Species on hyperplane arrangements

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1. Species and Hyperspecies

Hopf monoids in species

$\mathbf{Sp}$:= category of vector species (Joyal).

- An object P is a functor

$$\mathrm{P} : \mathbf{set}^{\times} \to \mathbf{Vec}$$

where $\mathbf{set}^{\times}$:= finite sets and bijections
$\mathbf{Vec}$:= vector spaces.

- Monoidal structure

$$(\mathrm{P} \cdot \mathrm{Q})[I] \; := \; \bigoplus_{I = S \sqcup T} \mathrm{P}[S] \otimes \mathrm{Q}[T].$$

(One summand for each ordered partition of I.)

- Symmetry

$$x \otimes y \mapsto y \otimes x.$$

We may consider Hopf monoids in $\mathbf{Sp}$.

Homogeneous components and ordered partitions

Species P: one vector space P[I] for each finite set I.

Let H be a Hopf monoid. The (co)product of H has components

$$\mathrm{H}[S] \otimes \mathrm{H}[T] \underset{\Delta_{S,T}}{\overset{\mu_{S,T}}{\rightleftarrows}} \mathrm{H}[I]$$

where $I = S \sqcup T$.

Note: in general $\mu_{S,T} \neq \mu_{T,S}$ and $\Delta_{S,T} \neq \Delta_{T,S}$.

Associativity axiom for Hopf monoids

For each $I = R \sqcup S \sqcup T$,

$$\begin{array}{ccc} \mathrm{H}[R] \otimes \mathrm{H}[S] \otimes \mathrm{H}[T] & \xrightarrow{\mathrm{id} \otimes \mu_{S,T}} & \mathrm{H}[R] \otimes \mathrm{H}[S \sqcup T] \\ {\scriptstyle \mu_{R,S} \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \mu_{R,S \sqcup T}} \\ \mathrm{H}[R \sqcup S] \otimes \mathrm{H}[T] & \xrightarrow[\mu_{R \sqcup S,T}]{} & \mathrm{H}[I] \end{array}$$

(and dual for Δ)

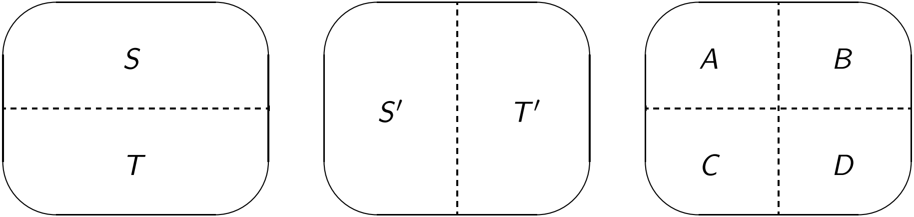
Consequence:

$$\mathrm{H}[R]\otimes\mathrm{H}[S]\otimes\mathrm{H}[T] \xrightarrow{\mu_{R,S,T}} \mathrm{H}[I] \quad \text{and} \quad \mathrm{H}[I] \xrightarrow{\Delta_{R,S,T}} \mathrm{H}[R]\otimes\mathrm{H}[S]\otimes\mathrm{H}[T]$$

are well-defined.

Compatibility axiom for Hopf monoids

Fix ordered partitions $S \sqcup T = I = S' \sqcup T'$, and let A, B, C, D be:



Then:

$$
\begin{array}{ccccc}
H[S] \otimes H[T] & \xrightarrow{\mu_{S,T}} & H[I] & \xrightarrow{\Delta_{S',T'}} & H[S'] \otimes H[T'] \\
\Big\downarrow {\scriptstyle \Delta_{A,B} \otimes \Delta_{C,D}} & & & & \Big\uparrow {\scriptstyle \mu_{A,C} \otimes \mu_{B,D}} \\
H[A] \otimes H[B] \otimes H[C] \otimes H[D] & & \xrightarrow[\mathrm{id} \otimes \mathrm{switch} \otimes \mathrm{id}]{} & & H[A] \otimes H[C] \otimes H[B] \otimes H[D]
\end{array}
$$

Reference

Monoidal functors, species, and Hopf algebras.
(With Swapneel Mahajan.)
CRM Monograph Series 29, AMS, 2010.

Hyperplane arrangements and hyperspecies

Let V be a real finite dimensional vector space.
A hyperplane arrangement in V is a finite set of hyperplanes in V.

$\mathbf{HSp}$:= category of hyperspecies.

