscr{A}^{op}, \mathrm{Set}]$ by a fully faithful $\widetilde{\gamma_{\mathscr{A}}} : \mathbb{C}(\mathscr{A}) \to [\mathscr{A}^{op}, \mathrm{Set}]$, $X \mapsto \mathbb{C}(\mathscr{A})(\gamma_{\mathscr{A}}-, X)$.
3. $\mathscr{I}$ by a general indexing category $\mathscr{M}$.
4. $\ulcorner F \urcorner$ by a general functor $G : \mathscr{M} \to [\mathscr{A}^{op}, \mathrm{Set}]$.
5. Set by a well-behaved base monoidal category $\mathscr{V}$.

Definition

A doctrine on $\mathscr{V}$-CAT is a pair $(\mathbb{C}, \gamma)$ consisting of a pseudofunctor $\mathbb{C} : \mathscr{V}\text{-CAT} \to \mathscr{V}\text{-CAT}$ and a pseudonatural $\gamma : \mathrm{Id} \to \mathbb{C}$ such that for each $\mathscr{A}$:

1. $\gamma_{\mathscr{A}} : \mathscr{A} \to \mathbb{C}(\mathscr{A})$ is fully faithful.
2. $\widetilde{\gamma_{\mathscr{A}}} : \mathbb{C}(\mathscr{A}) \to [\mathscr{A}^{op}, \mathscr{V}]$, $X \mapsto \mathbb{C}(\mathscr{A})(\gamma_{\mathscr{A}}-, X)$, is fully faithful (i.e., $\gamma_{\mathscr{A}}$ is dense).

Examples of Doctrines

1. $(\mathrm{Id}, \mathrm{id})$.
2. Any free cocompletion $\gamma_{\mathscr{A}} : \mathscr{A} \to \mathbb{C}(\mathscr{A})$ under a class $\mathbb{C}$ of colimits.
3. $\mathscr{V} = \mathrm{Set}$, $\mathbb{Q}(\mathscr{A})$ = quotients of representables.

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Definition

A functor $G : \mathscr{M} \to [\mathscr{A}^{op}, \mathscr{V}]$ is representable relative to $(\mathbb{C}, \gamma)$, if there exists a factorization

$$\begin{tikzcd} \mathscr{M} \arrow[d, dotted, "\mathrm{rep}(G)"'] \arrow[dr, "G"] & \\ \mathbb{C}(\mathscr{A}) \arrow[r, "\widetilde{\gamma_{\mathscr{A}}}"'] & {[\mathscr{A}^{op}, \mathscr{V}]} \end{tikzcd}$$

to within an isomorphism. The isomorphism $\alpha : G \to \widetilde{\gamma_{\mathscr{A}}} \cdot \mathrm{rep}(G)$ is called the representation.

This means:

$$\alpha_{M,A} : (GM)(A) \cong \mathbb{C}(\mathscr{A})(\gamma_{\mathscr{A}} A, \mathrm{rep}(G)M)$$

holds naturally in M and A.

Examples of Representability Relative to $(\mathbb{C}, \gamma)$

$\mathscr{V} = \mathsf{Set}$, $\mathscr{M} = \mathscr{I}$, $G = \ulcorner F \urcorner$ for $F : \mathscr{A}^{op} \to \mathscr{V}$.

1. Representability relative to $(\mathrm{Id}, \mathrm{id})$ is the usual representability.
2. Representability relative to $(\mathbb{Q}, \gamma)$ — the doctrine of quotients of representables — is the weak representability.
3. Representability relative to the doctrine of cocompletions under finite colimits is the plurirepresentability.
4. Etc. . .

