On closure operators and reflections in Goursat categories

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Outline

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Objects with closed diagonals

The story begins from the well known characterization of Hausdorff (or separated) topological spaces via the closure of the diagonal:

X is Hausdorff if and only if the diagonal

$\Delta_X : X \to X \times X$ is a closed subspace of the product

In turn, closed suspaces are completely determined by the class of Hausdorff spaces, since, given $M \subset X$,

M is closed if and only if M is the equalizer of two maps

$f, g : X \to A,$ with A Hausdorff

The Hausdorff case is just the paradigma of the more general notion of $\mathbb{A}$-closure, introduced by Salbany in 1976, where $\mathbb{A}$ is any class of topological spaces, and the definition of $\mathbb{A}$-closed is given by the above with $\mathbb{A}$ instead of Hausdorff spaces.

Using $\mathbb{A}$-closure for $\mathbb{A}$ an epireflective subcategory of topological spaces, Giuli and Hušek in 1986 characterized the objects of the quotient-reflective hull of $\mathbb{A}$ as those with an $\mathbb{A}$-closed diagonal.

In 1988 Giuli, M., Tholen proved that this diagonal theorem holds even in an arbitrary category $\mathbb{C}$ with finite limits, provided $\mathbb{A}$ reflective with a certain weak exactness property. The strongly epireflective hull of $\mathbb{A}$ is given by

$$S(\mathbb{A}) = \{X \in \mathbb{C} \mid \exists\, m : X \to A,\ A \in \mathbb{A}, m \,\text{monomorphism}\}$$

and

$$X \in S(\mathbb{A}) \text{ if and only } \Delta_X : X \to X \times X \quad \text{is } \mathbb{A}\text{-regular}$$

Outline

Closure operators

In 1995, Clementino-Tholen proved a diagonal theorem in a much more general case, using the notion of categorical closure operator, in the contest of a category $\mathbb{C}$ finitely complete which comes equipped with a proper $(\mathcal{E}, \mathcal{M})$-factorization system for morphisms. It is then possible to speak of ($\mathcal{E}$-) images and ($\mathcal{M}$-) subobjects, respectively:

$$\begin{tikzcd} A \arrow[rr, "f"'] \arrow[dr, "e"'] & & X \\ & f(A) \arrow[ur, "m"'] & \end{tikzcd}$$

This factorization systems has to be thought as a common frame for the two opposite cases given by the factorization (Epimorphisms, strong monomorphisms) used in the topological case and the (Strong epimorphisms, monomorphisms) used in the algebraic case (axiom of regularity).

Definition (Dikranjan-Giuli (1987), Dikranjan-Tholen (1995))

A *closure operator* $\overline{(\ )}$ in $\mathbb{C}$ associates, with any subobject

$M \xrightarrow{m} X$, another subobject $\overline{M} \xrightarrow{\overline{m}} X$, the closure of M in X.

This correspondence $\overline{(\ )} : Sub(X) \to Sub(X)$ satisfies:

1. $M \subseteq \overline{M}$
2. $M \subseteq N$ implies $\overline{M} \subseteq \overline{N}$
3. $\overline{f^{-1}(M)} \subseteq f^{-1}(\overline{M})$, for all $f : Y \to X$ in $\mathbb{C}$.

$m : M \to X$ is **closed** in X if $m = \overline{m}$, and it is **dense** if $\overline{m} = id_X$.

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The closure operator decomposes every $m : M \to X$ in $\mathcal{M}$ as:

$$M \xrightarrow{m/\overline{m}} \overline{M} \xrightarrow{\overline{m}} X$$

The closure operator $\overline{(\ )}$ is

- **idempotent** if $\overline{m}$ is closed
- **weakly hereditary** if $m/\overline{m}$ is dense

for every $m \in \mathcal{M}$.
Only under these two conditions the decomposition as above gives raise to a factorization system for morphisms as:

$$\begin{tikzcd}
A \arrow[rr, "f"'] \arrow[dr, "(m/\overline{m})\cdot e"'] & & X \\
& \overline{f(A)} \arrow[ur, "\overline{m}"'] &
\end{tikzcd}$$

Theorem (Clementino-Tholen 1995)