- An object P is a functor

$$\mathrm{P} : \mathbf{arr}^{\times} \to \mathbf{Vec}$$

  where $\mathbf{arr}^{\times}$:= hyperplane arrangements
  and isomorphisms.
- No monoidal structure on $\mathbf{HSp}$! (Extending that of $\mathbf{Sp}$.)
- However, we will define a notion of Hopf hypermonoid.

Distributive laws

There is a monad T on **HSp**, a comonad $T^\vee$ on **HSp**, and a mixed distributive law

$$\lambda : T \circ T^\vee \to T^\vee \circ T,$$

such that

$$\text{Hopf hypermonoids} = (T, T^\vee, \lambda)\text{-bialgebras.}$$

Mixed distributive laws: Beck (1969).
Also Burroni, Van Osdol, Wolff,. . . .

Different from the bimonads of Moerdijk and Bruguières-Virelizier.

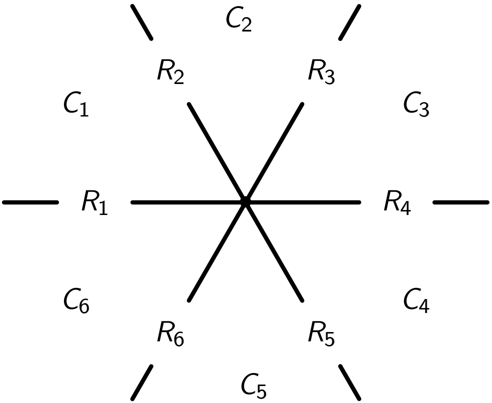
Faces

Let $\mathbb{A}$ be a hyperplane arrangement in V.

Fact. The hyperplanes in $\mathbb{A}$ split V into a collection $\Sigma(\mathbb{A})$ of convex sets called faces.

Example.

$\mathbb{A} = \{H_1, H_2, H_3\} \Rightarrow \Sigma(\mathbb{A}) = \{O, R_1, \dots, R_6, C_1, \dots, C_6\}$.

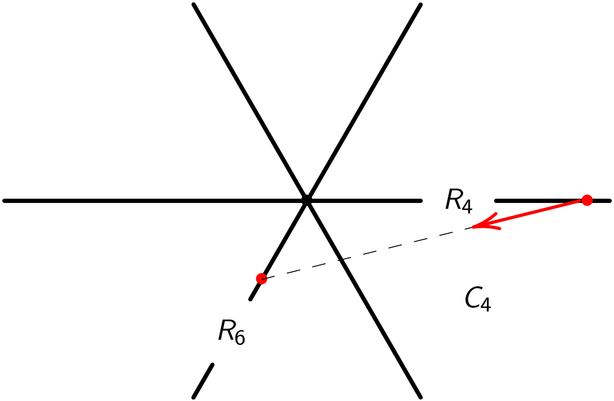


Tits projections

Let $\mathbb{A}$ be a hyperplane arrangement.

Fact. The set $\Sigma(\mathbb{A})$ is a monoid.

Example. $R_4 R_6 = C_4$.



Note. $FG \supseteq F$ for any faces $F, G \in \Sigma(\mathbb{A})$.

- Bland (1974), Tits (1974), Bidigare (1997).
- Brown-Diaconis (1998), Billera-Brown-Diaconis (1999).

Subarrangements

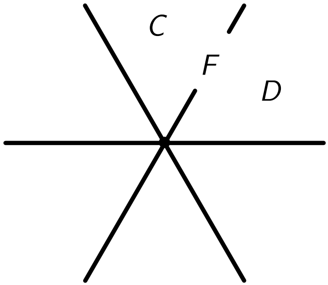
Let $\mathbb{A}$ be a hyperplane arrangement and $F \in \Sigma(\mathbb{A})$ a face.
The subarrangement determined by F is

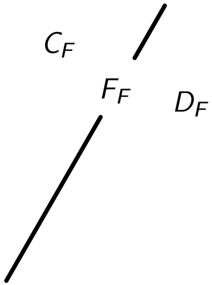
$$\mathbb{A}_F := \{H \in \mathbb{A} \mid H \supseteq F\}.$$

Fact. There is a canonical bijection

$$\Sigma(\mathbb{A}_F) \cong \{G \in \Sigma(\mathbb{A}) \mid G \supseteq F\}.$$

Let $G_F \in \Sigma(\mathbb{A}_F)$ denote the face of $\mathbb{A}_F$ corresponding to $G \supseteq F$.

