

Exact completion and small sheaves

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Abstract.

We observe that the exact completions of a regular category, a lex category, a weakly lex category, or a category with a factorization system are all special cases of the exact completion of a *unary site*. A unary site is a site whose covering families are generated by single morphisms, and which has weak finite limits “after passage to a cover”. Exact completion is then a reflection from unary sites into exact categories; this unifies the universal properties of all exact completions.

Similarly, sites of “higher arity” have pretopos and infinitary-pretopos completions. The infinitary-pretopos completion of a small site is equivalent to its topos of sheaves, and the universal property of the former is identical to the classifying property of the latter. The infinitary-pretopos completion of a *large* site is thus a “category of small sheaves,” which satisfies all the exactness conditions of Giraud’s theorem and has a natural universal property. For a large trivial site, it is equivalent to the category of “small presheaves”.