The case $G = \widetilde{F} : \mathscr{M} \to [\mathscr{A}^{op}, \mathscr{V}]$ for $F : \mathscr{A} \to \mathscr{M}$

Representability of $\widetilde{F}$ relative to $(\mathbb{C}, \gamma)$ is an isomorphism

$$\alpha_{M,A} : \mathbb{C}(\mathscr{A})(\gamma_{\mathscr{A}} A, \mathrm{rep}(\widetilde{F})M) \cong (\widetilde{F}M)(A) = \mathscr{M}(FA, M)$$

natural in M and A.
This means: F is a left adjoint along $\gamma_{\mathscr{A}}$, $F \dashv_{\gamma_{\mathscr{A}}} \mathrm{rep}(\widetilde{F})$, studied by Max Kelly, Walter Tholen, ...

$$\begin{tikzcd}
\mathscr{M} \arrow[d, dotted, "\mathrm{rep}(\widetilde{F})"'] \arrow[dr, "\widetilde{F}"] & \\
\mathbb{C}(\mathscr{A}) \arrow[r, "\widetilde{\gamma_{\mathscr{A}}}"'] & {[\mathscr{A}^{op}, \mathscr{V}]}
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathscr{M} \arrow[dr, "\mathrm{rep}(\widetilde{F})"] \arrow[d, phantom, "\Uparrow"] & \\
\mathscr{A} \arrow[ur, "F"] \arrow[rr, "\gamma_{\mathscr{A}}"'] & {} & \mathbb{C}(\mathscr{A})
\end{tikzcd}$$

Representability Relative to $(\mathbb{C}, \gamma)$ of $\widetilde{F}$, for $F : \mathscr{A} \to \mathscr{M}$

1. Any $\mathscr{V}$, $(\mathbb{C}, \gamma)$=identity doctrine. Then $F \dashv_{\mathrm{id}} \mathrm{rep}(\widetilde{F})$ is an honest adjuction:

$$\alpha_{M,A} : \mathscr{A}(\mathrm{id}_{\mathscr{A}} A, \mathrm{rep}(\widetilde{F})M) \cong (\widetilde{F} M)(A) = \mathscr{M}(FA, M)$$

natural in M and A.

2. $\mathscr{V}$=Set, $(\mathbb{C}, \gamma)$=free cocompletion under small colimits. Then $F \dashv_{\gamma_{\mathscr{A}}} \mathrm{rep}(\widetilde{F})$ asserts a solution set condition.

Example: The "Most General" Gabriel-Ulmer Duality

Suppose $(\mathbb{C}, \gamma)$ is a fixed free-cocompletion doctrine.
The obvious correspondence

$$
\begin{aligned}
G &: \mathscr{A} \to [\mathscr{B}^{op}, \mathscr{V}] \\
\mathrm{rep}(G) &: \mathscr{A} \to \mathbb{C}(\mathscr{B}) \\
\mathrm{Lan}_{\gamma_{\mathscr{A}}}(\mathrm{rep}(G)) &: \mathbb{C}(\mathscr{A}) \to \mathbb{C}(\mathscr{B})
\end{aligned}
$$

for $\mathscr{A}$, $\mathscr{B}$ small and G representable relative to $\mathbb{C}$ is a part of a biequivalence between certain profunctors and "$\mathbb{C}$-accessible functors".
This biequivalence restricts to duality of "theory morphisms" and "$\mathbb{C}$-accessible right adjoints".

Recollection of (Weighted) Limits

For a diagram $D : \mathscr{D} \to \mathscr{A}$ together with a weight $W : \mathscr{M} \to [\mathscr{D}, \mathscr{V}]^{op}$ form a cylinder functor

$$\mathrm{Cyl}(W, D) : \mathscr{M} \to [\mathscr{A}^{op}, \mathscr{V}], M \mapsto [\mathscr{D}, \mathscr{V}]^{op}(\widehat{D}-, WM)$$

where $\widehat{D} : A \mapsto \mathscr{A}(A, D-)$.
A limit of D weighted by W is a representation $\{W, D\} : \mathscr{M} \to \mathscr{A}$ of $\mathrm{Cyl}(W, D)$, i.e., we have a diagram

$$\begin{tikzcd} \mathscr{M} \arrow[d, "\{W,D\}"'] \arrow[dr, "\mathrm{Cyl}(W,D)"] & \\ \mathscr{A} \arrow[r, "Y_{\mathscr{A}}"'] & {[\mathscr{A}^{op}, \mathscr{V}]} \end{tikzcd}$$

commutative to within an isomorphism.

