

STONE SPACES VERSUS PRIESTLEY SPACES

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Priestley (\mathcal{PSp})/Stone as categories which arise from an equivalence induced by a "bigger" adjunction between \mathcal{OrdTop} and $\mathcal{Lat}/\mathcal{Top}$ and \mathcal{Lat} ;

$\mathcal{PSp} = \text{Profinite preorder} + \text{order} / \text{Stone} = \text{Profinite}$;

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Priestley ($\mathcal{P}Sp$)/Stone as categories which arise from an equivalence induced by a "bigger" adjunction between $\mathcal{O}rd\mathcal{T}op$ and $\mathcal{L}at/\mathcal{T}op$ and $\mathcal{L}at$;

$\mathcal{P}Sp = \text{Profinite preorder} + \text{order} / \text{Stone} = \text{Profinite}$;

$\mathcal{P}Sp = \mathcal{O}rd\mathcal{C}omp_{2_{do}} / \text{Stone} = \mathcal{C}omp_{2_d}$.

Recall that

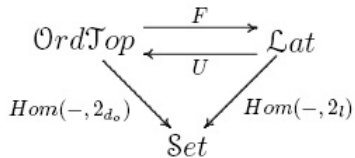
Every adjunction $F \dashv U : \mathcal{A} \rightarrow \mathcal{B}(\eta, \epsilon)$, induces a largest equivalence between the full subcategories \mathcal{A}_0 of \mathcal{A} and \mathcal{B}_0 of \mathcal{B} where

$$\mathcal{A}_0 = \text{Fix}\epsilon \equiv \{A \in \mathcal{A} \mid \epsilon_A \text{ is an isomorphism}\}$$

$$\mathcal{B}_0 = \text{Fix}\eta \equiv \{B \in \mathcal{B} \mid \eta_B \text{ is an isomorphism}\}$$

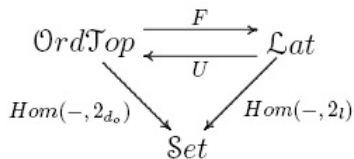
Priestley Duality

- $2_I = \{0 < 1\}$ $2_{do} = (\{0 < 1\}, \mathcal{D})$



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- $\mathcal{2}_l = \{0 < 1\}$ $\mathcal{2}_{do} = (\{0 < 1\}, \mathcal{D})$



Identifying:

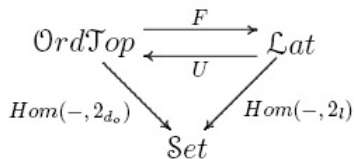
- $f \in \text{Hom}(L, \mathcal{2}_l)$ with $f^{-1}(1)$ (prime filter);

$$F(L) = (\mathcal{F}_p(L), \tau, \subseteq), L \in \text{Lat}$$

$$S = \{U_b | b \in L\} \cup \{\mathcal{F}_p(L) - U_b | b \in L\} \text{ with } U_b = \{F \in \mathcal{F}_p(L) | b \in F\}$$

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Identifying:

- $g \in \text{Hom}(X, 2_{do})$ with $g^{-1}(0)$ (decreasing clopen);

$$U(X) = (\text{DClopen}(X) \cap, \cup), \quad X \in \text{OrdTop}$$

The functor F is left adjoint to $U : \mathcal{O}rdTop^{op} \rightarrow \mathcal{L}at$

- $\eta_L : L \rightarrow UF(L)$, $\eta_L(a) = \Gamma_a = \{F \in \mathcal{F}_p(L) \mid a \in F\}$
- $\epsilon_X(x) = \Sigma_x = \{A \in DClopen(X) \mid x \in A\}$

Priestley Duality

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- $(\mathcal{F}_p(L), \tau, \subseteq)$ is a Priestley space for every distributive lattice L .
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- If X is a Priestley space then ϵ_X is an isomorphism.
- $(DClopen(X), \cap, \cup)$ is a distributive lattice for every space X .
- If L is a distributive lattice then η_L is an isomorphism.

We obtain the Priestley Duality as the equivalence induced by the adjunction above.

Priestley Duality

$$\begin{array}{ccc} \text{OrdTop}^{op} & \begin{array}{c} \xrightarrow{U} \\ \xleftarrow{F} \end{array} & \mathcal{L}at \\ \uparrow & & \uparrow \\ \mathcal{P}Sp^{op} & \sim & \mathcal{D}\mathcal{L}at \end{array}$$

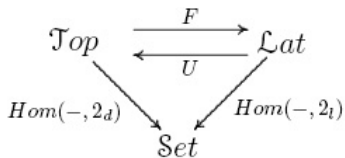
$$\text{Fix}\epsilon = \mathcal{P}Sp$$

$$\text{Fix}\eta = \mathcal{D}\mathcal{L}at$$

Stone Duality

In a similar way, from the facts

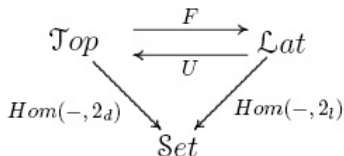
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Stone Duality

In a similar way, from the facts

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- $U(X) = (Clopen(X), \cap, \cup)$, $X \in \mathcal{T}op$, is a Boolean algebra.
- $F(L) = (\mathcal{F}_p(L), \tau)$, $L \in \mathcal{L}at$, is a Stone space.

