Exponentiability via Double Categories

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Abstract. For a small category B and a double category $\mathbb{D}$, let $\mathrm{Lax}_N(B, \mathbb{D})$ denote the category whose objects are vertical normal lax functors $B \to \mathbb{D}$ and morphisms are horizontal lax transformations. It is well known that $\mathrm{Lax}_N(B, \mathbb{C}\mathrm{at}) \simeq \mathrm{Cat}/B$, where $\mathbb{C}\mathrm{at}$ is the double category of small categories, functors, and profunctors. Last year, we generalized this equivalence to certain double categories (called framed bicategories with glueing), in the case where B is a finite poset. Other examples include the double categories $\mathbb{T}\mathrm{op}$, $\mathbb{L}\mathrm{oc}$, and $\mathbb{P}\mathrm{os}$, whose objects are topological spaces, locales, and posets, respectively.

In "Powerful Functors," Street showed that $X \to B$ is exponentiable in Cat/B if and only if the corresponding normal lax functor $B \to \mathbb{C}\mathrm{at}$ is a pseudo-functor. Using our general equivalence, we will show that a morphism $X \to B$ is exponentiable in $\mathbb{D}_0/B$ if and only if the corresponding normal lax functor $B \to \mathbb{D}$ is a pseudo-functor *plus* an additional condition that holds for all $X \to B$ in Cat. Thus, we obtain a single theorem characterizing exponentiable morphisms of small categories, topological spaces, locales, and posets; at least in the case where B is finite.