Distributive laws for Lawvere theories

Eugenia Cheng

University of Sheffield

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Plan

1. Introduction

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2. Lawvere theories

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3. Distributive laws for monads

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—So we can look for distributive laws between these monads.

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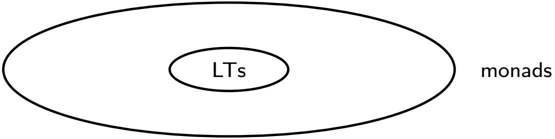

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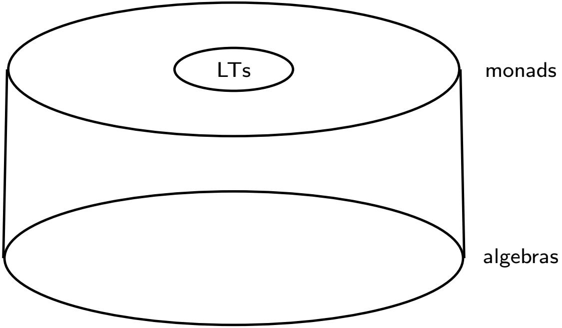

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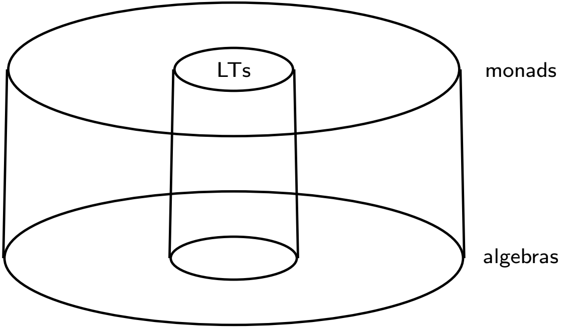

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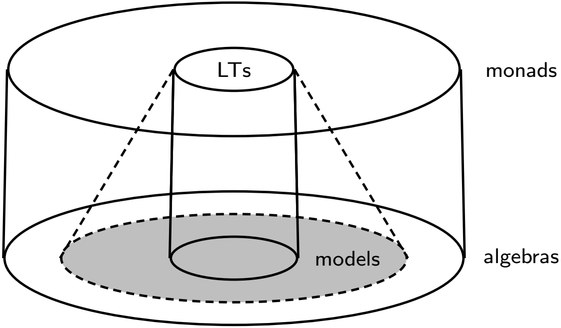

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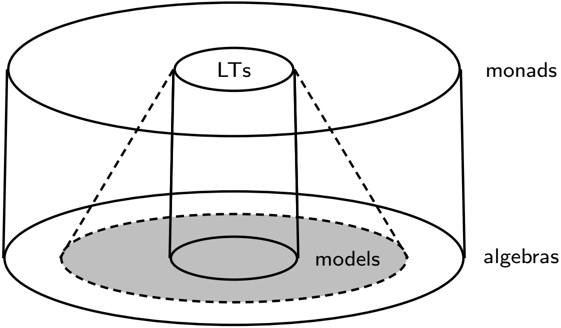

Example

Distributive law for monoids over abelian groups
$\longrightarrow$ rings internal to *any* finite-product category $\mathcal{V}$.

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Note: in $\mathbb{F}^{\mathrm{op}}$ the object m is the *product* of m copies of 1.

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We are allowed to forget and repeat variables.

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We have many arities for the “same” operation.

arity	operation
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$\vdots$			

$$\begin{tikzcd}
3 \arrow[r] & 1 \\
4 \arrow[u, dashed] \arrow[ur] & \\
5 \arrow[u, dashed] \arrow[uur] & \\
\vdots \arrow[u, dashed] &
\end{tikzcd}$$

These are all related by forgetting variables i.e. via projections in $\mathbb{F}^{\mathrm{op}}$.

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  —could use $\mathcal{P}\mathbb{A}$ to get “typed” theory
- could just say a Lawvere theory is *any* finite product category $\mathbb{C}$
- could do finite limits instead of just products.

