N-tuple Categories

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Category Theory 2011

Introduction

While the higher category lexicon is printed in more and more dictionaries, its paradigm seldomly transcends Dolan and Baez's globular setting. These slides are meant to convince you that there are much to gain to shift this paradigm to the cubical setting. This is a small account of the beauty I encountered.

Internalization

Definition

An **internal category** *in a category* $\underline{\mathbf{C}}$ *with pullbacks consists of :*

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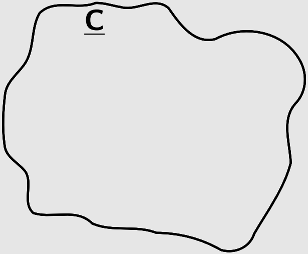

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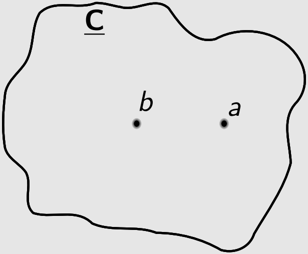

Two objects

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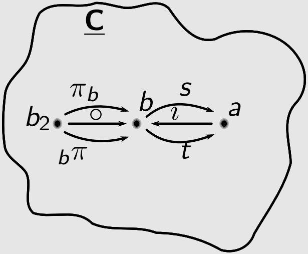

Two objects
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s — *source* $\imath$ — *identity*
t — *target* $\circ$ —*composition*

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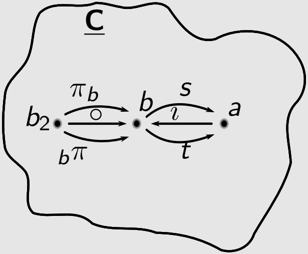

Two objects
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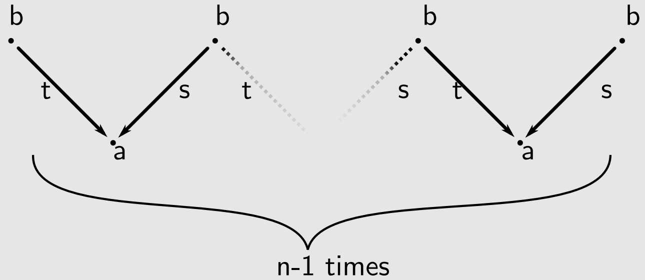
Relations

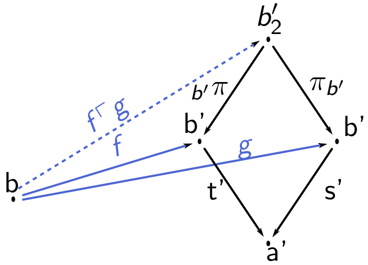
s — *source* $\imath$ — *identity*

t — *target* $\circ$ —*composition*

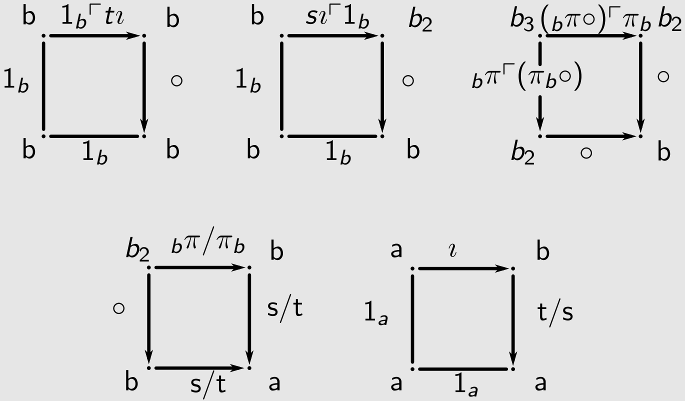
Where b_n is a chosen limit of :



and gives unique maps denoted by the symbol "⌜" :



together with :



