Stone Spaces versus Priestley Spaces

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$\mathcal{PS}p$ = Profinite preorder + order/ $\mathcal{S}tone$ = Profinite;

$\mathcal{PS}p{=}\mathcal{O}rd\mathcal{C}omp_{2_{do}}/\mathcal{S}tone{=}\mathcal{C}omp_{2_d}$.

Priestley Duality

Recall that

Every adjunction $F \dashv U : \mathcal{A} \to \mathcal{B}(\eta, \epsilon)$, induces a largest equivalence between the full subcategories $\mathcal{A}_0$ of $\mathcal{A}$ and $\mathcal{B}_0$ of $\mathcal{B}$ where

$$\mathcal{A}_0 = Fix\varepsilon \equiv \{A \in \mathcal{A} | \varepsilon_A \text{is an isomorphism}\}$$

$$\mathcal{B}_0 = Fix\eta \equiv \{B \in \mathcal{B} | \eta_B \text{ is an isomorphism}\}$$

Priestley Duality

- $2_l = \{0 < 1\}$ $\qquad$ $2_{do} = (\{0 < 1\}, \mathcal{D})$

$$\begin{tikzcd}
\mathcal{O}rd\mathcal{T}op \arrow[rr, shift left, "F"] \arrow[dr, "Hom(-,2_{d_o})"'] & & \mathcal{L}at \arrow[ll, shift left, "U"] \arrow[dl, "Hom(-,2_l)"] \\
& \mathcal{S}et &
\end{tikzcd}$$

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$$

Identifying:

- $f \in Hom(L, 2_l)$ with $f^{-1}(1)$ (prime filter);

$$F(L) = (\mathcal{F}_p(L), \tau, \subseteq),\ L \in \mathcal{L}at$$

$$S = \{U_b | b \in L\} \cup \{\mathcal{F}_p(L) - U_b | b \in L\} \text{ with } U_b = \{F \in \mathcal{F}_p(L) | b \in F\}$$

- $2_l = \{0 < 1\}$ $\qquad$ $2_{do} = (\{0 < 1\}, \mathcal{D})$

$$\begin{tikzcd} \mathfrak{OrdTop} \arrow[rr, shift left, "F"] \arrow[dr, "Hom(-,2_{d_o})"'] & & \mathfrak{Lat} \arrow[ll, shift left, "U"] \arrow[dl, "Hom(-,2_l)"] \\ & \mathfrak{Set} & \end{tikzcd}$$

Identifying:

- $g \in Hom(X, 2_{do})$ with $g^{-1}(0)$ (decreasing clopen);

$$U(X) = (DClopen(X) \cap, \cup),\ X \in \mathcal{O}rd\mathcal{T}op$$

The functor F is left adjoint to $U : \mathcal{O}rd\mathcal{T}op^{op} \rightarrow \mathcal{L}at$

- $\eta_L : L \rightarrow UF(L)$, $\eta_L(a) = \Gamma_a = \{F \in \mathcal{F}_p(L) | a \in F\}$
- $\epsilon_X(x) = \Sigma_x = \{A \in DClopen(X) | x \in A\}$

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- If X is a Priestley space then ϵ_X is an isomorphism.

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- $(\mathcal{F}_p(L), \tau, \subseteq)$ is a Priestley space for every distributive lattice L.
- If X is a Priestley space then ϵ_X is an isomorphism.
- $(DClopen(X), \cap, \cup)$ is a distributive lattice for every space X.
- If L is a distributive lattice then η_L is an isomorphism.

We obtain the Priestley Duality as the equivalence induced by the adjunction above.

Priestley Duality

$$
\begin{tikzcd}
\mathcal{O}rd\mathcal{T}op^{op} \arrow[r, shift left, "U"] & \mathcal{L}at \arrow[l, shift left, "F"] \\
\mathcal{P}\mathcal{S}p^{op} \arrow[u, hook] \arrow[r, phantom, "\sim"] & \mathcal{D}\mathcal{L}at \arrow[u, hook]
\end{tikzcd}
$$

$Fix\epsilon = \mathcal{PS}p$ $\qquad$ $Fix\eta = \mathcal{DL}at$

Stone Duality

In a similar way, from the facts

- $2_l = \{0 < 1\}$
- $2_d = (\{0,1\}, \mathcal{D})$

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& \mathfrak{Set} &
\end{tikzcd}$$

- $U(X) = (Clopen(X), \cap, \cup)$, $X \in \mathcal{T}op$, is a Boolean algebra.
- $F(L) = (\mathcal{F}_p(L), \tau)$, $L \in \mathcal{L}at$, is a Stone space.

Stone Duality

$$\begin{tikzcd}
\mathcal{T}op^{op} \arrow[r, shift left, "U"] & \mathcal{L}at \arrow[l, shift left, "F"] \\
\mathcal{S}tone^{op} \arrow[u, hook] \arrow[r, phantom, "\sim"] & \mathcal{B}ool \arrow[u, hook]
\end{tikzcd}$$

$Fix\epsilon = \mathcal{S}tone$ $\qquad$ $Fix\eta = \mathcal{B}ool$

Profinite orders are the Priestley spaces.

