# Some homotopy methods in the category of graphs

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A Quillen model structure on a category $\mathcal{E}$ involves three classes of morphisms - weak equivalences, fibrations, and cofibrations - which obey some weak factorization axioms. Quillen showed that these provide a good framework for describing and working with a universal homotopy functor $\mathcal{E} \to \mathrm{Ho}(\mathcal{E})$ which inverts all the weak equivalences.

Gph denotes our category of directed graphs. After describing graph covering morphisms and some related weak factorization systems, we describe two model structures on Gph - one based on cycles and the other on infinite walks. We calculate the associated homotopy categories and relate them to the study of zeta series, spectra, and dynamical systems of finite graphs.

Joint work with Aristide Tsemo.

# Acknowledgements

Coverings played a major role in REU projects at Canisius College for the last few years.

We learned about Coverings in algebraic graph theory from a paper by Boldi and Vigna, e.g.

**Proposition:** If $f : X \to Y$ is a Covering of finite graphs which is surjective on nodes, then the characteristic polynomial of $Y$ divides the characteristic polynomial of $X$.

Steve Schanuel suggested we look at coverings and factorizations in graph theory; discussions with J. Rosicky and W. Tholen, and papers by Stallings, and by Enochs and Herzog, and by Dress and Siebeneicher helped.

Thanks

# Categories of graphs?

There are many definitions of graph, designed to fit many different situations.

- What objects should be allowed?
  loops, multiple edges, finite, ...?
- What morphisms should be allowed?
  collapsing or not collapsing?

Since the definitions don't agree, it seems important to choose one with generality and abstraction, and stick with it.

We want a category of graphs
- close to many applications
- able to "emulate" other categories of graphs
- good relations with combinatorics
- with some of the "feel" of generalized spaces

Bill Lawvere has a nice paper from 1989
*Qualitative distinctions between
some toposes of generalized graphs*,
giving a good idea of the choices available.

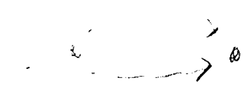
## The category Gph

- Directed (and possibly infinite) graphs, (with loops and multiple arcs allowed).
- A *graph* is a data-structure

$$X = (X_0, X_1, s, t)$$

with sets $X_0$ of *nodes* and $X_1$ of *arcs*, and with functions $s, t : X_1 \longrightarrow X_0$ which specify *source* node and *target* node of each arc.

- A *graph morphism* $f : X \longrightarrow Y$ is a pair of functions $f_1 : X_1 \longrightarrow Y_1$ and $f_0 : X_0 \longrightarrow Y_0$ such that $s \circ f_1 = f_0 \circ s$ and $t \circ f_1 = f_0 \circ t$.
- An arc $a \in X_1$ *leaves* $s(a)$ and *enters* $t(a)$.
- A *loop* is an arc $a$ with $s(a) = t(a)$.

## Gph is a presheaf topos, on site 

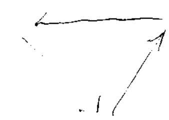
- The *terminal object* $1$ has one node and arc.
- The *initial object* $0$ has no nodes or arcs.
- etc.

# Examples and Concepts as Morphisms

- The *dot* $\mathbf{D}$ has one node and no arcs.
- The *arrow* $\mathbf{A}$ has one arc and its source and target nodes.
- The *path* $\mathbf{P}_n$ has nodes $0, \ldots, n$ and arcs $(0,1), \ldots, (n-1, n)$,
with $s((i-1,i)) = i-1$ and $t((i-1,i)) = i$.

- The *infinite walk* $\mathbf{N}$ has a node $n$ and arc $(n, n+1)$ for each non-negative integer.

- The *cycle* $\mathbf{C}_n$ identifies nodes $0$, $n$ in $\mathbf{P}_n$.

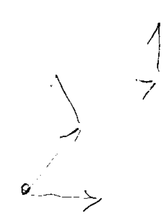


Define loops, paths, cycles, infinite walks, etc as graph morphisms.