(Weighted) Limits Relative to a Doctrine

A limit relative to $(\mathbb{C},\gamma)$ of $D : \mathscr{D} \to \mathscr{A}$ weighted by $W : \mathscr{M} \to [\mathscr{D},\mathscr{V}]^{op}$ is a representation $\{W,D\}_{(\mathbb{C},\gamma)} : \mathscr{M} \to \mathbb{C}(\mathscr{A})$ of $\mathrm{Cyl}(W,D)$ relative to $(\mathbb{C},\gamma)$, i.e., we have a diagram

$$\begin{tikzcd} \mathscr{M} \arrow[d, "\{W,D\}_{(\mathbb{C},\gamma)}"'] \arrow[dr, "\mathrm{Cyl}(W,D)"] & \\ \mathbb{C}(\mathscr{A}) \arrow[r, "\widetilde{\gamma_{\mathscr{A}}}"'] & {[\mathscr{A}^{op},\mathscr{V}]} \end{tikzcd}$$

commutative to within an isomorphism.

Or, in elementary terms:

$$\begin{aligned} \mathbb{C}(\mathscr{A})(\gamma_{\mathscr{A}}A,\{W,D\}_{(\mathbb{C},\gamma)}M) &\cong \mathrm{Cyl}(W,D)(M)(A) \\ &= [\mathscr{D},\mathscr{V}](WM,\widehat{D}A) \end{aligned}$$

naturally in M and A.

(Weighted) Limits of Some Class Relative to a Doctrine

Fix a limit doctrine $(\mathbb{L}, \lambda)$.
That is: for each $\mathscr{A}$, $\lambda_{\mathscr{A}} : \mathscr{A} \to \mathbb{L}(\mathscr{A})$ is a free completion under a class of limits.[a]

1. A weight $W : \mathscr{M} \to [\mathscr{D}, \mathscr{V}]^{op}$, $\mathscr{M}$, $\mathscr{D}$ small, is an $\mathbb{L}$-weight, if it factors through $\widehat{\lambda_{\mathscr{D}}} : \mathbb{L}(\mathscr{D}) \to [\mathscr{D}, \mathscr{V}]^{op}$.
2. A category $\mathscr{A}$ has $\mathbb{L}$-limits relative to $(\mathbb{C}, \gamma)$, if $\{W, D\}_{(\mathbb{C},\gamma)}$ exists for every $\mathbb{L}$-weight W and every diagram D.

[a] $(\mathbb{L}, \lambda)$ and $(\mathbb{C}, \gamma)$ are independent of each other.

Main Theorem

For any $\mathscr{A}$, the following are equivalent:

1. $\mathscr{A}$ has $\mathbb{L}$-limits relative to $(\mathbb{C}, \gamma)$.
2. $\mathbb{C}(\mathscr{A})$ has $\mathbb{L}$-limits of representables.
3. $\widetilde{\lambda_{\mathscr{A}}} : \mathbb{L}(\mathscr{A}) \to [\mathscr{A}^{op}, \mathscr{V}]$ is representable relative to $(\mathbb{C}, \gamma)$, i.e.,
$$\lambda_{\mathscr{A}} \dashv_{\gamma_{\mathscr{A}}} U : \mathbb{L}(\mathscr{A}) \to \mathbb{C}(\mathscr{A})$$
holds.

The Meaning of $U : \mathbb{L}(\mathscr{A}) \to \mathbb{C}(\mathscr{A})$

$$U : \mathrm{Ran}_D W \mapsto \{W, \gamma_{\mathscr{A}} D\}$$

for every diagram $D : \mathscr{D} \to \mathscr{A}$ and every $\mathbb{L}$-weight $W : \mathscr{D} \to \mathscr{V}$.

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for every diagram $D : \mathscr{D} \to \mathscr{A}$ and every $\mathbb{L}$-weight $W : \mathscr{D} \to \mathscr{V}$.