Let $\mathbb{C}$ be $\mathcal{M}$-complete. Given a full subcategory $\mathbb{A}$ of $\mathbb{C}$,

$\mathbb{A} = \{X | \overline{\Delta}_X = \Delta_X\}$ *for some closure operator* $\overline{(\ )}$ *in* $\mathbb{C}$ *if and only if* $\mathbb{A}$ *is closed under monosources in X.*

In particular, if $\mathbb{A}$ *is reflective,*

$\mathbb{A} = \{X | \overline{\Delta}_X = \Delta_X\}$ *for some closure operator* $\overline{(\ )}$ *in* $\mathbb{C}$ *if and only if* $\mathbb{A}$ *strong epireflective.*

$\overline{(\ )}$ can be chosen as the Salbany regular closure operator associated with $\mathbb{A}$, (opportunely defined in this more general context), but for any closure operator the corresponding subcategory of separated objects for a closure is strong epireflective, if $\mathbb{C}$ is complete and $\mathcal{E}$-cowellpowered.

In the same paper, they used again closure operators to characterize via another diagonal theorem, the so called right-constant subcategories (or disconnectedness), the non-pointed analog of the torsion free part of a torsion theory of abelian categories.

Outline

Torsion theories

Let $\mathbb{C}$ be a category with a chosen notion of constant morphism (obvious in the pointed case, much less obvious, otherwise). Given $\mathbb{A}$ and $\mathbb{B}$ subcategories of $\mathbb{C}$, let us define

$$\mathcal{T}(\mathbb{B}) = \{A \,|\, \forall\, B \in \mathbb{B}, \text{ any } f : A \to B \text{ is constant}\}$$

$$\mathcal{F}(\mathbb{A}) = \{B \,|\, \forall\, A \in \mathbb{A}, \text{ any } f : A \to B \text{ is constant}\}$$

This defines a Galois correspondence for full subcategories of $\mathbb{C}$, with the fixed elements being called **torsion** (left constant) and **torsion free** (right constant) subcategories.

Most of cases, torsion free subcategories are strongly epireflective and Clementino-Tholen characterized them among all the others, as the categories of separated objects for a particular closure operator.

Theorem (Clementino-Tholen 1995)

Let $\mathbb{C}$ be a $(\mathcal{E}, \mathcal{M})$-category complete and $\mathcal{E}$-cowellpowered, and let $\mathbb{C}$ have enough quasipoints. Then

- $\mathbb{F}$ is a torsion free subcategory if and only if $\mathbb{F} = \{X | \overline{\Delta}_X = \Delta_X\}$ for some coregular closure operator $\overline{(\ )}$.

(and dually

- $\mathbb{T}$ is a torsion subcategory if and only if $\mathbb{T} = \{X | \overline{\Delta}_X = X \times X\}$ for some regular closure operator $\overline{(\ )}$.)

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The abelian case

Also in the abelian case, there is a classical result relating closure operators to **hereditary** torsion free subcategories, where the torsion part is closed under monomorphisms.

Theorem

*If $\mathbb{C}$ be an abelian category, there is a bijection between the hereditary torsion free subcategories of $\mathbb{C}$ and the **universal** closure operators for monomorphisms in $\mathbb{C}$, where a closure operator $\overline{(\ )}$ is universal if it is idempotent and*

$$f^{-1}(\overline{N}) = \overline{f^{-1}(N)},$$

for any $N \rightarrowtail Y$ and any $f : X \to Y$.

Note that universal closure operator are weakly hereditary.

This bijection can be realized as follow:

- Given in $\mathbb{C}$ an universal closure operator $\overline{(\ )}$, $\mathbb{F} = \{X | \overline{0}_X = 0_X\}$ is an hereditary torsion free subcategory

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- Given an hereditary torsion free subcategory $\mathbb{F}$ and any monomorphism $m : M \to X$, on can define $\overline{m} : \overline{M} \to X$ as the inverse image of the kernel $K[\eta_{X/M}]$ along the quotient q of X by M, i.e. $\overline{M} = q^{-1}(K[\eta_{X/M}])$:

$$\begin{tikzcd}
 & \overline{M} \arrow[r] \arrow[d] & K[\eta_{X/S}] \arrow[d, "i"'] \\
M \arrow[r] \arrow[ur, dotted, "i_M"] & X \arrow[r, "q"'] & X/M \arrow[d, "\eta_{X/M}"] \\
 & & F(X/M)
\end{tikzcd}$$

where $\eta_Z : Z \to F(Z)$ is the unit of the epireflection of $\mathbb{C}$ in $\mathbb{F}$.

