Weak Yang-Baxter operators and weak braided Hopf algebras

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Categorical Methods in Algebra, Topology and Computer Science

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Articles

(1) Alonso Álvarez, J.N., González Rodríguez, R., Crossed products for weak Hopf algebras with coalgebra splitting, J. of Algebra, **281** (2004), 731-752.

(2) Alonso Álvarez, J.N., González Rodríguez, R., Fernández Vilaboa, J.M.: Yetter-Drinfeld modules and projections of weak Hopf algebras, J. of Algebra, **315** (2007), 396-418.

(3) Alonso Álvarez, J.N., González Rodríguez, R., Fernández Vilaboa, J.M.: Weak Hopf algebras and weak Yang-Baxter operators, preprint (2007).

(4) Alonso Álvarez, J.N., González Rodríguez, R., Fernández Vilaboa, J.M.: Weak braided Hopf algebras, preprint (2007).

Preliminaries

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We assume that algebras are associative with unity and the coalgebras coassociative with counity. Given an algebra A and a coalgebra C:

$$\eta_A : K \to A, \quad \mu_A : A \otimes A \to A, \quad \varepsilon_C : C \to K, \quad \delta_C : C \to C \otimes C$$

denote the unity, the product, the counity, and the coproduct respectively.

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If A is an algebra, B is a coalgebra and $\alpha : B \to A$, $\beta : B \to A$ are morphisms, we denote the convolution product by

$$\alpha \wedge \beta = \mu_A \circ (\alpha \otimes \beta) \circ \delta_B.$$

Weak Yang-Baxter operators

Definition. (Joyal and Street, **Adv. in Math., 1993**)

Let $D \in Obj(\mathcal{C})$. A Yang-Baxter operator is an isomorphism $t_{D,D} : D \otimes D \to D \otimes D$ in $\mathcal{C}$ satisfying the Yang-Baxter equation

$$(t_{D,D} \otimes D) \circ (D \otimes t_{D,D}) \circ (t_{D,D} \otimes D) = (D \otimes t_{D,D}) \circ (t_{D,D} \otimes D) \circ (D \otimes t_{D,D})$$

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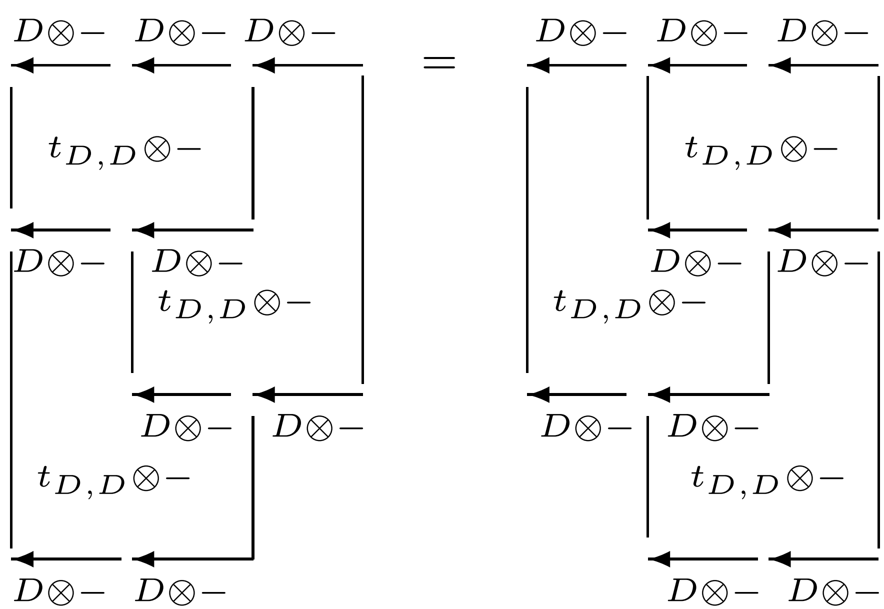

Weak Yang-Baxter operators

Definition. Let $D \in Obj(\mathcal{C})$. A weak Yang-Baxter operator is a morphism $t_{D,D} : D \otimes D \rightarrow D \otimes D$ in $\mathcal{C}$ satisfying the following conditions:

(1) $t_{D,D}$ satisfies the Yang-Baxter equation.

(2) There exists an idempotent morphism $\nabla_{D \otimes D} : D \otimes D \rightarrow D \otimes D$ such that:

(2-1) $(\nabla_{D \otimes D} \otimes D) \circ (D \otimes \nabla_{D \otimes D}) = (D \otimes \nabla_{D \otimes D}) \circ (\nabla_{D \otimes D} \otimes D)$,

(2-2) $(\nabla_{D \otimes D} \otimes D) \circ (D \otimes t_{D,D}) = (D \otimes t_{D,D}) \circ (\nabla_{D \otimes D} \otimes D)$,

(2-3) $(t_{D,D} \otimes D) \circ (D \otimes \nabla_{D \otimes D}) = (D \otimes \nabla_{D \otimes D}) \circ (t_{D,D} \otimes D)$,

(2-4) $t_{D,D} \circ \nabla_{D \otimes D} = \nabla_{D \otimes D} \circ t_{D,D} = t_{D,D}$.

(3) There exists a morphism $t'_{D,D} : D \otimes D \rightarrow D \otimes D$ such that:

(3-1) $t'_{D,D}$ satisfies the Yang-Baxter equation.