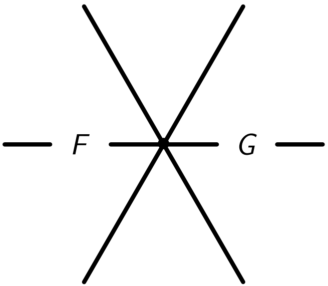




Fact. For any faces $G \supseteq F$ of $\mathbb{A}$, $(\mathbb{A}_F)_{G_F} = \mathbb{A}_G$.

Subarrangements and the product of faces

The product of faces is not commutative.



However,

$$\mathbb{A}_{FG} = \mathbb{A}_{GF}$$

for any faces $F, G \in \Sigma(\mathbb{A})$.

Hopf hypermonoids

A Hopf hypermonoid $(\mathrm{H}, \mu, \Delta)$ consists of:

- A functor $\mathrm{H} : \mathbf{arr}^{\times} \to \mathbf{Vec}$ (a hyperspecies).
- For each arrangement $\mathbb{A}$ and each face $F \in \Sigma(\mathbb{A})$, maps

$$\mathrm{H}[\mathbb{A}_F] \underset{\Delta_F}{\overset{\mu_F}{\rightleftarrows}} \mathrm{H}[\mathbb{A}]$$

subject to the following axioms.

Associativity. For each $\mathbb{A}$ and $F \subseteq G$ in $\Sigma(\mathbb{A})$,

$$\begin{array}{ccc} \mathrm{H}[\mathbb{A}_F] & \xrightarrow{\mu_F} & \mathrm{H}[\mathbb{A}] \\ \uparrow{\scriptstyle \mu_{G_F}} & & \uparrow{\scriptstyle \mu_G} \\ \mathrm{H}[(\mathbb{A}_F)_{G_F}] & = & \mathrm{H}[\mathbb{A}_G] \end{array} \qquad \text{(and dual for } \Delta\text{)}$$

Compatibility. For each $\mathbb{A}$ and any faces F and G in $\Sigma(\mathbb{A})$,

$$\begin{array}{ccccccc} \mathrm{H}[\mathbb{A}_F] & & \xrightarrow{\mu_F} & \mathrm{H}[\mathbb{A}] & \xrightarrow{\Delta_G} & & \mathrm{H}[\mathbb{A}_G] \\ \downarrow{\scriptstyle \Delta_{(FG)_F}} & & & & & & \uparrow{\scriptstyle \mu_{(GF)_F}} \\ \mathrm{H}[(\mathbb{A}_F)_{(FG)_F}] & = & \mathrm{H}[\mathbb{A}_{FG}] & = & \mathrm{H}[\mathbb{A}_{GF}] & = & \mathrm{H}[(\mathbb{A}_G)_{(GF)_F}] \end{array}$$

Perspective

finite set I	finite hyperplane arrangement $\mathbb{A}$
ordered partition of I	face of $\mathbb{A}$
Joyal species	hyperspecies
Hopf monoid	Hopf hypermonoid

Connection through the braid arrangement.

Examples: L and E

For each arrangement $\mathbb{A}$, let

$$\mathsf{L}[\mathbb{A}] = \Bbbk\{C \mid C \text{ is a chamber of } \mathbb{A}\}.$$

For each face F of $\mathbb{A}$, define

$$\begin{aligned} \mu_F : \mathsf{L}[\mathbb{A}_F] &\to \mathsf{L}[\mathbb{A}] \\ C_F &\mapsto C \end{aligned} \qquad \begin{aligned} \Delta_F : \mathsf{L}[\mathbb{A}] &\to \mathsf{L}[\mathbb{A}_F] \\ C &\mapsto (FC)_F. \end{aligned}$$

Then L is a Hopf hypermonoid.

For each arrangement $\mathbb{A}$, let

$$\mathsf{E}[\mathbb{A}] = \Bbbk\{*_{\mathbb{A}}\}.$$

For each face F of $\mathbb{A}$, define

$$\begin{aligned} \mu_F : \mathsf{E}[\mathbb{A}_F] &\to \mathsf{E}[\mathbb{A}] \\ *_{\mathbb{A}_F} &\mapsto *_{\mathbb{A}} \end{aligned} \qquad \begin{aligned} \Delta_F : \mathsf{E}[\mathbb{A}] &\to \mathsf{E}[\mathbb{A}_F] \\ *_{\mathbb{A}} &\mapsto *_{\mathbb{A}_F}. \end{aligned}$$

Then E is a Hopf hypermonoid.