Torsion theory in Homological categories

In 2006 Bourn-Gran revisited these topics in the non-additive context of homological categories, where

Definition (Borceux-Bourn, 2004)

A finitely complete category $\mathbb{C}$ is **homological** if

- $\mathbb{C}$ is regular
- $\mathbb{C}$ is pointed
- $\mathbb{C}$ is protomodular, i.e. any change-of-base functor $f^* : Pt_Y(\mathbf{C}) \to Pt_X(\mathbf{C})$ reflects isomorphisms.

They showed that, also in the homological context, the same construction gives a bijection between hereditary torsion free subcategories and universal closure operators, **but** just for normal monomorphisms, i.e. kernels.

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They were able also to show what happens for ordinary torsion free subcategories, namely:

Theorem

Let $\mathbb{C}$ be a homological category. There is a bijection between the torsion free subcategories of $\mathbb{C}$ and the idempotent closure operators for kernels in $\mathbb{C}$ such that

1. *they are weakly hereditary*
2. $$f^{-1}(\overline{N}) = \overline{f^{-1}(N)},$$
   *for any $n : N \to Y$ and any **regular epimorphism** $f : X \to Y$.*

Splitting universality in three less restrictive conditions, allows us to understand also what happens just for regular epireflective subcategories.

Outline

Objects with closed diagonals

Closure operators

Torsion theories

Regular Epireflections in Homological categories

Back to diagonals: the regular case

And now?

Theorem (Bourn-Gran 2006)

If $\mathbb{C}$ be a homological category, there is a bijection between the regular epireflective subcategories of $\mathbb{C}$ and the idempotent closure operators for kernels in $\mathbb{C}$ such that

$$f^{-1}(\overline{N}) = \overline{f^{-1}(N)},$$

for any $n : N \to Y$ and any regular epimorphism $f : X \to Y$.

A regular epireflective $\mathbb{B}$ subcategory of $\mathbb{C}$ is closed under subobjects and products and when it is closed also under quotiens, it is called Birkhoff subcategory.

Theorem (Bourn-Gran 2006)

If $\mathbb{C}$ is a semi-abelian category, i.e. an exact homological category with binary coproducts, there is a bijection between the Birkhoff subcategories of $\mathbb{C}$ and the idempotent closure operators for kernels in $\mathbb{C}$ such that

1. $f^{-1}(\overline{N}) = \overline{f^{-1}(N)}$,
2. $f(\overline{M}) = \overline{f(M)}$

for any $n : N \to Y$, any $m : M \to X$ and any regular epimorphism $f : X \to Y$.

Outline

In trying to face the same problem in the non-pointed case, we realized that protomodularity is needed **only** to represent quotients by kernels and that all the procedure works perfectly well in the regular context, just by using the natural substitutes of kernels, i.e. kernel pairs (= effective equivalence relations).

Definition (Bourceux-Gran-M. 2007)

Let $\mathbb{C}$ be a finitely complete category. An effective closure operator $\overline{(\ )}$ on effective equivalence relations consists in giving for every effective equivalence relation $s : S \rightarrowtail X \times X$ another effective equivalence relation $\overline{s} : \overline{S} \rightarrowtail X \times X$ such that

1. $S \subseteq \overline{S}$
2. $S \subseteq T$ implies $\overline{S} \subseteq \overline{T}$
3. $\overline{\overline{S}} = \overline{S}$
4. $\overline{f^{-1}(S)} \subseteq f^{-1}(\overline{S})$, if f is any map
5. $\overline{f^{-1}(S)} = f^{-1}(\overline{S})$, if f is a regular epimorphism.

Theorem (Bourceux-Gran-M. 2007)

Let $\mathcal{C}$ be a regular category. There is a bijection between the regular epireflective subcategories of $\mathcal{C}$ and the effective closure operators on effective equivalence relation.