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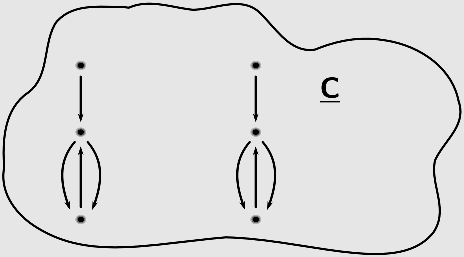

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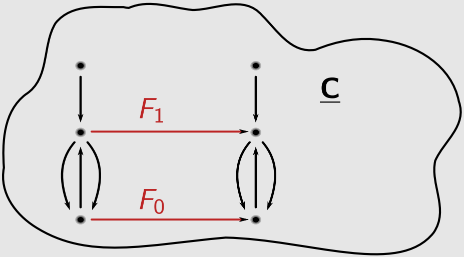

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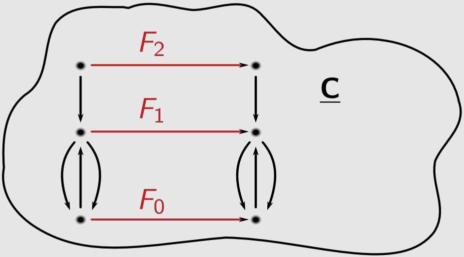

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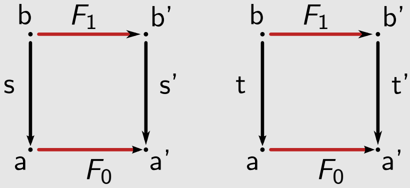

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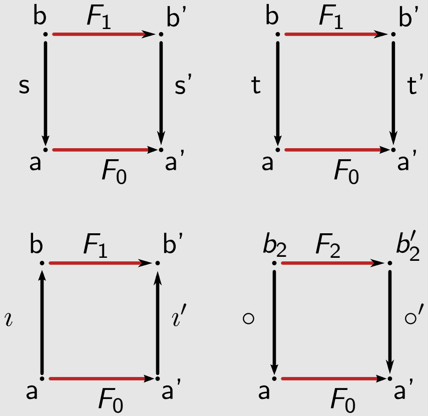

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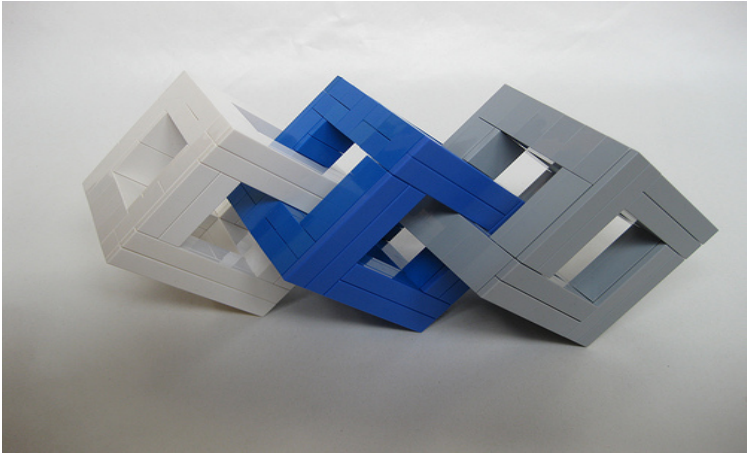
What happens if one repeats this process iteratively ?
We will see that it adds one dimension to the objects. It yields things called Cubical categories or **N-tuple categories**.

But we would just be playing Lego would it just be for that.