(3-2) The morphism $p_{D \otimes D} \circ t_{D,D} \circ i_{D \otimes D} : D \times D \rightarrow D \times D$ is an isomorphism with inverse $p_{D \otimes D} \circ t'_{D,D} \circ i_{D \otimes D} : D \times D \rightarrow D \times D$, where $p_{D \otimes D}$ and $i_{D \otimes D}$ are the morphisms such that $i_{D \otimes D} \circ p_{D \otimes D} = \nabla_{D \otimes D}$ and $p_{D \otimes D} \circ i_{D \otimes D} = id_{D \times D}$ being $D \times D$ the image of $\nabla_{D \otimes D}$.

(3-3) $t'_{D,D} \circ \nabla_{D \otimes D} = \nabla_{D \otimes D} \circ t'_{D,D} = t'_{D,D}$.

Weak Yang-Baxter Operators

Definition. **(Böhm, Nill and Szlachányi, J. of Algebra, 1999)**
A weak Hopf algebra (or quantum groupoid) in a strict symmetric monoidal category $\mathcal{C}$ is by definition an algebra (H, η_H, μ_H) and coalgebra $(H, \varepsilon_H, \delta_H)$ such that the following axioms hold:

(1) $\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H)$.

(2) $\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) = (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes \delta_H \otimes H)$
$= (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes (c_{H,H} \circ \delta_H) \otimes H)$.

(3) $(\delta_H \otimes H) \circ \delta_H \circ \eta_H = (H \otimes \mu_H \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$
$= (H \otimes (\mu_H \circ c_{H,H}) \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$.

(4) There exists a morphism $\lambda_H : H \to H$ in $\mathcal{C}$ (called antipode of H) satisfying:

(4-1) $id_H \wedge \lambda_H = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)$.

(4-2) $\lambda_H \wedge id_H = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$.

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A weak Hopf algebra is a Hopf algebra if an only if the morphism δ_H (comultiplication) is unit-preserving (if and only if the counit is a homomorphism of algebras).

Weak Yang-Baxter Operators

If H is a weak Hopf algebra, the antipode λ_H is unique, antimultiplicative, anticomultiplicative and leaves the unit η_H and the counit ε_H invariant:

$$\lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H}, \quad \delta_H \circ \lambda_H = c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H,$$

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The morphisms Π_H^L (target), Π_H^R (source), $\overline{\Pi}_H^L$ and $\overline{\Pi}_H^R$ defined by

$$\Pi_H^L = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H) : H \to H,$$

$$\Pi_H^R = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)) : H \to H,$$

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are idempotent.

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In this talk we denote by H_L the image of Π_H^L and by $p_L : H \to H_L$, $i_L : H_L \to H$ the morphisms such that $i_L \circ p_L = \Pi_H^L$ and $i_L \circ p_L = id_{H_L}$.

Weak Yang-Baxter Operators

Let H be a weak Hopf algebra. We say that (M, φ_M) is a left H-module if M is an object in $\mathcal{C}$ and $\varphi_M : H \otimes M \to M$ is a morphism in $\mathcal{C}$ satisfying:

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Given two left H-comodules (M, ϱ_M) and (N, ϱ_N), $f : M \to N$ is a morphism of left H-comodules if $\varrho_N \circ f = (H \otimes f) \circ \varrho_M$.
Let $(M, \varphi_M, \varrho_M)$, $(N, \varphi_N, \varrho_N)$ be left H-modules-comodules.

$$\boxed{\begin{aligned} \varphi_{M\otimes N} &= (\varphi_M \otimes \varphi_N) \circ (H \otimes c_{H,M} \otimes N) \circ (\delta_H \otimes M \otimes N) : H \otimes M \otimes N \to M \otimes N \\ \varrho_{M\otimes N} &= (\mu_H \otimes M \otimes N) \circ (H \otimes c_{M,H} \otimes N) \circ (\varrho_M \otimes \varrho_N) : M \otimes N \to H \otimes M \otimes N. \end{aligned}}$$

Weak Yang-Baxter Operators

The morphisms

$$\nabla_{M\otimes N} = \varphi_{M\otimes N} \circ (\eta_H \otimes M \otimes N) : M \otimes N \to M \otimes N$$

$$\nabla'_{M\otimes N} = (\varepsilon_H \otimes M \otimes N) \circ \varrho_{M\otimes N} : M \otimes N \to M \otimes N$$

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$$M \otimes N \xrightarrow{\nabla_{M\otimes N}} M \otimes N \qquad M \otimes N \xrightarrow{\nabla'_{M\otimes N}} M \otimes N$$

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\end{tikzcd}
\qquad
\begin{tikzcd}
M \otimes N \arrow[rr, "\nabla'_{M\otimes N}"] \arrow[dr, "p'_{M\otimes N}"'] & & M \otimes N \\
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If H is a Hopf algebra then $\nabla_{M\otimes N} = id_{M\otimes N} = \nabla'_{M\otimes N}$ and $M \times N = M \otimes N = M \odot N$.

Definition. **(Böhm, Comm. in Algebra, 2000)**

Let H be a weak Hopf algebra. We shall denote by ${}^{H}_{H}\mathcal{YD}$ the category of left-left Yetter-Drinfeld modules over H. That is, $M = (M, \varphi_M, \varrho_M)$ is an object in ${}^{H}_{H}\mathcal{YD}$ if (M, φ_M) is a left H-module, (M, ϱ_M) is a left H-comodule and

(1) $$(\mu_H \otimes M) \circ (H \otimes c_{M,H}) \circ ((\varrho_M \circ \varphi_M) \otimes H) \circ (H \otimes c_{H,M}) \circ (\delta_H \otimes M)$$
$$= (\mu_H \otimes \varphi_M) \circ (H \otimes c_{H,H} \otimes M) \circ (\delta_H \otimes \varrho_M),$$

