Countable meets of opens in coherent spaces*

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*For full paper, with details, see:
ftp.math.mcgill.ca/pub/barr/pdffiles/countabl.pdf

A topological space is called **coherent** if is sober, if any finite (including empty) meet of compact open sets is compact, and the compact open sets are a base for the topology.

Examples of coherent spaces include 2^T and S^T ($S =$ Sierpinski space) for any set T. We will see more examples later.

Coherent spaces are characterized by the following: If $\mathcal{M}$ is the class of compact open sets and $\mathcal{N}$ consists of the complements of the sets in $\mathcal{M}$, then the topology for which $\mathcal{M} \cup \mathcal{N}$ is a subbase is compact Hausdorff (and totally disconnected). [Hochster]

If X is coherent, $\mathcal{M}$ the family of compact opens and $\mathcal{N}$ their complements, it follows from the characterization above that the topology for which $\mathcal{N}$ is a basis is also coherent. This might be called the dual topology.

For any partial equational theory $\mathbf{Th}$ (it needn't be essentially algebraic) and any model A both the space of subobjects with the "exclusion topology" or the "inclusion topology" and the space of quotient objects with the similar topologies (called the "Zariski" and "Scott" topologies, resp.), are coherent.

By the exclusion topology, we mean that the sets

$$M(a) = \{B \subseteq A \mid a \notin B\}$$

form a base, while the inclusion topology is the dual. The Zariski topology has for basis

$$M(a, a') = \{(a, a') \in A \times A \mid (a.a') \notin E\}$$

where E is the congruence kernel of the quotient mapping. The Scott topology is dual.

The prime ideal spectrum of a ring is also coherent with either the Zariski or Scott topology.

In the topology with basis $\mathcal{M} \cup \mathcal{N}$, the sets in $\mathcal{M}$ are now closed as well as compact and hence satisfy the finite intersection property. If U is open in $\mathcal{M}$-topology it remains open in the stronger one and hence we have:

PROPOSITION. *Suppose that* $\{M_\alpha\} \subseteq \mathcal{M}$ *and that* U *is open in* X. *If* $\bigcap M_\alpha \subseteq U$, *then there is a finite set* $\alpha_1, \cdots, \alpha_n$ *of indices such that*

$$\bigcap_{i=1}^{n} M_{\alpha_i} \subseteq U$$

If X is any topological space and $\{A_\alpha\}$ is any collection of subspaces, we have an ordinary intersection $\bigcap A_\alpha$ and their localic intersection $\bigwedge A_\alpha$. The latter comes about as follows.

If $\mathcal{O}(\mathcal{X})$ is the open set lattice of X, every subset $A \subseteq X$ gives an equivalence relation E_A on $\mathcal{O}(\mathcal{X})$ defined by UE_AV if $U \cap A = V \cap A$. Then E_A is a congruence on the frame ($= \vee$-$\bigwedge$ lattice) $\mathcal{O}(\mathcal{X})$. The sup of the E_{A_α} in the lattice of congruences is the localic meet of the A_α.

It is trivial to see that $\bigwedge A_\alpha \subseteq \bigcap A_\alpha$, but the inclusion is often strict.

The main result of this talk is:

THEOREM. *Suppose* X *is coherent and* $\{U_n\}$ *is a countable collection of open sets in* X*. Then* $\bigcap U_n = \bigwedge U_n$.

In the rest of this talk, I will sketch as much of the proof as I can. The main tool in the proof is the fact that there is a one-one correspondence between frame congruences on $\mathcal{O}(\mathcal{X})$ and nuclei.

A **nucleus** is a function $j : \mathcal{O}(\mathcal{X}) \to \mathcal{O}(\mathcal{X})$ that is expansive ($U \subseteq j(U)$), idempotent, and preserves finite intersection. Given a congruence E, you get a nucleus by

$$j(U) = \bigcup \{V \mid UEV\}$$

and, given a nucleus j, you get a congruence E by letting UEV when $j(U) = j(V)$ [Johnstone, Stone Spaces].

To prove the main theorem, suppose $A = \bigcap U_n$ and that $U_n = \bigcup_{\sigma \in \Sigma_n} M_{n,\sigma}$ with each $M_{n,\sigma} \in \mathcal{M}$. Let $L = \bigwedge U_n$ and denote by j_n, j_A, and j_L, resp. the nuclei corresponding to U_n, A, and L. By definition, $j_L = \bigvee j_n$, the sup taken in the lattice of nuclei. Since $A \subseteq U_n$ for all n, we see that $j_n \le j_A$ whence $j_L \le j_A$.

By a **choice function**, we mean a map $\xi : \mathbf{N} \to \bigcup \Sigma_n$ such that $\xi(n) \in \Sigma_n$ for all $n > 0$. If ξ is a choice function, then from $M_{n,\xi(n)} \subseteq U_n$, it follows that $\bigcap_{n=1}^{\infty} M_{n,\xi(n)} \subseteq A$.

If we suppose that $L \subsetneqq A$, then $j_L \lneqq j_A$. Thus there is an open set V such that $j_L(V) \subsetneqq j_A(V)$ and hence there is an $M_0 \in \mathcal{M}$ with $M_0 \subseteq j_A(V)$ while $M_0 \not\subseteq j_L(V)$. This last implies that for all $n > 0$, $M_0 \not\subseteq j_n(V)$ which, we will show, leads to a contradiction.

LEMMA. *Suppose that $M \in \mathcal{M}$ with $M \not\subseteq j_L(V)$. Then for each $n > 0$, there is a $\sigma \in \Sigma_n$ such that $M \cap M_{n,\sigma} \not\subseteq j_L(V)$.*

Since $M \not\subseteq j_L(V) = j_L^2(V)$ and $j_L = \bigvee j_n$, we see that $M \not\subseteq j_n(j_L(V))$ and hence $M \cap U_n \not\subseteq j_L(V)$. But $U_n = \bigcup_{\sigma \in \Sigma_n} M_{n,\sigma}$ so there must be some $\sigma \in \Sigma_n$ with $M \cap M_{n,\sigma} \not\subseteq j_L(V)$.

By using this lemma we can inductively construct a sequence $\xi(1), \xi(2), \ldots, \xi(n), \ldots$ of elements of Σ such that $\xi(n) \in \Sigma_n$ and that for each $n > 0$,

$$M_0 \cap M_{1,\xi(1)} \cap M_{2,\xi(2)} \cap \cdots \cap M_{n,\xi(n)} \not\subseteq j_L(V)$$

Finish the proof by applying the proposition on Page 4 to conclude that $M_0 \cap \bigcap_{n=1}^{\infty} M_{n,\xi(n)} \not\subseteq j_L(V)$ and therefore $M_0 \cap \bigcap_{n=1}^{\infty} M_{n,\xi(n)} \not\subseteq V$. It follows that $M_0 \cap A \not\subseteq V$ and hence $M_0 \not\subseteq j_A(V)$, contrary to assumption.