Main Theorem when $(\mathbb{C}, \gamma)$ is a Colimit Doctrine

For any $\mathscr{A}$, the following are equivalent:

1. $\mathscr{A}$ has $\mathbb{L}$-limits relative to $(\mathbb{C}, \gamma)$.
2. For every X in $\mathbb{L}(\mathscr{A})$ there exists a pair $W_X : \mathscr{K}_X^{op} \to \mathscr{V}$, $J_X : \mathscr{K}_X \to \mathscr{A}$, with W_X a $\mathbb{C}$-weight such that there is an isomorphism
$$\mathbb{L}(\mathscr{A})(\lambda_{\mathscr{A}} A, X) \cong \int^{K \in \mathscr{K}_X^{op}} W_X K \otimes \mathscr{A}(A, J_X K)$$
natural in A.
3. $\mathbb{C}(\lambda_{\mathscr{A}}) : \mathbb{C}(\mathscr{A}) \to \mathbb{C}\mathbb{L}(\mathscr{A})$ has a right adjoint that preserves $\mathbb{C}$-colimits.

Examples

1. $\mathscr{V}$=Set, $\mathbb{L}$=finite limits, $\mathbb{C}$=finite colimits. $\mathscr{A}$ has $\mathbb{L}$-limits relative to $\mathbb{C}$ iff it has finite plurilimits of finite diagrams (P.Karazeris, J.Rosický,J.V., JPAA, 2005). Exploit the coend formula

$$\mathbb{L}(\mathscr{A})(\lambda_{\mathscr{A}}A, X) \cong \int^{K \in \mathscr{K}_X^{op}} W_X K \times \mathscr{A}(A, J_X K)$$

   - $\mathscr{K}_X$=finite category
   - $\mathbb{L}(\mathscr{A})(\lambda_{\mathscr{A}}A, X)$=set of cones for a finite diagram X with vertex A
   - $W_X K$=set of distinguished cones for a finite diagram X with vertex K
   - the coend provides the factorizations up to a zig-zag

Examples, cont.

2. The same $\mathbb{L}$ and $\mathbb{C}$ as above (i.e., finite) but with $\mathscr{V}$=Abelian groups. One-object $\mathscr{A}$ has $\mathbb{L}$-limits relative to $\mathbb{C}$ iff it is a left coherent ring.

   A ring A is left coherent iff the dualization functor

$$\mathrm{Hom}(-, A) : \mathrm{Mod}\text{-}A \to A\text{-}\mathrm{Mod}$$

   restricts to categories of f.p. A-modules (R.R.Colby, J.Algebra, 1975).

3. Any $\mathscr{V}$, $\mathbb{L}$=small limits, $\mathbb{C}$=small colimits. $\mathscr{A}$ has $\mathbb{L}$-limits relative to $\mathbb{C}$ iff $\mathbb{C}(\mathscr{A})$ (the category of small presheaves) has small limits of representables (B.Day, S.Lack, JPAA, 2007).

4. ...

Corollary of the Main Theorem

The following are equivalent:

1. Every $\mathbb{C}(\mathscr{A})$ has $\mathbb{L}$-limits whenever $\mathscr{A}$ has $\mathbb{L}$-limits.
2. Every $\mathbb{C}(\mathscr{A})$ has $\mathbb{L}$-limits whenever $\mathscr{A}$ has $\mathbb{L}$-limits relative to $\mathbb{C}$.
3. Every $\mathbb{C}(\mathbb{L}(\mathscr{A}))$ has $\mathbb{L}$-limits.

These equivalent conditions are satisfied in the presence of a distributive law $\delta : \mathbb{L}\mathbb{C} \to \mathbb{C}\mathbb{L}$.

Promonoidal Structures

A promonoidal structure on $\mathscr{A}$ is given by

$$J : \mathscr{I} \to [\mathscr{A}^{op}, \mathscr{V}] \qquad P : \mathscr{A} \otimes \mathscr{A} \to [\mathscr{A}^{op}, \mathscr{V}]$$

such that P is "associative" and J is a "unit" (to within isomorphisms) — see B.Day, 1974.