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Let H be a weak Hopf algebra. We shall denote by ${}^H_H\mathcal{YD}$ the category of left-left Yetter-Drinfeld modules over H. That is, $M = (M, \varphi_M, \varrho_M)$ is an object in ${}^H_H\mathcal{YD}$ if (M, φ_M) is a left H-module, (M, ϱ_M) is a left H-comodule and

(1) $$(\mu_H \otimes M) \circ (H \otimes c_{M,H}) \circ ((\varrho_M \circ \varphi_M) \otimes H) \circ (H \otimes c_{H,M}) \circ (\delta_H \otimes M)$$
$$= (\mu_H \otimes \varphi_M) \circ (H \otimes c_{H,H} \otimes M) \circ (\delta_H \otimes \varrho_M),$$

(2) $$(\mu_H \otimes \varphi_M) \circ (H \otimes c_{H,H} \otimes M) \circ ((\delta_H \circ \eta_H) \otimes \varrho_M) = \varrho_M.$$

Let M, N in ${}^H_H\mathcal{YD}$. The morphism $f : M \to N$ is a morphism of left-left-Yetter-Drinfeld modules if

$$f \circ \varphi_M = \varphi_N \circ (H \otimes f), \quad (H \otimes f) \circ \varrho_M = \varrho_N \circ f.$$

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${}^H_H\mathcal{YD}$ is a non-strict braided monoidal category

Weak Yang-Baxter Operators

For two left-left Yetter-Drinfeld modules $M = (M, \varphi_M, \varrho_M)$, $N = (N, \varphi_N, \varrho_N)$ we have $\nabla_{M\otimes N} = \nabla'_{M\otimes N}$ and then the tensor product is defined as object by

$$Im(\nabla_{M\otimes N}) = M \times N = M \circledcirc N = Im(\nabla'_{M\otimes N}).$$

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$M \times N$ is a left-left Yetter-Drinfeld module with the following action and coaction:

$$\varphi_{M\times N} = p_{M\otimes N} \circ \varphi_{M\otimes N} \circ (H \otimes i_{M\otimes N}),$$
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The structure of left-left Yetter-Drinfeld module for H_L is

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The unit constrains are:

$$l_M = \varphi_M \circ (i_L \otimes M) \circ i_{H_L\otimes M} : H_L \times M \to M,$$
$$r_M = \varphi_M \circ c_{M,H} \circ (M \otimes (\overline{\Pi}_H^L \circ i_L)) \circ i_{M\otimes H_L} : M \times H_L \to M.$$

Weak Yang-Baxter Operators

These morphisms are isomorphisms with inverses:

$$l_M^{-1} = p_{H_L \otimes M} \circ (p_L \otimes \varphi_M) \circ ((\delta_H \circ \eta_H) \otimes M) : M \to H_L \times M,$$
$$r_M^{-1} = p_{M \otimes H_L} \circ (\varphi_M \otimes p_L) \circ (H \otimes c_{H,M}) \circ ((\delta_H \circ \eta_H) \otimes M) : M \to M \times H_L.$$

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If M, N, P are objects in the category ${}^H_H\mathcal{YD}$, the associativity constrains are defined by

$$a_{M,N,P} : M \times (N \times P) \to (M \times N) \times P,$$
$$a_{M,N,P} = p_{(M \times N) \otimes P} \circ (p_{M \otimes N} \otimes P) \circ (M \otimes i_{N \otimes P}) \circ i_{M \otimes (N \times P)}$$

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where the inverse are the morphisms:

$$a_{M,N,P}^{-1} : (M \times N) \times P \to M \times (N \times P).$$
$$a_{M,N,P}^{-1} = p_{M \otimes (N \times P)} \circ (M \otimes p_{N \otimes P}) \circ (i_{M \otimes N} \otimes P) \circ i_{(M \times N) \otimes P}$$

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If $\gamma : M \to M'$ and $\phi : N \to N'$ are morphisms in the category, then

$$\gamma \times \phi = p_{M' \otimes N'} \circ (\gamma \otimes \phi) \circ i_{M \otimes N} : M \times N \to M' \times N'$$

is a morphism in ${}_H^H\mathcal{YD}$ and

$$(\gamma' \times \phi') \circ (\gamma \times \phi) = (\gamma' \circ \gamma) \times (\phi' \circ \phi).$$

Weak Yang-Baxter Operators

Finally, the braiding is

$$\tau_{M,N} = p_{N\otimes M} \circ t_{M,N} \circ i_{M\otimes N} : M \times N \longrightarrow N \times M,$$

where

$$t_{M,N} = (\varphi_N \otimes M) \circ (H \otimes c_{M,N}) \circ (\varrho_M \otimes N) : M \otimes N \longrightarrow N \otimes M.$$

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The morphism $\tau_{M,N}$ is a natural isomorphism with inverse:

$$\tau^{-1}_{M,N} = p_{M\otimes N} \circ t'_{M,N} \circ i_{N\otimes M} : N \times M \to M \times N$$

where

$$t'_{M,N} = c_{N,M} \circ (\varphi_N \otimes M) \circ (c_{N,H} \otimes M) \circ (N \otimes \lambda_H^{-1} \otimes M) \circ (N \otimes \varrho_M).$$

Weak Yang-Baxter Operators

Proposition. Let H be a weak Hopf algebra with invertible antipode. If $(M, \varphi_M, \varrho_M)$ is a left-left Yetter-Drinfeld module over H, the morphism $t_{M,M} : M \otimes M \to M \otimes M$ defined by

$$t_{M,M} = (\varphi_M \otimes M) \circ (H \otimes c_{M,M}) \circ (\varrho_M \otimes M)$$

is a weak Yang-Baxter operator where

$$\nabla_{M \otimes M} = \varphi_{M \otimes M} \circ (\eta_H \otimes M \otimes M),$$

$$t'_{M,M} = c_{M,M} \circ (\varphi_M \otimes M) \circ (c_{M,H} \otimes M) \circ (M \otimes \lambda_H^{-1} \otimes M) \circ (M \otimes \varrho_M).$$