This bijection is realized by defining, given an effective closure operator $\overline{(\ )}$

$$\mathbb{A} = \{X | \overline{\Delta}_X = \Delta_X\},$$

and, given a regular epireflective subcategory $\mathbb{A}$, $\overline{S} = q^{-1}(R[\eta_{X/S}])$:

$$\begin{tikzcd}
& \overline{S} \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & R[\eta_{X/S}] \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] \\
S \arrow[ur, dotted, "i_S"] \arrow[r, shift left] \arrow[r, shift right] & X \arrow[r, "q"'] & X/S \arrow[d, "\eta_{X/S}"] \\
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As a corollary, we then obtain

Theorem

Let $\mathbb{C}$ be a regular category.
$\mathbb{A}$ is a regular epireflective subcategory of $\mathbb{C}$ if and only if $\mathbb{A}$ is the subcategory of separated objects for an effective closure operator.

Example

As expected, the effective closure arising from the reflection of the regular category $\mathbb{T}$(**Top**) of Mal'tsev topological algebras into its subcategory $\mathbb{T}$(**Haus**) of Hausdoff topological algebras coincides with the usual topological closure.

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If we want to extend to the non-pointed context also the axiom on the closure operators characterizing Birkhoff case, we have to face the fact that in a regular category the regular image $f(S)$ of an effective equivalence relation S is not in general an effective equivalence relation. But this condition on equivalence relation is exactly what characterizes Goursat regular categories:

Definition (Carboni-Kelly-Pedicchio 1993)

An exact category $\mathcal{C}$ is a *Goursat category* when one of the two following equivalent condition holds:

1. the regular image $f(S)$ of an equivalence relation S is an equivalence relation.
2. any pair of equivalence relations R, S on the same object X in $\mathcal{C}$ satisfies the condition

$$R \circ S \circ R = S \circ R \circ S,$$

Then in an exact Goursat category makes sense the following:

Definition

A *Birkhoff closure operator* on equivalence relations $\overline{(\ )}$ is an effective closure operator satisfying the following additional property: for any regular epi $f \colon X \to Y$

$$\overline{f(S)} = f(\overline{S}).$$

Theorem

Let $\mathcal{C}$ be an exact Goursat category. There is a bijection between the Birkhoff subcategories of $\mathcal{C}$ and the Birkhoff closure operators.

In this case, we can also describe how to obtain the closure of an equivalence relation, namely

$$\overline{S} = \overline{\Delta_X} \circ S \circ \overline{\Delta_X} = S \circ \overline{\Delta_X} \circ S$$

Example

If $\mathbb{T}$ is a Mal'tsev theory, $\mathbb{T}(\mathbf{Profin})$ is a Birkhoff subcategory of $\mathbb{T}(\mathbf{HComp})$ and the corresponding closure of an equivalence relation S on X is given by

$$\overline{S} = S \circ R_X,$$

where R_X is the congruence on X that identifies two points when they are in the same connected component.

If $\mathcal{C}$ is an exact Goursat category and $\overline{(\ )}$ a Birkhoff closure operator, we have also that

1. given a regular epimorphism $f \colon X \to Y$ and two equivalence relations R and S on X, one has that

$$\overline{f(R \vee S)} = \overline{f(R)} \vee \overline{f(S)};$$

2. for any equivalence relations R and S on X, one has

$$\overline{R \vee S} = \overline{R} \vee \overline{S}.$$

Thanks to these properties, we characterized the exact Goursat categories having the property that the lattice of equivalence relations is distributive:

Theorem

For an exact Goursat category $\mathcal{C}$ the following conditions are equivalent:

1. *the lattice of equivalence relations on any object X in $\mathcal{C}$ is distributive;*
2. *any Birkhoff closure operator satisfies the axiom*

$$f(\overline{R \wedge S}) = \overline{f(R)} \wedge \overline{f(S)}$$

   for any regular epimorphism $f : X \to Y$.

Outline

Back to the future!

Now we are studying closure operator properties of torsion free subcategories in the non-pointed regular case, where the problem of defining constant morphisms is determinant to develop the theory. But in this (and not only in this!), I had a good teacher: this was one of the topics of my (old!) Ph.D. thesis, so

THANKS WALTER, THANKS AGAIN!

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