- The bouquet of loops $B(S)$ on a set $S$.

An arc-labeling is a graph morphism to $B(S)$.

- A *rooted tree* $T$ has one root node $r$, and a unique path from $r$ to $x$, for each node $x$.



## Some classes of morphisms

Attaching a rooted forest $F$ (a sum of rooted trees) to $X$ means forming a new graph $X_F$ as the pushout along a map of the roots to $X$.

**Definition:** A *Whiskering* is a graph morphism formed by attaching some rooted forest.

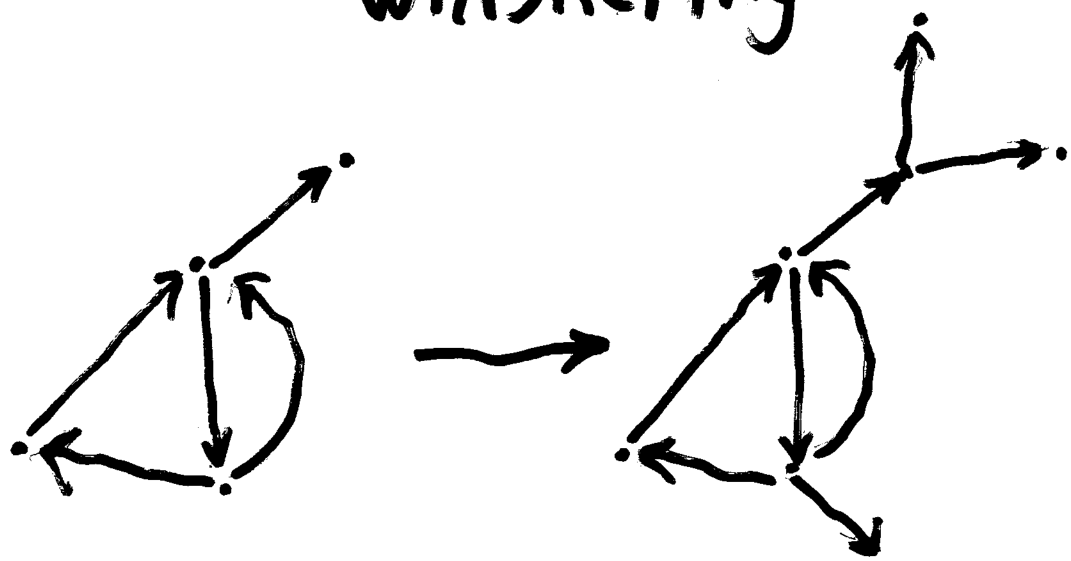
For node $x$ in graph $X$, let $X(x, *)$ denote the set of arcs with source $x$. Note that a graph morphism $f : X \to Y$ gives a function $f : X(x, *) \to Y(f(x), *)$, etc.

**Definition:** A *Covering* is a graph morphism $f : X \to Y$ so that $f : X(x, *) \to Y(f(x), *)$ is bijective for all $x \in X_0$.

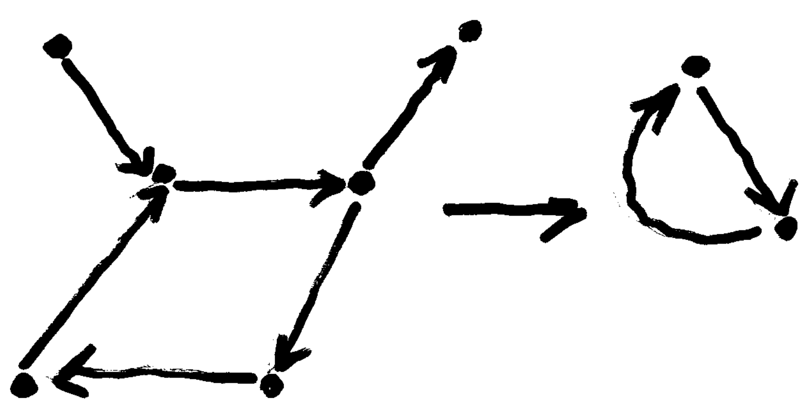
They are important in algebraic graph theory

**Also:** A *Surjecting* is a graph morphism $f : X \to Y$ so that $f : X(x, *) \to Y(f(x), *)$ is surjective for all $x$.