Weak braided Hopf algebras

Definition. A weak braided Hopf algebra (**WBHA**) D is an object in $\mathcal{C}$ with an algebra structure (D, η_D, μ_D) and a coalgebra structure $(D, \varepsilon_D, \delta_D)$ such that there exists a weak Yang-Baxter operator $t_{D,D} : D \otimes D \to D \otimes D$ with associated idempotent $\nabla_{D\otimes D}$ satisfying the following conditions:

(1)

(1-1) $\mu_D \circ \nabla_{D\otimes D} = \mu_D,$

(1-2) $\nabla_{D\otimes D} \circ (\mu_D \otimes D) = (\mu_D \otimes D) \circ (D \otimes \nabla_{D\otimes D}),$

(1-3) $\nabla_{D\otimes D} \circ (D \otimes \mu_D) = (D \otimes \mu_D) \circ (\nabla_{D\otimes D} \otimes D).$

(2)

(2-1) $\nabla_{D\otimes D} \circ \delta_D = \delta_D,$

(2-2) $(\delta_D \otimes D) \circ \nabla_{D\otimes D} = (D \otimes \nabla_{D\otimes D}) \circ (\delta_D \otimes D),$

(2-3) $(D \otimes \delta_D) \circ \nabla_{D\otimes D} = (\nabla_{D\otimes D} \otimes D) \circ (D \otimes \delta_D).$

Weak braided Hopf algebras

(3) The morphisms η_D, μ_D, ε_D and δ_D commute with $t_{D,D}$, i.e.,

(3-1) $t_{D,D} \circ (\eta_D \otimes D) = \nabla_{D\otimes D} \circ (D \otimes \eta_D)$,

(3-2) $t_{D,D} \circ (D \otimes \eta_D) = \nabla_{D\otimes D} \circ (\eta_D \otimes D)$,

(3-3) $t_{D,D} \circ (\mu_D \otimes D) = (D \otimes \mu_D) \circ (t_{D,D} \otimes D) \circ (D \otimes t_{D,D})$,

(3-4) $t_{D,D} \circ (D \otimes \mu_D) = (\mu_D \otimes D) \circ (D \otimes t_{D,D}) \circ (t_{D,D} \otimes D)$,

(3-5) $(\varepsilon_D \otimes D) \circ t_{D,D} = (D \otimes \varepsilon_D) \circ \nabla_{D\otimes D}$,

(3-6) $(D \otimes \varepsilon_D) \circ t_{D,D} = (\varepsilon_D \otimes D) \circ \nabla_{D\otimes D}$,

(3-7) $(\delta_D \otimes D) \circ t_{D,D} = (D \otimes t_{D,D}) \circ (t_{D,D} \otimes D) \circ (D \otimes \delta_D)$,

(3-8) $(D \otimes \delta_D) \circ t_{D,D} = (t_{D,D} \otimes D) \circ (D \otimes t_{D,D}) \circ (\delta_D \otimes D)$.

Weak braided Hopf algebras

(4) $\delta_D \circ \mu_D = (\mu_D \otimes \mu_D) \circ (D \otimes t_{D,D} \otimes D) \circ (\delta_D \otimes \delta_D)$.

(5) $\varepsilon_D \circ \mu_D \circ (\mu_D \otimes D) = (\varepsilon_D \otimes \varepsilon_D) \circ (\mu_D \otimes \mu_D) \circ (D \otimes \delta_D \otimes D)$

$= (\varepsilon_D \otimes \varepsilon_D) \circ (\mu_D \otimes \mu_D) \circ (D \otimes (t'_{D,D} \circ \delta_D) \otimes D)$.

(6) $(\delta_D \otimes D) \circ \delta_D \circ \eta_D = (D \otimes \mu_D \otimes D) \circ (\delta_D \otimes \delta_D) \circ (\eta_D \otimes \eta_D)$

$= (D \otimes (\mu_D \circ t'_{D,D}) \otimes D) \circ (\delta_D \otimes \delta_D) \circ (\eta_D \otimes \eta_D)$.

(7) There exists a morphism $\lambda_D : D \to D$ in $\mathcal{C}$ (called the antipode of D) satisfying:

(7-1) $id_D \wedge \lambda_D = ((\varepsilon_D \circ \mu_D) \otimes D) \circ (D \otimes t_{D,D}) \circ ((\delta_D \circ \eta_D) \otimes D)$,

(7-2) $\lambda_D \wedge id_D = (D \otimes (\varepsilon_D \circ \mu_D)) \circ (t_{D,D} \otimes D) \circ (D \otimes (\delta_D \circ \eta_D))$,

(7-3) $\lambda_D \wedge id_D \wedge \lambda_D = \lambda_D$.

Weak braided Hopf algebras

- If $\mathcal{C}$ is symmetric, $t_{D,D} = c_{D,D} = t'_{D,D}$, i.e. the weak Yang-Baxter operator is the twist of the symmetric category $\mathcal{C}$, then $\nabla_{D\otimes D} = id_{D\otimes D}$ and the last definition is the usual definition of weak Hopf algebra (**Böhm, Nill and Szlachányi**).

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- If $\mathcal{C}$ is braided, $t_{D,D} = c_{D,D}$, $t'_{D,D} = c^{-1}_{D,D}$, i.e. the weak Yang-Baxter operator is the braiding of the braided category $\mathcal{C}$, then $\nabla_{D\otimes D} = id_{D\otimes D}$ and we introduce the definition of weak Hopf algebra in a braided category.