Stone Duality

$$\begin{array}{ccc} \mathcal{T}op^{op} & \begin{array}{c} \xrightarrow{U} \\ \xleftarrow{F} \end{array} & \mathcal{L}at \\ \uparrow & & \uparrow \\ Stone^{op} & \sim & \mathcal{B}ool \end{array}$$

$$Fix\epsilon = Stone$$

$$Fix\eta = \mathcal{B}ool$$

Profinite orders are the Priestley spaces.

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We are going to show that the **Priestley spaces are limits of finite topologically-discrete preordered spaces.**

- $X \in Stone$

\mathcal{R} - Set of all equivalence relations R of X such that X/R is finite and topologically-discrete.

$$D : \mathcal{R} \rightarrow Stone, D(R) = X/R$$

Profinite Preorders / Profinite Spaces

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$$D : \mathcal{R} \rightarrow Stone, D(R) = X/R$$

$$\begin{array}{ccc} X & & \\ \downarrow \varphi & \searrow p_R & \\ Lim D & \xrightarrow{\lambda_R} & X/R \end{array}$$

The unique morphism φ , for every $R \in \mathcal{R}$, is an homeomorphism.

We remark that:

- $X \in \mathcal{PSp}$

The relation induced in X/R by transitive closure of the image of the relation on X is not, in general, an order relation.

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- The full subcategory of $\mathcal{P}reordTop$:

$\mathcal{P}reord\mathcal{P}$ - Stone spaces equipped with a preorder with respect to which they are totally preordered-disconnected.

Profinite Preorders / Profinite Spaces

- $X \in \mathcal{Preord}\mathcal{P}$

\mathcal{R} - Set of all equivalence relations R of X such that X/R is finite, topologically-discrete and equipped with the **preorder induced by image of the preorder of X** .

$$D_O : \mathcal{R} \rightarrow \mathcal{Preord}\mathcal{P}, D_O(R) = X/R$$

Profinite Preorders / Profinite Spaces

- $X \in \mathcal{P}reord\mathcal{P}$

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$$D_O : \mathcal{R} \rightarrow \mathcal{P}reord\mathcal{P}, D_O(R) = X/R$$

A commutative triangle diagram with X at the top vertex, $Lim D_O$ at the bottom-left vertex, and X/R at the bottom-right vertex. A vertical arrow labeled φ points from X down to $Lim D_O$. A diagonal arrow labeled p_R points from X down and to the right to X/R . A horizontal arrow labeled λ_R points from $Lim D_O$ to the right to X/R .

We just have to prove that the morphism φ is an order isomorphism, for every $R \in \mathcal{R}$.

For that it is enough to show that if

- $x \not\leq x'$
- U is the clopen decreasing subset of X such that $x' \in U$ and $x \notin U$
- R_U the equivalence relation on X corresponding to the partition

$$X = U \cup (X - U)$$

then p_{R_U} separates x and x' and so $\varphi(x) \not\leq \varphi(x')$

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PreordP is the category of profinite preorders.

- $\mathcal{P}Sp$, being a full regular-epireflective subcategory of $\mathcal{P}reord\mathcal{P}$, is closed under limits.

$X \in \mathcal{P}reord\mathcal{T}op$

X is a Priestley space if and only if the limit object of $D_O : \mathcal{R} \rightarrow \mathcal{P}reord\mathcal{P}$ is an ordered space.

Stone Spaces are the 2_d - compact spaces

- Let E be an Hausdorff space.

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Using the terminology of Engelking and Mrówka:

- The E - *completely regular* space are the subspaces of some power of E .
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Full subcategories of $\mathcal{T}op$:

- $\mathcal{C}Reg_E$: E - completely regular spaces.
- $\mathcal{C}omp_E$: E - compact spaces.

$$\mathcal{C}Reg_E \begin{array}{c} \xrightarrow{\quad} \\ \perp \\ \xleftarrow{\quad} \end{array} \mathcal{C}omp_E$$

Stone Spaces are the 2- compact spaces

- $E = 2_d$ (The two point discrete topological space)

$$\mathcal{C}\mathcal{R}eg_{2_d} \begin{array}{c} \xrightarrow{\zeta} \\ \perp \\ \xleftarrow{\quad} \end{array} \mathcal{C}omp_{2_d}$$

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$$\mathcal{CR}eg_{2_d} \begin{array}{c} \xrightarrow{\zeta} \\ \perp \\ \xleftarrow{\quad} \end{array} \mathcal{C}omp_{2_d}$$

$\mathcal{CR}eg_{2_d}$ - category of Hausdorff zero-dimensional spaces,

$\mathcal{C}omp_{2_d}$ - category of Stone spaces,

as proved by B. Banaschewski.

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We consider the full subcategories of $OrdTop$:

- $OrdCReg_E$: E - completely regular spaces with the induced order.
- $OrdComp_E$: E - compact spaces with the induced order.

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$OrdComp_E$ is a reflective subcategory of $OrdCReg_E$.

$$OrdCReg_E \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} OrdComp_E$$

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*OrdComp*_{2_{do}} is the category *PSp*