# Whiskering



# Covering      Surjecting



## 'Homotopical Algebra" for Gph

• Quillen (1967) gave abstract axioms to work with homotopy concepts in general categories.

But not necessarily homotopy as topological deformation of structure.

• "Giving a model structure" means satisfying these axioms.

• First choose a class of "weak equivalences".

In most cases, they are chosen to preserve some interesting invariant.

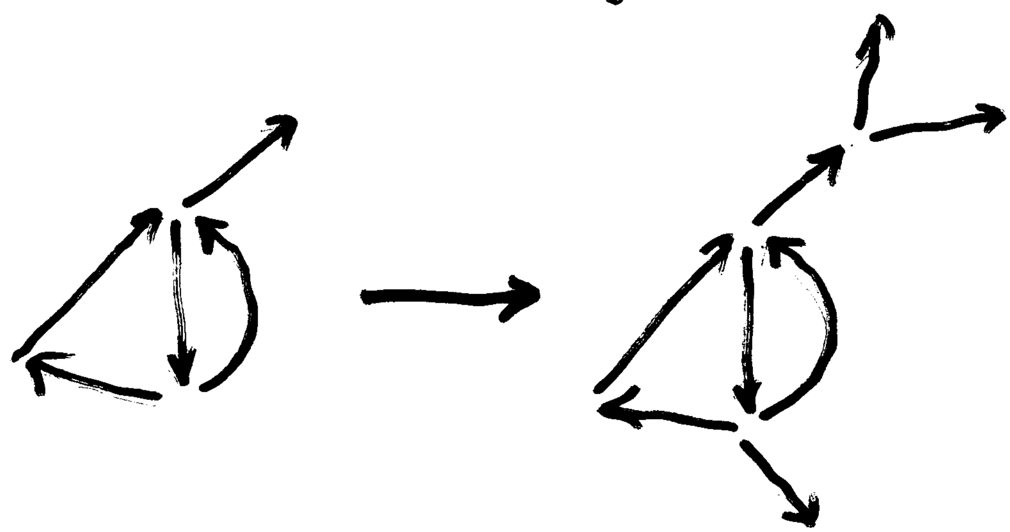
• For our "cycles" model structure we take the Acyclic graph morphisms which neither create nor destroy cycles.

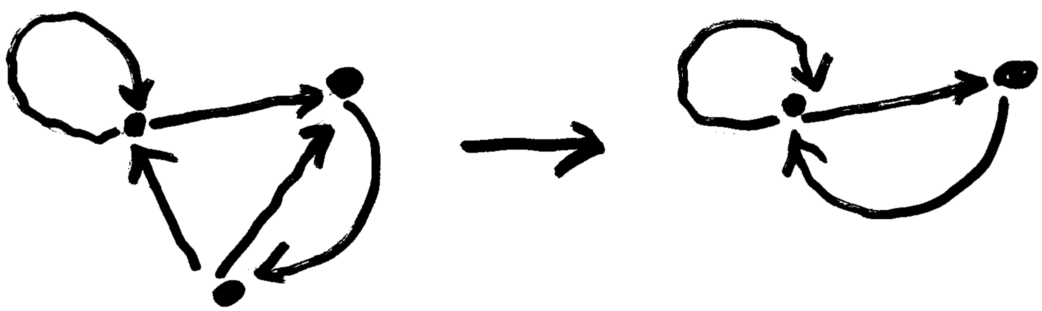
• For our "walks" model structure we take the graph morphisms which neither create nor destroy walks.

• Goal: relevance to algebraic graph theory.