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- If $\mathcal{C}$ is a category of vector spaces (symmetric), $t_{D,D}$ is a Yang-Baxter operator, $t'_{D,D} = t^{-1}_{D,D}$ and $\eta_D \otimes \eta_D = \delta_D \circ \eta_D$, D is a braided Hopf algebra (**Takeuchi**).

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Weak braided Hopf algebras

Definition. Let D, B be weak braided Hopf algebras. We will say that $f : D \rightarrow B$ is a morphism of weak braided Hopf algebras if f is an algebra coalgebra morphism and $t_{B,B} \circ (f \otimes f) = (f \otimes f) \circ t_{D,D}$ and $t'_{B,B} \circ (f \otimes f) = (f \otimes f) \circ t'_{D,D}$.
If $f : D \rightarrow B$ is a morphism of weak braided Hopf algebras, then $f \circ \lambda_D = \lambda_B \circ f$.

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If $f : D \to B$ is a morphism of weak braided Hopf algebras, then $f \circ \lambda_D = \lambda_B \circ f$.

Proposition. Let H be a weak Hopf algebra in $\mathcal{C}$ such that λ_H is an isomorphism. Let $(D, u_D, m_D, e_D, \Delta_D, \lambda_D)$ be a Hopf algebra in ${}_H^H\mathcal{YD}$ with action φ_D and coaction ϱ_D. Let $t_{D,D} = (\varphi_D \otimes D) \circ (H \otimes c_{D,D}) \circ (\varrho_D \otimes D)$ be the weak Yang-Baxter operator and $\nabla_{D\otimes D} = i_{D\otimes D} \circ p_{D\otimes D}$ the associated idempotent. Then

$$D = (D, \eta_D = u_D \circ p_L \circ \eta_H, \mu_D = m_D \circ p_{D\otimes D}, \varepsilon_D = \varepsilon_H \circ i_L \circ e_D, \delta_D = i_{D\otimes D} \circ \Delta_D, \lambda_D)$$

is a WBHA in $\mathcal{C}$.

Weak braided Hopf algebras

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- By an analogous calculus, if $\eta_D \otimes \eta_D = \delta_D \circ \eta_D$, we obtain that H is a Hopf algebra.

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- If $\varepsilon_D \circ \mu_D = \varepsilon_D \otimes \varepsilon_D$ then $\Pi_H^L = \varepsilon_H \otimes \eta_H$, or equivalently, H is a Hopf algebra in $\mathcal{C}$.
- By an analogous calculus, if $\eta_D \otimes \eta_D = \delta_D \circ \eta_D$, we obtain that H is a Hopf algebra.
- If $\lambda_D \wedge id_D = \varepsilon_D \otimes \eta_D$ we have $u_D \circ e_D = \eta_D \circ \varepsilon_D$ and then

$$id_{H_L} = p_L \circ \eta_H \circ \varepsilon_H \circ i_L.$$

Therefore, $\Pi_H^L = \varepsilon_H \otimes \eta_H$ and we obtain that H also is a Hopf algebra.

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- By an analogous calculus, if $\eta_D \otimes \eta_D = \delta_D \circ \eta_D$, we obtain that H is a Hopf algebra.
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  Therefore, $\Pi_H^L = \varepsilon_H \otimes \eta_H$ and we obtain that H also is a Hopf algebra.
- Finally, D is not a weak Hopf algebra since the condition

$$\delta_D \circ \mu_D = (\mu_D \otimes \mu_D) \circ (D \otimes t_{D,D} \otimes D) \circ (\delta_D \otimes \delta_D)$$

  does not imply $\delta_D \circ \mu_D = (\mu_D \otimes \mu_D) \circ (D \otimes c_{D,D} \otimes D) \circ (\delta_D \otimes \delta_D)$ where $c_{D,D}$ is the symmetric braiding of $\mathcal{C}$.

Proposition. In a WBHA D the following assertions are equivalent.

(1) The equality

$$\varepsilon_D \circ \mu_D \circ (\mu_D \otimes D) = ((\varepsilon_D \circ \mu_D) \otimes (\varepsilon_D \circ \mu_D)) \circ (D \otimes (t'_{D,D} \circ \delta_D) \otimes D)$$

holds.

(2) There exists a morphism $\Pi_D^L : D \to D$ such that

$$\mu_D \circ (D \otimes \Pi_D^L) = ((\varepsilon_D \circ \mu_D) \otimes D) \circ (D \otimes t_{D,D}) \circ (\delta_D \otimes D).$$

(3) There exists a morphism $\Pi_D^R : D \to D$ such that

$$\mu_D \circ (\Pi_D^R \otimes D) = (D \otimes (\varepsilon_D \circ \mu_D)) \circ (t_{D,D} \otimes D) \circ (D \otimes \delta_D).$$

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holds.

(2) There exists a morphism $\overline{\Pi}_D^L : D \to D$ such that

$$\mu_D \circ (D \otimes \overline{\Pi}_D^L) = (D \otimes (\varepsilon_D \circ \mu_D)) \circ (\delta_D \otimes D).$$

(3) There exists a morphism $\overline{\Pi}_D^R : D \to D$ such that

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Proposition. In a WBHA D the following assertions are equivalent.

(1) The equality

$$(\delta_D \otimes D) \circ \delta_D \circ \eta_D = (D \otimes (\mu_D \circ t'_{D,D}) \otimes D) \circ ((\delta_D \circ \eta_D) \otimes (\delta_D \circ \eta_D))$$

holds.

