Profinite Relational Structures

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(joint work with George Janelidze)

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J. Adámek and W. Tholen

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Introduction

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This can either be deduced from results of Carboni, Janelidze and Magid in *A note on the Galois correspondence for commutative rings*, J. of Algebra 183 (1996) 266-272, or seen as a special case of our Theorem 2.4.

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These suggest to investigate relational structures in general.

Stone and Priestley Spaces

Extending Stone duality, Priestley duality is an equivalence

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Priestley spaces are the ordered compact topological spaces in which every two distinct points can be separated by a clopen decreasing subset.

Profinite Preorders

As shown by Margarida Dias in
Priestley spaces: the threefold way, Preprint 07-45, DMUC,
this can be repeated for preorders as follows:

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(a) $(X, \leq)$ is a limit of finite topologically-discrete preordered spaces;

(b) X is a Stone space in which for every two points x and x' with $x' \nleq x$, there exists a clopen decreasing subset U in X such that $x \in U$ and $x' \notin U$.

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Proof:

If $(X, \leq) = lim_{i \in I}(X_i, \leq_i)$, with $(X_i, \leq_i)$ finite preorders, then $\leq = \cap R_i$ where the R_i's are the inverse images of $\leq_i$ under the induced maps $X \times X \to X_i \times X_i$.

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Conversely ,
- if $\leq$ is clopen, for each $x \in X$, $\downarrow x = \{u \in U | u \leq x\}$ is a clopen decreasing subset: it is the inverse image of $\leq$ under the continuous map $X \longrightarrow X \times X$ sending u to (u, x).

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- if $(X, \leq) = (X, \cap_{i \in I} R_i)$ then it is a limit of clopen preorders (on Stone spaces): the limit of the diagram formed by all identity maps $(X, R_i) \longrightarrow (X, X \times X)$.

Profinite Orders via Preorders

Theorem 1.3. *An ordered topological space* $(X, \leq)$ *belongs to* $\mathcal{P}ro\mathcal{F}in(\mathcal{O}rd)$, *i.e. is a Priestley space, if only if X is compact and the relation* $\leq$ *is inter-clopen (as a preorder).*

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Proof: We just need to show that, whenever $\leq$ is inter-clopen, every two distinct points x and x' in X can be separated by a clopen decreasing subset, or, equivalently, by an inter-clopen subset: if $x' \nleq x$, using again the map

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Proof: We just need to show that, whenever $\leq$ is inter-clopen, every two distinct points x and x' in X can be separated by a clopen decreasing subset, or, equivalently, by an inter-clopen subset: if $x' \not\leq x$, using again the map

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Profinite Orders

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But, there are no such relations!

Ordered Topological Spaces

The following four conditions on an ordered topological space $(X, \leq)$ are equivalent:

1. $\leq$ is an inter-clopen subset in $X \times X$;
2. $\leq$ is an clopen subset in $X \times X$;
3. $\leq$ is an open subset in $X \times X$;
4. X is discrete as a topological space.

First Order Language

A first order (finitary and one-sorted) languague $\mathbb{L}$ is determined by the set $\mathcal{F} = F(\mathbb{L})$ of its functional symbols and the set $\mathcal{P} = P(\mathbb{L})$ of its set predicate symbols.

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and morphisms all maps $f : A_0 \longrightarrow B_0$ such that

$$fF_A = F_B f^n \quad \text{and} \quad f^n(P_A) \subseteq P_B$$

for all natural n, n-ary F in $F(\mathbb{L})$, and n-ary P in $P(\mathbb{L})$.

The Forgetful Functor

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- $f^*(A)_0$ and all $F_{f*(A)}$ with $F \in F(\mathbb{L})$ are as in A;
- $P_{f^*(A)} = (f^n)^{-1}(P_B)$ for all natural n and each n-ary P in $P(\mathbb{L})$.

A Suitable Subcategory

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This makes the forgetful functor $U_{\mathcal{C}} : \mathcal{C} \longrightarrow Alg(\mathbb{L})$ a *topological functor*, a concept first introduced by H. Herrlich in *Topological functors*, General topl. and Appl. 4 (1974),125-142.

Setting Notation

We will also consider:

- the category $\mathcal{T}op(\mathcal{C})$ whose objects are objects A in $\mathcal{C}$ equipped with a topology on A_0, making F_A continuous for each F in $F(\mathbb{L})$;
- the full subcategory $\mathcal{F}in(\mathcal{C})$ in $\mathcal{T}op(\mathcal{C})$ with objects all A in $\mathcal{T}op(\mathcal{C})$ with finite discrete A_0.
- the full subcategory $\mathcal{P}ro\mathcal{F}in(\mathcal{C})$ in $\mathcal{T}op(\mathcal{C})$ defined as the limit completion of $\mathcal{F}in(\mathcal{C})$ in $\mathcal{T}op(\mathcal{C})$.

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(c) A is clopen if it is closed and open at the same time;
(d) A is inter-clopen if A^* has a set $\mathcal{S}$ of clopen subobjects, such that

$$S_0 = A_0, \text{ for all } S \in \mathcal{S}, \text{ and } P_A = \cap_{S \in \mathcal{S}} P_S$$

for each P in $P(\mathbb{L})$.

Profinite implies Inter-clopen

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For each object x in X, let $A[x]$ be the object in $\mathcal{C}$ defined via the cartesian lifting $A[x] \longrightarrow D(x)$ of $U_{\mathcal{C}}(p_x) : U_{\mathcal{C}}(A) \longrightarrow U_{\mathcal{C}}D(x)$.

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Assuming $A[x]$ to be equipped with the topology of A, we just take $\mathcal{S}$ to be the set of all such objects $A[x], (x \in X)$.

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(b)$\Leftrightarrow$ (c) is obvious;

(b) implies (a)

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Since $\mathcal{P}ro\mathcal{F}in(\mathcal{C})$ is closed under limits in $\mathcal{T}op(\mathcal{C})$, this implies that A belongs to $\mathcal{P}ro\mathcal{F}in(\mathcal{C})$.

Final Remarks

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Condition 2.3 holds for preorders

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It also holds for equivalence relations and, in Stone spaces, the inter-clopen equivalence relations are exactly the effective equivalence relations.

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Finding good sufficient conditions for Condition 2.3 seems to be an interesting problem.

The concept of topological functor, is relevant here. It is exactly the relevant difference between preorders and orders: the forgetful functor from $\mathcal{P}\boldsymbol{reord}$ to $\mathcal{S}\boldsymbol{et}$ is topological and the one from $\mathcal{O}\boldsymbol{rd}$ to $\mathcal{S}\boldsymbol{et}$ is not.