Revisiting Topological Descent Theory

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Effective descent morphisms

The pullback functor

$$f^* : \mathbf{Top}/Y \to \mathbf{Top}/X$$

is monadic.

Theorem. [Reiterman-Tholen1994]

A continuous map $f : X \to Y$ is of effective descent iff:

- for each family of ultrafilters $(\mathfrak{b}_i)_{i\in I}$ and ultrafilter $\mathfrak{u}$ on I,
- whenever $\mathfrak{b}_i \to y_i$, for $i \in I$, and $y_i \xrightarrow{\mathfrak{u}} y$ in Y,
- there exists an ultrafilter $\mathfrak{a}$ on X such that:
  - $\mathfrak{a} \to x \in f^{-1}(y)$ and,
  - for each $U \in \mathfrak{u}$, $\bigcup_{i\in U}(f^{-1}(y_i) \cap \mathrm{adh}(f^{-1}(\mathfrak{b}_i))) \in \mathfrak{a}$.

Theorem. [Plewe1995]

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A continuous map $f : X \to Y$ is a triquotient map if there exists a map $(\ )^\sharp : OX \to OY$ such that, for $U, V \in OX$:

(T1) $U^\sharp \subseteq f(U)$

(T2) $X^\sharp = Y$

(T3) $U \subseteq V \Rightarrow U^\sharp \subseteq V^\sharp$

(T4) $\forall y \in U^\sharp \ \ \forall \Sigma \subseteq OX$ directed

$$f^{-1}(y) \cap U \subseteq \bigcup \Sigma \ \Rightarrow \ \ \exists S \in \Sigma : y \in S^\sharp.$$

Janelidze-Sobral 2002 (CT99)

For each space X, consider:

- $\mathrm{Conv}(X) = \{(\mathfrak{x}, x) \,|\, \mathfrak{x} \to x$ in $X\}$;
- the projection $p : \mathrm{Conv}(X) \to X$, with $p(\mathfrak{x}, x) := x$;
- $\mathfrak{X}$ converges to $(\mathfrak{x}, x)$ in $\mathrm{Conv}(X)$ if $p(\mathfrak{X}) = (\mathfrak{x}, x)$.

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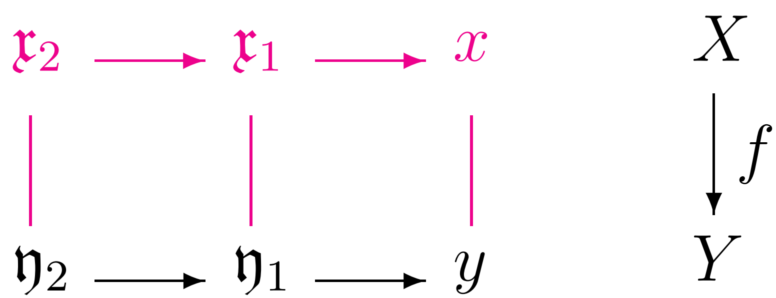

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$$\begin{array}{ccccccccc} \mathfrak{x}_{\alpha+1} & \to & \mathfrak{x}_{\alpha} & \longrightarrow & \cdots & \longrightarrow & \mathfrak{x}_{1} & \longrightarrow & x \\ | & & | & & | & & | & & | \\ \mathfrak{y}_{\alpha+1} & \to & \mathfrak{y}_{\alpha} & \longrightarrow & \cdots & \longrightarrow & \mathfrak{y}_{1} & \longrightarrow & y \end{array} \qquad \begin{array}{c} X \\ \downarrow f \\ Y \end{array}$$

Using $(\ )^\sharp := X \to OY$*: A cont. map* $f : X \to Y$ *is:*

f triquotient map $\Leftrightarrow$ (T1)-(T2)-(T3)-(T4)

f universal quotient map $\Leftrightarrow$ (T1)-(T2)-(T3)-(U4)

f proper surjection $\Leftrightarrow$ (T1)-(T2)-(T3)-(P4)

f open surjection $\Leftrightarrow$ (T1)-(T2)-(T3)-(O4)

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(P4) $\forall y \in U^\sharp\ \forall \Sigma \subseteq OX$ directed $f^{-1}(y) \cap U \subseteq \bigcup \Sigma \Rightarrow \exists S \in \Sigma : y \in S^\sharp$ and $f^{-1}(y) \subseteq S$.