(2) There exists a morphism $\Pi_D^L : D \to D$ such that

$$(D \otimes \Pi_D^L) \circ \delta_D = (\mu_D \otimes D) \circ (D \otimes t_{D,D}) \circ ((\delta_D \circ \eta_D) \otimes D).$$

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$$(\Pi_D^R \otimes D) \circ \delta_D = (D \otimes \mu_D) \circ (t_{D,D} \otimes D) \circ (D \otimes (\delta_D \circ \eta_D)).$$

Weak braided Hopf algebras

Proposition. In a WBHA D the following assertions are equivalent.

(1) The equality

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holds.

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$$(\overline{\Pi}_D^L \otimes D) \circ \delta_D = (D \otimes \mu_D) \circ ((\delta_D \circ \eta_D) \otimes D).$$

(3) There exists a morphism $\overline{\Pi}_D^R : D \to D$ such that

$$(D \otimes \overline{\Pi}_D^R) \circ \delta_D = (\mu_D \otimes D) \circ (D \otimes (\delta_D \circ \eta_D)).$$

Weak braided Hopf algebras

Proposition. If D is a WBHA the morphisms Π_D^L (target), Π_D^R (source), $\overline{\Pi}_D^L$ and $\overline{\Pi}_D^R$ defined by

$$\Pi_D^L = ((\varepsilon_D \circ \mu_D) \otimes D) \circ (D \otimes t_{D,D}) \circ ((\delta_D \circ \eta_D) \otimes D) : D \to D,$$

$$\Pi_D^R = (H \otimes (\varepsilon_D \circ \mu_D)) \circ (t_{D,D} \otimes D) \circ (D \otimes (\delta_D \circ \eta_D)) : D \to D,$$

$$\overline{\Pi}_D^L = (D \otimes (\varepsilon_D \circ \mu_D)) \circ ((\delta_D \circ \eta_D) \otimes D) : D \to D,$$

$$\overline{\Pi}_D^R = ((\varepsilon_D \circ \mu_D) \otimes D) \circ (D \otimes (\delta_D \circ \eta_D)) : D \to D.$$

are idempotent and leave the unit and the counit invariant.

Weak braided Hopf algebras

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$$\Pi_D^R = (H \otimes (\varepsilon_D \circ \mu_D)) \circ (t_{D,D} \otimes D) \circ (D \otimes (\delta_D \circ \eta_D)) : D \to D,$$

$$\overline{\Pi}_D^L = (D \otimes (\varepsilon_D \circ \mu_D)) \circ ((\delta_D \circ \eta_D) \otimes D) : D \to D,$$

$$\overline{\Pi}_D^R = ((\varepsilon_D \circ \mu_D) \otimes D) \circ (D \otimes (\delta_D \circ \eta_D)) : D \to D.$$

are idempotent and leave the unit and the counit invariant.

Proposition. In a WBHA D the following identities hold.

(1) $\Pi_H^L \circ \overline{\Pi}_D^L = \Pi_D^L, \quad \Pi_D^L \circ \overline{\Pi}_D^R = \overline{\Pi}_D^R, \quad \overline{\Pi}_D^L \circ \Pi_D^L = \overline{\Pi}_D^L, \quad \overline{\Pi}_D^R \circ \Pi_D^L = \Pi_D^L.$

(2) $\Pi_D^R \circ \overline{\Pi}_D^L = \overline{\Pi}_D^L, \quad \Pi_D^R \circ \overline{\Pi}_D^R = \Pi_D^R, \quad \overline{\Pi}_D^L \circ \Pi_D^R = \Pi_D^R, \quad \overline{\Pi}_D^R \circ \Pi_D^R = \overline{\Pi}_D^R.$

(3) $\Pi_D^L \circ \lambda_D = \Pi_D^L \circ \Pi_D^R = \lambda_D \circ \Pi_D^R, \quad \Pi_D^R \circ \lambda_D = \Pi_D^R \circ \Pi_D^L = \lambda_D \circ \Pi_D^L.$

(4) $\Pi_D^L = \overline{\Pi}_D^R \circ \lambda_D = \lambda_D \circ \overline{\Pi}_D^L, \quad \Pi_D^R = \overline{\Pi}_D^L \circ \lambda_D = \lambda_D \circ \overline{\Pi}_D^R.$

Proposition. In a WBHA D the following identities hold.

(1) $t_{D,D}\circ(\Pi_D^L\otimes D)=(D\otimes\Pi_D^L)\circ t_{D,D},\quad t'_{D,D}\circ(\Pi_D^L\otimes D)=(D\otimes\Pi_D^L)\circ t'_{D,D}.$

(2) $t_{D,D}\circ(D\otimes\Pi_D^L)=(\Pi_D^L\otimes D)\circ t_{D,D},\quad t'_{D,D}\circ(D\otimes\Pi_D^L)=(\Pi_D^L\otimes D)\circ t'_{D,D}.$

(3) $t_{D,D}\circ(\Pi_D^R\otimes D)=(D\otimes\Pi_D^R)\circ t_{D,D},\quad t'_{D,D}\circ(\Pi_D^R\otimes D)=(D\otimes\Pi_D^R)\circ t'_{D,D}.$

(4) $t_{D,D}\circ(D\otimes\Pi_D^R)=(\Pi_D^R\otimes D)\circ t_{D,D},\quad t'_{D,D}\circ(D\otimes\Pi_D^R)=(\Pi_D^R\otimes D)\circ t'_{D,D}.$

(5) $\nabla_{D\otimes D}\circ(\Pi_D^L\otimes D)=(\Pi_D^L\otimes D)\circ\nabla_{D\otimes D},\quad \nabla_{D\otimes D}\circ(\Pi_D^R\otimes D)=(\Pi_D^R\otimes D)\circ\nabla_{D\otimes D},$