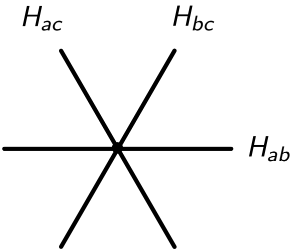
The braid arrangement

Let I be a finite set.

$$\mathbb{R}^I := \{\text{ functions } x : I \to \mathbb{R}\}; \quad \mathbb{R}^I_0 := \{x \in \mathbb{R}^I \mid \sum_{i \in I} x_i = 0\};$$

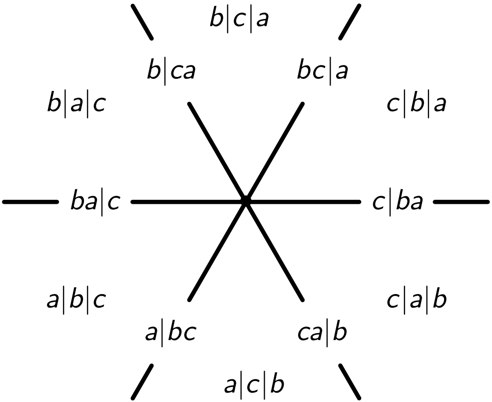
$$H_{ij} := \{x \in \mathbb{R}^I_0 \mid x_i = x_j\}; \quad \mathbb{B}^I := \{H_{ij} \mid i, j \in I,\ i \neq j\}.$$

Example. $I = \{a, b, c\}$.



Faces of the braid arrangement

Fact. Faces are in bijection with ordered partitions of I.



Fact. If the face F corresponds to the partition $(S_1, \ldots, S_k)$, then

$$(\mathbb{B}^I)_F \cong \mathbb{B}^{S_1} \times \cdots \times \mathbb{B}^{S_k}.$$

Tits projections for the braid arrangement

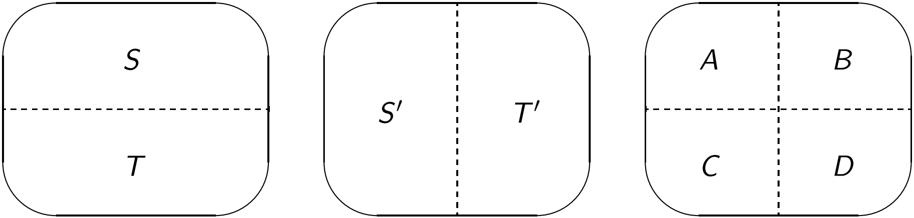
If $F = (S_1, \ldots, S_h)$ and $G = (T_1, \ldots, T_k)$, then

$$FG = (S_1 \cap T_1, \ldots, S_1 \cap T_k, \ldots, S_h \cap T_1, \ldots, S_h \cap T_k)\widehat{\ }.$$

Example. Let $F = (S, T)$ and $G = (S', T')$. Then

$$FG = (A, B, C, D) \quad \text{and} \quad GF = (A, C, B, D),$$

where



From Hopf hypermonoids to Hopf monoids

There is a functor

$$\mathbf{set}^\times \to \mathbf{arr}^\times, \quad I \mapsto \mathbb{B}^I$$

and hence a functor

$$\mathbf{HSp} \to \mathbf{Sp}\,.$$

Proposition. Let $\mathrm{P} : \mathbf{arr}^\times \to \mathbf{Vec}$ be a hyperspecies. Suppose that

$$\mathrm{P}[\mathbb{A}_1 \times \mathbb{A}_2] \cong \mathrm{P}[\mathbb{A}_1] \otimes \mathrm{P}[\mathbb{A}_2]$$

for any arrangements $\mathbb{A}_1$ and $\mathbb{A}_2$. If P is a Hopf hypermonoid, then $\mathrm{P} : \mathbf{set}^\times \to \mathbf{Vec}$ is a Hopf monoid in species.

The theory of Hopf hypermonoids

- Antipode and basic properties.
- Commutative monoids.
- Lie monoids.
- Primitive elements and coradical filtration.
- Poincaré-Birkhoff-Witt theorem.
- Cartier-Milnor-Moore theorem.

1. **Species and Hyperspecies**
2. **Operads and Hyperoperads**

Operads

Let $\Pi(I)$ denote the set of (unordered) partitions of a finite set I.