E+

same cycles

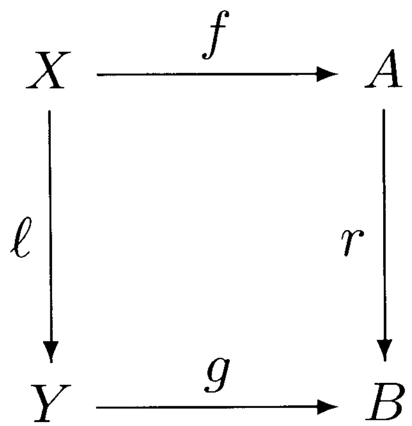




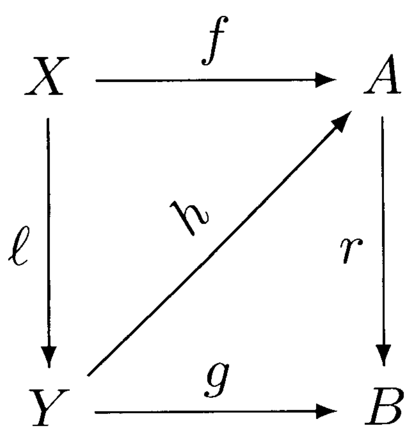
same walks (and cycles)

## "Homotopy" factorization.

**Definition:** For morphisms $\ell$ and $r$, $\ell \dagger r$ ($\ell$ is *weakly orthogonal* to $r$) means, for all $f$ and $g$,

if 

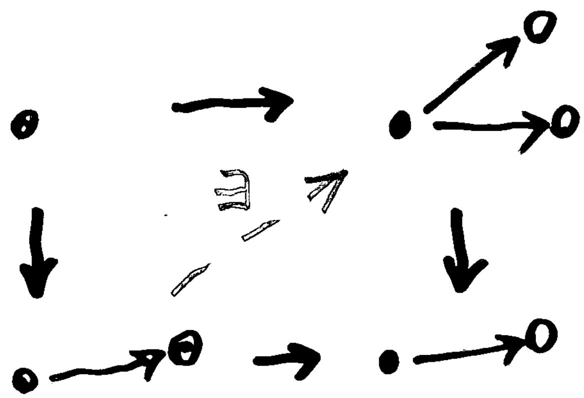
commutes, then

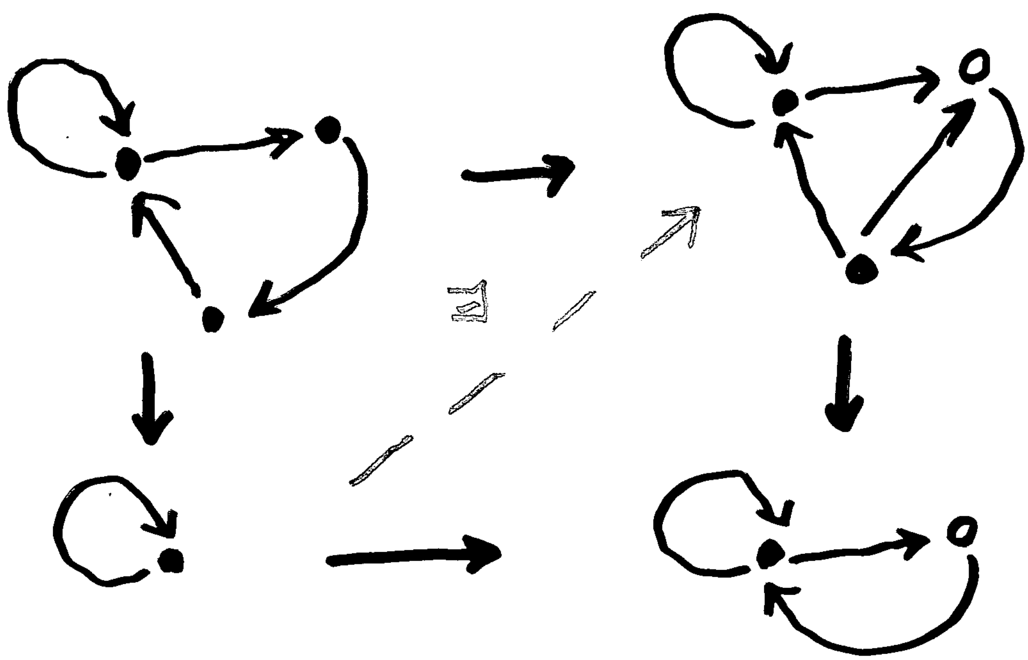


commutes, some $h$.