Example

A promonoidal structure $(\mathscr{A}, J, P)$ with J, P representable is precisely a monoidal structure on $\mathscr{A}$:

$$\begin{tikzcd} \mathscr{I} \arrow[d, dotted] \arrow[dr, "J"] & \\ \mathscr{A} \arrow[r, "Y_{\mathscr{A}}"'] & {[\mathscr{A}^{op}, \mathscr{V}]} \end{tikzcd} \qquad \begin{tikzcd} \mathscr{A} \otimes \mathscr{A} \arrow[d, dotted] \arrow[dr, "P"] & \\ \mathscr{A} \arrow[r, "Y_{\mathscr{A}}"'] & {[\mathscr{A}^{op}, \mathscr{V}]} \end{tikzcd}$$

Promonoidal Structures

A promonoidal structure on $\mathscr{A}$ is given by

$$J : \mathscr{I} \to [\mathscr{A}^{op}, \mathscr{V}] \qquad P : \mathscr{A} \otimes \mathscr{A} \to [\mathscr{A}^{op}, \mathscr{V}]$$

such that P is "associative" and J is a "unit" (to within isomorphisms) — see B.Day, 1974.

Example

A promonoidal structure $(\mathscr{A}, J, P)$ with J, P representable is precisely a monoidal structure on $\mathscr{A}$:

$$\begin{tikzcd}
\mathscr{I} \arrow[d, dotted] \arrow[dr, "J"] & \\
\mathscr{A} \arrow[r, "Y_{\mathscr{A}}"'] & {[\mathscr{A}^{op}, \mathscr{V}]}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{A} \otimes \mathscr{A} \arrow[d, dotted] \arrow[dr, "P"] & \\
\mathscr{A} \arrow[r, "Y_{\mathscr{A}}"'] & {[\mathscr{A}^{op}, \mathscr{V}]}
\end{tikzcd}$$

Promonoidal Structures Representable Relative to $(\mathbb{C}, \gamma)$

$$
\begin{tikzcd}
\mathscr{I} \arrow[d, dotted] \arrow[dr, "J"] & \\
\mathbb{C}(\mathscr{A}) \arrow[r, "\widetilde{\gamma_{\mathscr{A}}}"'] & {[\mathscr{A}^{op}, \mathscr{V}]}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{A} \otimes \mathscr{A} \arrow[d, dotted] \arrow[dr, "P"] & \\
\mathbb{C}(\mathscr{A}) \arrow[r, "\widetilde{\gamma_{\mathscr{A}}}"'] & {[\mathscr{A}^{op}, \mathscr{V}]}
\end{tikzcd}
$$

are precisely monoidal structures on $\mathbb{C}(\mathscr{A})$, if $\mathbb{C}$ is a doctrine of free cocompletions.
(B.Day, S.Lack, JPAA, 2007 for $\mathbb{C}$=small colimits).

Example: Flatness and Merging

Suppose $\mathscr{A}$ has $\mathbb{L}$-limits relative to $(\mathbb{C}, \gamma)$ and let $\mathscr{B}$ have $\mathbb{L}$-limits and $\widetilde{\gamma_{\mathscr{A}}}$-colimits. A functor $H : \mathscr{A} \to \mathscr{B}$ is called

1. $\mathbb{L}$-flat relative to $(\mathbb{C}, \gamma)$ if $\mathrm{Lan}_{\gamma_{\mathscr{A}}} H : \mathbb{C}(\mathscr{A}) \to \mathscr{B}$ preserves $\mathbb{L}$-limits of representables.
2. merging $\mathbb{L}$-limits relative to $(\mathbb{C}, \gamma)$ if the canonical comparison

$$\widetilde{\gamma_{\mathscr{A}}}(\{W, \gamma_{\mathscr{A}} D\}) * H \to \{W, HD\}$$

is an isomorphism for every $D : \mathscr{D} \to \mathscr{A}$ and every $\mathbb{L}$-weight $W : \mathscr{M} \to [\mathscr{D}, \mathscr{V}]^{op}$ (introduced by H.Hu, W.Tholen).

Result: these concepts are equivalent.

Hear Panagis' talk for further applications.