Given species P and Q, define a new species $\mathrm{P} \circ \mathrm{Q}$ by

$$(\mathrm{P} \circ \mathrm{Q})[I] := \bigoplus_{X \in \Pi(I)} \mathrm{P}[X] \otimes \bigotimes_{S \in X} \mathrm{Q}[S].$$

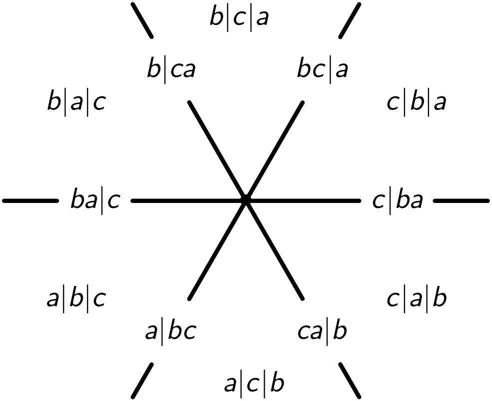
Then (Day, Joyal):

- $(\mathbf{Sp}, \circ)$ is a monoidal category.
- A monoid in $(\mathbf{Sp}, \circ)$ is a (symmetric) operad.

Flats

Let $\mathbb{A}$ be a hyperplane arrangement.
A flat of $\mathbb{A}$ is an intersection of any number of hyperplanes in $\mathbb{A}$.

Fact. The flats of $\mathbb{B}^I$ are in bijection with partitions of I.



Hyperoperads

Let $\Pi(\mathbb{A})$ denote the set of flats of a hyperplane arrangement $\mathbb{A}$. Given $X \in \Pi(\mathbb{A})$, define

$$\mathbb{A}_X := \{H \mid H \supseteq X\} \quad \text{and} \quad \mathbb{A}^X := \{H \cap X \mid H \notin \mathbb{A}_X\}.$$

Given hyperspecies P and Q, define a new hyperspecies $\mathrm{P} \circ \mathrm{Q}$ by

$$(\mathrm{P} \circ \mathrm{Q})[\mathbb{A}] := \bigoplus_{X \in \Pi(\mathbb{A})} \mathrm{P}[\mathbb{A}^X] \otimes \mathrm{Q}[\mathbb{A}_X].$$

Then $(\mathbf{HSp}, \circ)$ is a monoidal category.

Definition. A monoid in $(\mathbf{HSp}, \circ)$ is a hyperoperad.

The associative and commutative hyperoperads

The hyperspecies L is a hyperoperad:

$$\mathsf{L}[\mathbb{A}^X] \otimes \mathsf{L}[\mathbb{A}_X] \to \mathsf{L}[\mathbb{A}], \quad F \otimes C \mapsto FC.$$

Proposition. An L-module in $(\mathbf{HSp}, \circ)$ is the same as a hypermonoid.

The hyperspecies E is a hyperoperad:

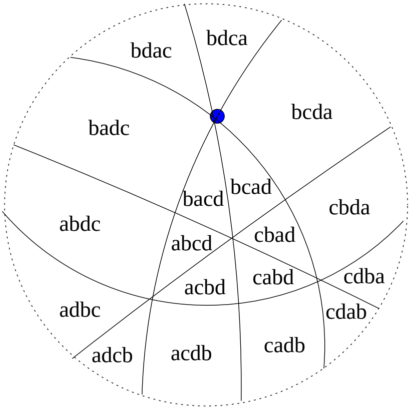
$$\mathsf{E}[\mathbb{A}^X] \otimes \mathsf{E}[\mathbb{A}_X] \to \mathsf{E}[\mathbb{A}], \quad *_{\mathbb{A}^X} \otimes *_{\mathbb{A}_X} \mapsto *_{\mathbb{A}}.$$

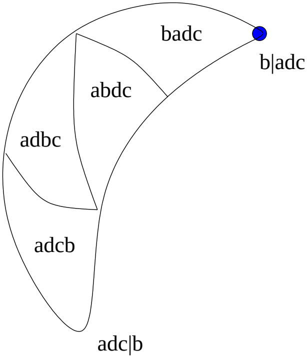
Proposition. An E-module in $(\mathbf{HSp}, \circ)$ is the same as a commutative hypermonoid.

Lunes

Let F be a face and C be a chamber of $\mathbb{A}$ with $F \subseteq C$.
The corresponding lune is

$$\Psi(F, C) := \{D \mid D \text{ is a chamber and } FD = C\}.$$





The Lie hyperoperad

Let Lie be the subhyperspecies of L defined by

$$\mathrm{Lie}[\mathbb{A}] := \Big\{ \sum_{D} a_D D \in \mathrm{L}[\mathbb{A}] \ \Big| \sum_{D \in \Psi(F,C)} a_D = 0 \text{ for all } F \subseteq C,\ F \neq O \Big\}.$$

Proposition. Lie is a subhyperoperad of L.

Definition. A Lie hypermonoid is a Lie-module in $(\mathbf{HSp}, \circ)$.

Thank you.