For a class $\mathcal{F}$ of morphisms, let

$$\mathcal{F}^\dagger = \{r : \mathcal{F} \dagger r\} \text{ and } {}^\dagger\mathcal{F} = \{\ell : \ell \dagger \mathcal{F}\}.$$



**Definition:** A *weak factorization system* is two classes $\mathcal{L}$ and $\mathcal{R}$ such that $\mathcal{L}^{\dagger} = \mathcal{R}$ and $\mathcal{L} = {}^{\dagger}\mathcal{R}$ and such that, for any morphism $c$, there exist $\ell \in \mathcal{L}$ and $r \in \mathcal{R}$ with $c = r \circ \ell$.

The notion of weak factorization system has become a part of homotopical algebra.

For a category with finite limits and colimits. Quillen's notion of "model category" can be expressed via the following axioms
(learned from a paper of Joyal and Tierney).

**Definition:** A *model structure* is a triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ of classes of morphisms satisfying

1) "three for two": if two of the three morphisms $a, b, a \circ b$ belong to $\mathcal{W}$ then so does the third,

2) the pair $(\underline{\mathcal{C}}, \mathcal{F})$ is a weak factorization system (where $\underline{\mathcal{C}} = \mathcal{C} \cap \mathcal{W}$),

3) the pair $(\mathcal{C}, \underline{\mathcal{F}})$ is a weak factorization system (where $\underline{\mathcal{F}} = \mathcal{W} \cap \mathcal{F}$).

# Two model structures for Gph

According to Hovey in *Model Categories*,

> "It tends to be quite difficult to prove that a category admits a model structure. The axioms are always hard to check."

**Theorem:** The "cycles" model structure on Gph has: $\mathcal{W}$ the Acyclics, $\underline{\mathcal{C}}$ the Whiskerings, and $\mathcal{F}$ the Surjectings.

Then $\underline{\mathcal{F}}$ is Acyclic Surjectings, and $\mathcal{C}$ is ${}^{\dagger}\underline{\mathcal{F}}$ .

$f$ Acylic means $C_n(f) : C_n(X) \to C_n(Y)$ is bijective for all $n > 0$, where $C_n(X)$ is the set of graph morphisms from $\mathbf{C}_n \to X$.

**Theorem:** There is also a "walks" model structure on Gph with $\mathcal{W}$ inducing bijections on walks, and with $\mathcal{F} = \mathrm{All}$.

# Generating our "cycles" model

• $\mathcal{F} = J^\dagger$ where $J = \{\mathrm{s}\}$ with
$\mathrm{s} : \mathbf{D} \to \mathbf{A}$ the "source" graph morphism.

• $\mathcal{W} = K^\dagger$ where $K = \{\mathbf{i}_n, \mathbf{j}_n : n > 0\}$ with
$\mathbf{i}_n : \mathbf{0} \to \mathbf{C}_n$ the initial graph morphism, and
$\mathbf{j}_n : \mathbf{C}_n + \mathbf{C}_n \to \mathbf{C}_n$ the natural morphism.

• $\underline{\mathcal{F}} = I^\dagger$ with $I = J \cup K$.

So $\mathcal{F} = \text{Surjectings}$, and $\underline{\mathcal{C}} = \text{Whiskerings}$.

Cofibrations for our "cycles" model structure:

**Proposition:**

a) Whiskerings are cofibrations (and Acyclic).

b) If $C$ is a sum of cycle graphs, then the inclusion $X \to X + C$ is a cofibration.

c) All graph morphisms between sums of cycle graphs are cofibrations.

d) $0 \to \mathbf{Z}$ is **not** a cofibration.

