Classifying Fiber Bundles with Fiber $K(\pi, n)$

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Abstract. This is joint work with André Joyal. A longstanding problem in algebraic topology is to define correctly the k-invariants for a non-simply connected space. We will present a solution which we think is simple, elegant, and conceptually correct. In fact, if $p : E \to X$ is a fiber bundle in simplicial sets with fiber $K(\pi, n)$ for $n \geq 2$, and π_1 denotes the fundamental groupoid of X, then π_1 acts on the n-dimensional homotopy of the fibers of p, producing a local coefficient system $\tilde{\pi}$. We show that then $E \to X$ is a torsor for the group $L(\pi_1, 1) \otimes_{\pi_1} K(\tilde{\pi}, n)$ over $K(\pi_1, 1)$. The torsor $L(\pi_1, 1) \otimes_{\pi_1} L(\tilde{\pi}, n+1) \to L(\pi_1, 1) \otimes_{\pi_1} K(\tilde{\pi}, n+1)$ is universal for such torsors, so $E \to X$ is induced by a map $X \to L(\pi_1, 1) \otimes_{\pi_1} K(\tilde{\pi}, n+1)$, representing a class in $H^{n+1}_{\pi_1}(X, \tilde{\pi})$, the cohomology of X with local coefficients $\tilde{\pi}$. Thus it is this cohomology that classifies such bundles, and this theorem that produces the desired k-invariant.