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Profinite spaces are exactly the Stone spaces.

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We are going to show that the Priestley spaces are limits of finite topologically-discrete preordered spaces.

Profinite Preorders / Profinite Spaces

- $X \in \mathcal{S}tone$

  $\mathcal{R}$ - Set of all equivalence relations R of X such that X/R is finite and topologically-discrete.

  $D : \mathcal{R} \to \mathcal{S}tone$, $D(R) = X/R$

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\begin{tikzcd}
X \arrow[d, "\varphi"] \arrow[dr, "p_R"] & \\
LimD \arrow[r, "\lambda_R"] & X/R
\end{tikzcd}
$$

The unique morphism φ, for every $R \in \mathcal{R}$, is an homeomorphism.

We remark that:

- $X \in \mathcal{PS}p$

The relation induced in X/R by transitive closure of the image of the relation on X is not, in general, an order relation.

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  The relation induced in X/R by transitive closure of the image of the relation on X is not, in general, an order relation.

- The full subcategory of $\mathcal{P}reord\mathcal{T}op$:

  $\mathcal{P}reord\mathcal{P}$ - Stone spaces equipped with a preorder with respect to which they are totally preordered-disconnected.

Profinite Preorders / Profinite Spaces

- $X \in \mathcal{P}reord\mathcal{P}$

  $\mathcal{R}$ - Set of all equivalence relations R of X such that X/R is finite, topologically-discrete and equipped with the preorder induced by image of the preorder of X.

$$D_O : \mathcal{R} \to \mathcal{P}reord\mathcal{P}, \; D_O(R) = X/R$$

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$$\begin{tikzcd}
X \arrow[d, "\varphi"] \arrow[dr, "p_R"] & \\
LimD_O \arrow[r, "\lambda_R"] & X/R
\end{tikzcd}$$

We just have to prove that the morphism φ is an order isomorphism, for every $R \in \mathcal{R}$.

For that it is enough to show that if

- $x \not\preceq x'$
- U is the clopen decreasing subset of X such that $x' \in U$ and $x \notin U$
- R_U the equivalence relation on X corresponding to the partition

$$X = U \cup (X - U)$$

then p_{R_U} separates x and x' and so $\varphi(x) \not\preceq \varphi(x')$

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$\mathcal{P}reord\mathcal{P}$ is the category of profinite preorders.

- $\mathcal{PS}p$, being a full regular-epireflective subcategory of $\mathcal{P}reord\mathcal{P}$, is closed under limits.

$X \in \mathcal{P}reord\mathcal{T}op$

X is a Priestley space if and only if the limit object of $D_O : \mathcal{R} \to \mathcal{P}reord\mathcal{P}$ is an ordered space.

Stone Spaces are the 2_d - *compact spaces*

- Let E be an Hausdorff space.

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  Full subcategories of $\mathcal{T}op$:

  - $\mathcal{CR}eg_E$: E - *completely regular* spaces.
  - $\mathcal{C}omp_E$: E - *compact* spaces.

$$\begin{tikzcd} \mathcal{CR}eg_E \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{C}omp_E \arrow[l, shift left] \end{tikzcd}$$

Stone Spaces are the 2- *compact spaces*

- $E = 2_d$ (The two point discrete topological space)

$$\begin{tikzcd} \mathcal{CR}eg_{2_d} \arrow[r, shift left, "\zeta"] \arrow[r, phantom, "\perp"] & \mathcal{C}omp_{2_d} \arrow[l, shift left] \end{tikzcd}$$

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$$\mathcal{CR}eg_{2_d} \underset{\longleftarrow}{\overset{\zeta}{\underset{\perp}{\longrightarrow}}} \mathcal{C}omp_{2_d}$$

$\mathcal{CR}eg_{2_d}$ - category of Hausdorff zero-dimensional spaces,

$\mathcal{C}omp_{2_d}$ - category of Stone spaces,

as proved by B. Banaschewski.

- Let E be an ordered Hausdorff space.

Priestley Spaces are the 2_{do} - *ordered compact spaces*

- Let E be an ordered Hausdorff space.

  We consider the full subcategories of $\mathcal{O}rd\mathcal{T}op$:

  - $\mathcal{O}rd\mathcal{C}\mathcal{R}eg_E$: E - *completely regular* spaces with the induced order.
  - $\mathcal{O}rd\mathcal{C}omp_E$: E - *compact* spaces with the induced order.

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$\mathcal{O}rd\mathcal{C}omp_E$ is a reflective subcategory of $\mathcal{O}rd\mathcal{CR}eg_E$.

$$\mathcal{O}rd\mathcal{CR}eg_E \rightleftarrows \mathcal{O}rd\mathcal{C}omp_E$$

- $E = 2_{do}$ (The two chain with discrete topology)

$$\begin{tikzcd} \mathcal{O}rd\mathcal{CR}eg_{2_{do}} \arrow[r, shift left, "\zeta_o"] \arrow[r, phantom, "\perp"] & \mathcal{O}rd\mathcal{C}omp_{2_{do}} \arrow[l, shift left] \end{tikzcd}$$

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$\mathcal{O}rd\mathcal{C}omp_{2_{do}}$ is the category $\mathcal{PS}p$