2. Lawvere theories

Models $\equiv$ algebras

A model for $\mathbb{L}$ in a finite-product category $\mathcal{C}$ is a finite-product preserving functor

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\mathbb{L} \arrow[r] & \mathcal{C}
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$$1 \longmapsto A \in \mathcal{C} \quad \text{underlying data}$$

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$$\begin{tikzcd} k \arrow[r, mapsto] & A^k \end{tikzcd}$$

operation of arity k

$$\begin{tikzcd} k \arrow[d] & & A^k \arrow[d] \\ 1 \arrow[rr, phantom, "\longmapsto" description, pos=0.5, yshift=4ex] & & A \end{tikzcd}$$

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Idea

Lawvere theories are related to monads via the Kleisli category.

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Definition

Monad T on $\mathbf{Set} \longrightarrow$ Lawvere theory $\mathbb{L}_T$

$$\mathbb{L}_T \;=\; \text{full subcategory of } (\mathbf{Kl}T)^{\mathrm{op}} \text{ whose objects are \emph{finite} sets.}$$

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Lawvere theory $\mathbb{L}$ ⟶ monad $T_{\mathbb{L}}$ on **Set**

$$T_{\mathbb{L}}X = \int^{n \in \mathbb{F}^{\mathrm{op}}} \mathbb{L}(n,1) \times X^n.$$

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Theorem

This gives a correspondence between Lawvere theories and *finitary* monads on **Set**.

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- **Iterated distributive laws (Cheng)**
  Combine n monads with distributive laws and Yang-Baxter condition.

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Examples

monoid + abelian group → ring

horizontal composition + vertical composition → 2-category

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Or combining more structures:

$$\left.\begin{array}{c}\text{0-composition}\\ +\\ \text{1-composition}\\ +\\ \vdots\\ +\\ (n-1)\text{-composition}\end{array}\right\}\longrightarrow n\text{-category}$$

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3. Kleisli bicategory of $\mathcal{P}$ on profunctors

   —Hyland, Distributive laws (CLP 2010)

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—factorisations are only unique up to morphisms in $\mathbb{F}^{\mathrm{op}}$.

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It is the pullback

$$\begin{tikzcd}
 & & \cdot \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee", very near start] & & \\
 & A \arrow[dl] \arrow[dr] & & B \arrow[dl] \arrow[dr] & \\
\mathbf{ob}\,\mathbb{F} & & \mathbf{ob}\,\mathbb{F} & & \mathbf{ob}\,\mathbb{F}
\end{tikzcd}$$

composable pairs

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The distributive law tells us how to re-express a pair

$$k \xrightarrow{\in\mathbb{B}} l \xrightarrow{\in\mathbb{A}} m \qquad \text{as} \qquad k \xrightarrow{\in\mathbb{A}} l' \xrightarrow{\in\mathbb{B}} m$$

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ignored the fact that $\mathbb{F}^{\mathrm{op}}$ is in both $\mathbb{A}$ and $\mathbb{B}$.

So we want a coequaliser

$$\begin{tikzcd}
\mathbb{B} \otimes \mathbb{F}^{\mathrm{op}} \otimes \mathbb{A} \arrow[r, shift left, "\textsf{absorb } \mathbb{F}^{\mathrm{op}} \textsf{ into } \mathbb{A}"] \arrow[r, shift right, "\textsf{absorb } \mathbb{F}^{\mathrm{op}} \textsf{ into } \mathbb{B}"'] & \mathbb{B} \otimes \mathbb{A} \arrow[r] & \mathbb{B} \otimes_{\mathbb{F}^{\mathrm{op}}} \mathbb{A}
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—looks like bimodules.

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$$\begin{tikzcd}
 & & \mathbf{Set} \arrow[rr, "T"] & & \mathbf{Set} & & \\
 & & & {} \arrow[d, mapsto] & & & \\
 \mathbb{F} \arrow[rr, "I_*", "\shortmid" marking] & & \mathbf{Set} \arrow[rr, "T_*", "\shortmid" marking] & {} & \mathbf{Set} \arrow[rr, "I^*", "\shortmid" marking] & & \mathbb{F}
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- Moreover the functors are full and faithful, so given Lawvere theories on the right, *any* distributive law between them corresponds to one on the left.

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So we have three equivalent notions of distributive laws for Lawvere theories, which correspond to distributive laws between the associated monads.