## Homotopy categories.

**Definition:** A *homotopy functor* on a model category $\mathcal{E}$ takes each weak equivalence to an isomorphism.

Quillen describes a category and functor

$$\gamma : \mathcal{E} \to \mathrm{Ho}(\mathcal{E})$$

which is *initial* for homotopy functors on $\mathcal{E}$. Here, $\mathrm{Ho}(\mathcal{E})$ has the same objects as $\mathcal{E}$, and so the universal definition determines $\mathrm{Ho}(\mathcal{E})$ up to isomorphism of categories.

The universal definition of $\mathrm{Ho}(\mathcal{E})$ does not involve the fibrations and cofibrations, but they provide a kind of "scaffolding" to describe the *set* of morphisms $\mathrm{Ho}(X, Y)$ for $X$ and $Y$ in $\mathcal{E}$.

The function $\mathcal{E}(X, Y) \to \mathrm{Ho}(X, Y)$ is not always surjective; general morphisms in $\mathrm{Ho}(\mathcal{E})$ are zig-zag compositions of homotopy classes of morphisms in $\mathcal{E}$.

Two objects in $\mathcal{E}$ are *homotopy-equivalent* when they become isomorphic in $\mathrm{Ho}(\mathcal{E})$.

## Fibrant/Cofibrant objects.

Object $X$ in a model category $\mathcal{E}$ is *fibrant* when $X \to 1$ is in $\mathcal{F}$; dually, it is *cofibrant* when $0 \to X$ is in $\mathcal{C}$.

A *cofibrant replacement* for $X$ is $f : X' \to X$ in $\mathcal{W}$ with $X'$ cofibrant. It is a *full cofibrant replacement* when $f$ is in $\mathcal{F}$. Dually for *fibrant* and *full fibrant replacements*.

Each $X$ has a full cofibrant and full fibrant replacement. Also, a full fibrant replacement of a cofibrant object is fibrant-cofibrant, etc.

Quillen uses cofibrant/fibrant replacements to describe $\mathrm{Ho}(\mathcal{E})$.

Fibrant/Cofibrant in the "cycles" model:

- Graph $X$ is fibrant when $X$ is *walkable* (every node has at least one arc leaving).
- Graph $X$ is cofibrant when $X$ is a sum of whiskered cycles.

This seems related to certain sub-categories of Gph which are equivalent to NSet and/or ZSet.

# A model structure for ZSet

Let Z be the group of integers under addition. Consider the presheaf topos ZSet; its objects are $(S, \sigma)$ with set $S$ and bijection $\sigma : S \to S$; maps of Z-sets are those commuting with $\sigma$.

There is a model structure on ZSet where:

• all Z-maps are fibrations

• cofibrations are generated by $i_n : 0 \to Z/n$ and $j_n : Z/n + Z/n \to Z/n$.

• weak equivalences are Z-maps which are isomorphic on periodic elements.

In ZSet, every object is fibrant, but an object is cofibrant iff every element has finite period.

The full subcategory of cofibrant objects in ZSet is cZSet, the category of periodic Z-sets. Give cZSet the trivial model structure.

# Relating Gph and ZSet and cZSet

We have a "presheaf triple" $(F, G, H)$ relating Gph and ZSet, made up of adjunctions

$$F : \mathrm{Gph} \rightleftharpoons \mathrm{ZSet} : G \quad G : ZSet \rightleftharpoons \mathrm{Gph} : H$$

Here $H(X) = [\mathbf{Z}, X]$, bi-infinite walks with shift $\sigma$, and $F(X) = \pi_0(\mathbf{Z} \times X)$, while $G$ is a Cayley graph construction.