$\nabla_{D\otimes D}\circ(D\otimes\Pi_D^L)=(D\otimes\Pi_D^L)\circ\nabla_{D\otimes D},\quad \nabla_{D\otimes D}\circ(D\otimes\Pi_D^R)=(D\otimes\Pi_D^R)\circ\nabla_{D\otimes D}.$

(6) $t_{D,D}\circ(D\otimes\overline{\Pi}_D^L)=(\overline{\Pi}_D^L\otimes D)\circ t_{D,D},\quad t'_{D,D}\circ(D\otimes\overline{\Pi}_D^L)=(\overline{\Pi}_D^L\otimes D)\circ t'_{D,D}.$

(7) $t_{D,D}\circ(\overline{\Pi}_D^R\otimes D)=(D\otimes\overline{\Pi}_D^R)\circ t_{D,D},\quad t'_{D,D}\circ(\overline{\Pi}_D^R\otimes D)=(D\otimes\overline{\Pi}_D^R)\circ t'_{D,D}.$

Proposition. Let D be a WBHA. If the antipode of D is an isomorphism the following identities hold.

(1) $t_{D,D} \circ (\overline{\Pi}_D^L \otimes D) = (D \otimes \overline{\Pi}_D^L) \circ t_{D,D}, \quad t'_{D,D} \circ (\overline{\Pi}_D^L \otimes D) = (D \otimes \overline{\Pi}_D^L) \circ t'_{D,D}.$

(2) $t_{D,D} \circ (D \otimes \overline{\Pi}_D^R) = (\overline{\Pi}_D^R \otimes D) \circ t_{D,D}, \quad t'_{D,D} \circ (D \otimes \overline{\Pi}_D^R) = (\overline{\Pi}_D^R \otimes D) \circ t'_{D,D}.$

(3) $\nabla_{D\otimes D} \circ (\overline{\Pi}_D^L \otimes D) = (\overline{\Pi}_D^L \otimes D) \circ \nabla_{D\otimes D}, \quad \nabla_{D\otimes D} \circ (\overline{\Pi}_D^R \otimes D) = (\overline{\Pi}_D^R \otimes D) \circ \nabla_{D\otimes D},$

$\nabla_{D\otimes D} \circ (D \otimes \overline{\Pi}_D^L) = (D \otimes \overline{\Pi}_D^L) \circ \nabla_{D\otimes D}, \quad \nabla_{D\otimes D} \circ (D \otimes \overline{\Pi}_D^R) = (D \otimes \overline{\Pi}_D^R) \circ \nabla_{D\otimes D}.$

Proposition. Let D be a WBHA. If the antipode of D is an isomorphism the following identities hold.

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(3) $\nabla_{D\otimes D}\circ(\overline{\Pi}_D^L\otimes D)=(\overline{\Pi}_D^L\otimes D)\circ\nabla_{D\otimes D}, \quad \nabla_{D\otimes D}\circ(\overline{\Pi}_D^R\otimes D)=(\overline{\Pi}_D^R\otimes D)\circ\nabla_{D\otimes D},$

$\nabla_{D\otimes D}\circ(D\otimes\overline{\Pi}_D^L)=(D\otimes\overline{\Pi}_D^L)\circ\nabla_{D\otimes D}, \quad \nabla_{D\otimes D}\circ(D\otimes\overline{\Pi}_D^R)=(D\otimes\overline{\Pi}_D^R)\circ\nabla_{D\otimes D}.$

Proposition. Let D be a WBHA. The following identities hold.

(1) $t_{D,D}\circ(\lambda_D\otimes D)=(D\otimes\lambda_D)\circ t_{D,D}, \quad t'_{D,D}\circ(D\otimes\lambda_D)=(\lambda_D\otimes D)\circ t'_{D,D}.$

(2) $t_{D,D}\circ(D\otimes\lambda_D)=(\lambda_D\otimes D)\circ t_{D,D}, \quad t'_{D,D}\circ(\lambda_D\otimes D)=(D\otimes\lambda_D)\circ t'_{D,D}.$

(3) $\nabla_{D\otimes D}\circ(\lambda_D\otimes D)=(\lambda_D\otimes D)\circ\nabla_{D\otimes D}, \quad \nabla_{D\otimes D}\circ(D\otimes\lambda_D)=(D\otimes\lambda_D)\circ\nabla_{D\otimes D}.$

Proposition. Let D be a WBHA. The antipode is unique, antimultiplicative, anticomultiplicative and leaves the unit and the counit invariant,i.e.:

$$\lambda_D \circ \mu_D = \mu_D \circ t_{D,D} \circ (\lambda_D \otimes \lambda_D),$$

$$\delta_D \circ \lambda_D = (\lambda_D \otimes \lambda_D) \circ t_{D,D} \circ \delta_D,$$

$$\lambda_D \circ \eta_D = \eta_D, \;\; \varepsilon_D \circ \lambda_D = \varepsilon_D.$$

Hopf modules for WBHA

Definition. (Caenepeel and De Groot, **Cont. Math., 2000**)

A right-right weak entwining structure on $\mathcal{C}$ consists of a triple (A, C, ψ), where A is an algebra, C a coalgebra, and $\psi : C \otimes A \to A \otimes C$ a morphism satisfying the relations

(e1) $\psi \circ (C \otimes \mu_A) = (\mu_A \otimes C) \circ (A \otimes \psi) \circ (\psi \otimes A)$,

(e2) $(A \otimes \delta_C) \circ \psi = (\psi \otimes C) \circ (C \otimes \psi) \circ (\delta_C \otimes A)$,

(e3) $\psi \circ (C \otimes \eta_A) = (e_{RR} \otimes C) \circ \delta_C$,

(e4) $(A \otimes \varepsilon_C) \circ \psi = \mu_A \circ (e_{RR} \otimes A)$,

where $e_{RR} : C \to A$ is the morphism defined by $e_{RR} = (A \otimes \varepsilon_C) \circ \psi \circ (C \otimes \eta_A)$. The morphism ψ is known as entwining morphism.