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(O4)
$\forall y \in f(U)\ \forall \Sigma \subseteq OX$ directed $f^{-1}(y) \cap U \subseteq \bigcup \Sigma \Rightarrow \exists S \in \Sigma : y \in S^\sharp$.

Theorem

If X and Y are finite, for any effective descent map $f : X \to Y$ there exists a map $(\ )^\sharp : LC(X) \to LC(Y)$ such that, for $A, B \in LC(X)$,

(1) $A^\sharp \subseteq f(A)$

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Moreover, effective descent maps are those that *stably* have this property.

Exponentiable maps

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Or equivalently, the change of base functor $f^* : \mathbf{Top}/Y \to \mathbf{Top}/X$ has a right adjoint.

Theorem. [C-Hofmann-Tholen2003]

A continuous map $f : X \to Y$ is exponentiable iff

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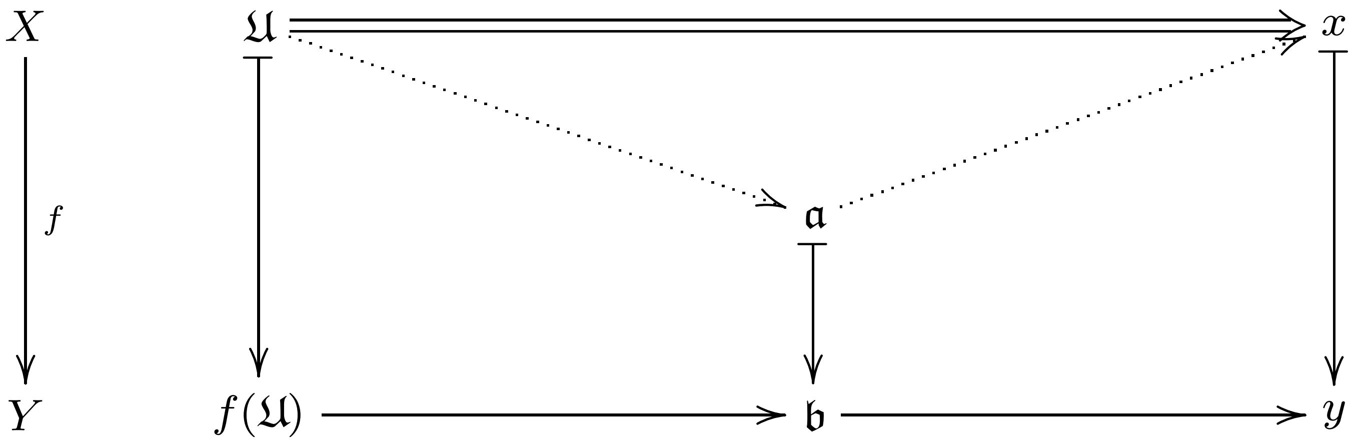
whenever $\mathfrak{U} \Rightarrow x$ in X and $f(\mathfrak{U}) \to \mathfrak{b}$ in UY and $\mathfrak{b} \to f(x)$ in Y,
there is $\mathfrak{a} \in UX$ with $f(\mathfrak{a}) = \mathfrak{b}$, $\mathfrak{U} \to \mathfrak{a}$ in UX, and $\mathfrak{a} \to x$ in X.



Theorem. [Richter2002]

A continuous map $f : X \to Y$ is exponentiable iff

- for each $x \in X$, there is $Y_x \subseteq Y$ locally closed s.t.:
  - $f^{-1}(Y_x)$ is a neighbourhood of x in X
  - the restriction $f^{-1}(Y_x) \to Y_x$ of f is exponentiable and a triquotient map.

Theorem.

For a continuous map $f : X \to Y$ the following cond. are equiv.:

(i) for each $x \in X$, there is $Y_x \subseteq Y$ locally closed such that:

- $f^{-1}(Y_x)$ is a neighbourhood of x in X
- the restriction $f^{-1}(Y_x) \to Y_x$ of f is exponentiable and a triquotient map.

(ii) for each $x \in X$, there is $Y_x \subseteq Y$ locally closed such that:

- $f^{-1}(Y_x)$ is a neighbourhood of x in X
- the restriction $f^{-1}(Y_x) \to Y_x$ of f is exponentiable and an effective descent morphism.

(iii) f is exponentiable.