Consider the functor $j : \mathrm{ZSet} \to \mathrm{cZSet}$ which forgets non-periodic elements. It is half of a further adjunction:

$$i : \mathrm{cZSet} \rightleftharpoons \mathrm{ZSet} : j$$

## Quillen equivalences and homotopy

Adjoint functors between model categories induce derived functors between the homotopy categories. Quillen gave conditions for these to form an equivalence of homotopy categories.

A simple instance of this shows that

$$H : \mathrm{Gph} \to \mathrm{ZSet} \quad \text{and} \quad jH : \mathrm{Gph} \to \mathrm{cZSet}$$

are such *Quillen equivalences.*

So, the homotopy category for
our "cycles" model structure on Gph
has a simple description.

**Theorem:** Ho(Gph) is equivalent to cZSet.

This has a nice combinatorial interpretation.

**Theorem:** Two finite graphs are homotopy equivalent for the "cycles" model iff they are almost isospectral.

## Zeta series for Gph

**Definition:** A *finite* directed graph $X$ (with finitely many nodes and arcs) has *zeta series*

$$Z(u) = \exp(\sum_{m=1}^{\infty} c_m \frac{u^m}{m}),$$

where $c_m = |C_m(X)|$ for $m > 0$.

**Proposition:** If $X$ is a finite graph with $n$ nodes then the zeta series of $X$ satisfies

$$Z(u) = \det(I - uA)^{-1} = \frac{1}{u^n a(u^{-1})}$$

where $A$ is the adjacency operator of $X$ and $a(x)$ is the characteristic polynomial of $X$.

**Definition:** Finite graphs $X$ and $Y$ with the same characteristic polynomial are *isospectral*. They are *almost isospectral* if they have the same zeta series.

# A "walks" model for Gph

- $\mathcal{F} = \mathrm{All}$

- $\mathcal{W} = L^{\dagger}$ where $L = \{\mathbf{i}, \mathbf{j}\}$ with

$\mathbf{i} : \mathbf{0} \to \mathbf{N}$ the initial graph morphism, and
$\mathbf{j} : \mathbf{N} + \mathbf{N} \to \mathbf{N}$ the natural morphism.

So $\underline{\mathcal{C}} = {}^{\dagger}\mathcal{W}$.

**Theorem:** For the "walks" model, Ho(Gph) is equivalent to NSet.

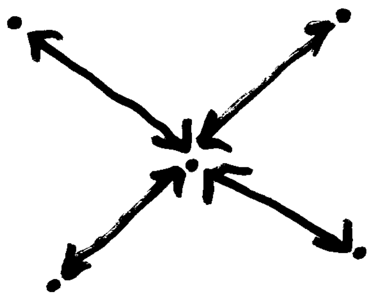
**Corollary:** Two finite graphs are homotopy equivalent for walks

$\not\Uparrow$ $\Downarrow$

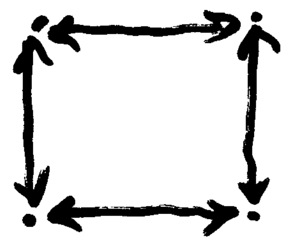
they are homotopy equivalent for cycles.

But: the walk space $N(X)$ has a natural topology, and many N-set maps are not continuous.

In our new paper on the arXiv we present some results on when walk spaces are topologically equivalent.



"cycles" equivalent



same cycles and zeta series,
not "walks" equivalent

# References


M. Hovey,
Model Categories

S. Mac Lane and I. Moerdijk,
Sheaves in Geometry and Logic: a first introduction to topos theory

F.W.Lawvere,
Qualitative distinctions between some toposes of generalized graphs

A. W. M. Dress and C. Siebeneicher,
The Burnside ring of the infinite cyclic group and its relations to the necklace algebra, $\lambda$-rings, and the universal ring of Witt vectors

E. E. Enochs and I. Herzog,
A homotopy of quiver morphisms with applications to representations

P. Boldi and S. Vigna,
Fibrations of graphs

J. R. Stallings,
Topology of finite graphs

A. Joyal and M. Tierney,
Quasi-categories vs Segal spaces