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(e4) $(A \otimes \varepsilon_C) \circ \psi = \mu_A \circ (e_{RR} \otimes A)$,

where $e_{RR} : C \to A$ is the morphism defined by $e_{RR} = (A \otimes \varepsilon_C) \circ \psi \circ (C \otimes \eta_A)$. The morphism ψ is known as entwining morphism.

Proposition. Let D be a WBHA. If ψ is the morphism defined by

$$\psi = (D \otimes \mu_D) \circ (t_{D,D} \otimes D) \circ (D \otimes \delta_D),$$

(D, D, ψ) is a right-right weak entwining structure.

Hopf modules for WBHA

Definition. Let (A, C, ψ) be a right-right weak entwining structure in $\mathcal{C}$. We denote by $\mathcal{M}_A^C(\psi)$ the category whose objects are triples (M, ϕ_M, ρ_M), where (M, ϕ_M) is a right A-module , (M, ρ_M) is a right C-comodule and

$$\rho_M \circ \phi_M = (\phi_M \otimes C) \circ (M \otimes \psi) \circ (\rho_M \otimes A).$$

The morphisms in $\mathcal{M}_A^C(\psi)$ are morphisms of A-modules and C-comodules.

If D is a WBHA, a right D-Hopf module is an object in $\mathcal{M}_D^D(\psi)$ for the right-right weak entwining structure (D, D, ψ). The category of right D-Hopf modules is denoted by $\mathcal{M}_D^D$.

For example, D itself is an right-right D-Hopf module via $\phi_D = \mu_D$ and $\rho_D = \delta_D$.

Hopf modules for WBHA

Proposition. Let D be a weak braided Hopf algebra with target morphism Π_D^L. Put $D_L = Im(\Pi_D^L)$ and let $p_L : D \to D_L$ and $i_L : D_L \to D$ be the morphisms such that $\Pi_D^L = i_L \circ p_L$ and $p_L \circ i_L = id_{D_L}$. Then,

$$
\begin{tikzcd}
D_L \arrow[r, "i_L"] & D \arrow[r, shift left, "\delta_D"] \arrow[r, shift right, "(D\otimes \Pi_D^L)\circ\delta_D"'] & D\otimes D
\end{tikzcd}
$$

is an equalizer diagram and

$$
\begin{tikzcd}
D\otimes D \arrow[r, shift left, "\mu_D"] \arrow[r, shift right, "\mu_D\circ(D\otimes \Pi_D^L)"'] & D \arrow[r, "p_L"] & D_L
\end{tikzcd}
$$

is a coequalizer diagram. As a consequence, $(D_L, \eta_{D_L} = p_L \circ \eta_D, \mu_D = p_L \circ \mu_D \circ (i_L \otimes i_L))$ is an algebra in $\mathcal{C}$ and $(D_L, \varepsilon_{D_L} = \varepsilon_D \circ i_L, \delta_D = (p_L \otimes p_L) \circ \delta_D \circ i_L)$ is a coalgebra in $\mathcal{C}$.

Hopf modules for WBHA

Proposition. Let D be a WBHA. We have the following:

(1) If $(M, \phi_M, \rho_M) \in \mathcal{M}_D^D$ then $q_D^M = \phi_M \circ (M \otimes \lambda_D) \circ \rho_M : M \rightarrow M$ is an idempotent morphism with factorization $q_D^M = i_D^M \circ p_D^M$.

(2) If we denote by M_D the image of q_D^M, then

$$\begin{tikzcd} M_D \arrow[r, "i_D^M"] & M \arrow[rr, shift left, "\rho_M"] \arrow[rr, shift right, "\zeta_M = (\phi_M \otimes D) \circ (M \otimes (\delta_D \circ \eta_D))"'] & & M \otimes D \end{tikzcd}$$

is an equalizer diagram.

(3) The pair (M_D, ϕ_{M_D}) is a right D_L-module, where $\phi_{M_D} : M_D \otimes D_L \rightarrow M_D$ is the factorization of $\phi_M \circ (i_D^M \otimes i_L)$ through the equalizer i_D^M.

Hopf modules for WBHA

Theorem. Let D be a WBHA. Let M be a right D-Hopf module and M_D the right D_L-module defined previously. Let $\Omega_{M_D} : M_D \otimes D \to M_D \otimes D$ be the morphism defined by $\Omega_{M_D} = (p_D^M \otimes D) \circ \rho_M \circ \phi_M \circ (i_D^M \otimes D)$. We have the following assertions.

(1) The morphism Ω_{M_D} is idempotent.

(2) If $M_D \times D$ is the image of Ω_{M_D} and $p_{M_D \otimes D}$, $i_{M_D \otimes D}$ are the morphisms such that

$$p_{M_D \otimes D} \circ i_{M_D \otimes D} = id_{M_D \times D}, \quad i_{M_D \otimes D} \circ p_{M_D \otimes D} = \Omega_{M_D},$$

we obtain that $M_D \times D$ is a right D-Hopf module via

$$\phi_{M_D \times D} = p_{M_D \otimes D} \circ (M_D \otimes \mu_D) \circ (i_{M_D \otimes D} \otimes D),$$

$$\rho_{M_D \times D} = (p_{M_D \otimes D} \otimes D) \circ (M_D \otimes \delta_D) \circ i_{M_D \otimes D},$$

and there exists an isomorphism $\alpha : M \to M_D \times D$ of right D-Hopf modules.