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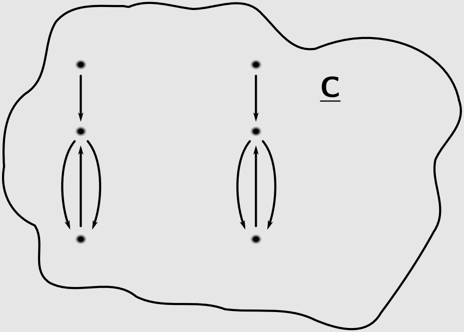

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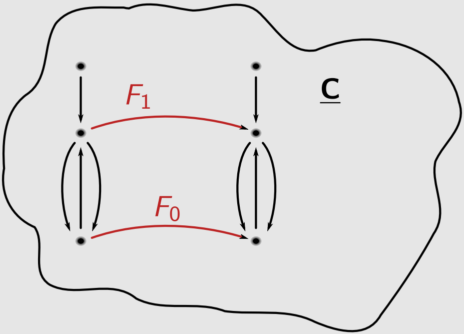

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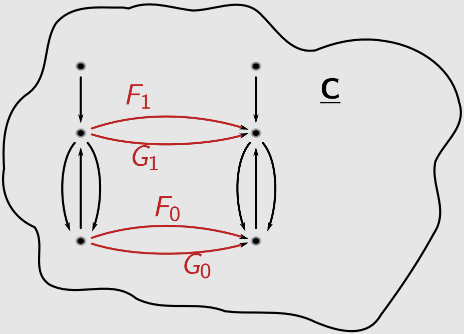

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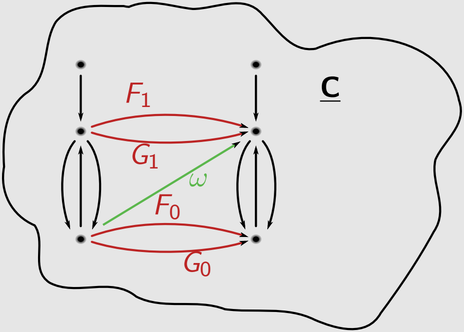

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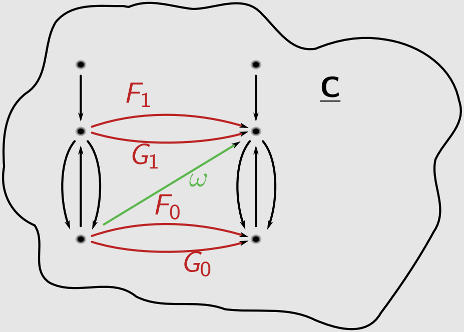

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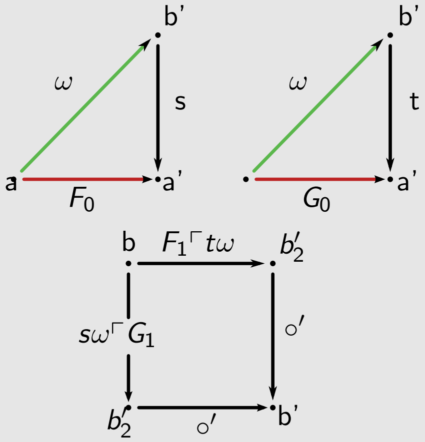
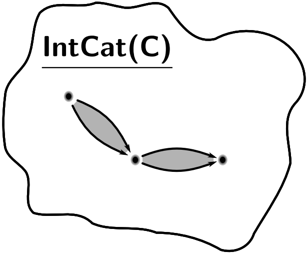
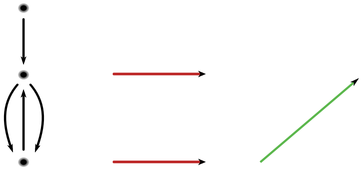


For a fixed category $\underline{\mathbf{C}}$, the collection of internal categories, internal functors and internal natural transformations forms a 2-category, i.e. something of the form :



In a 2-category, the list of previous list of internal objects becomes :



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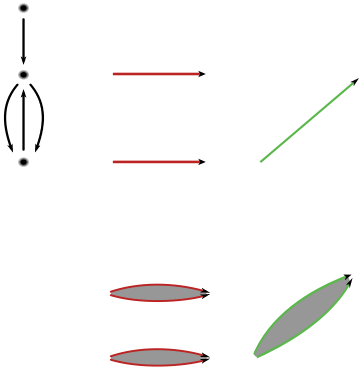
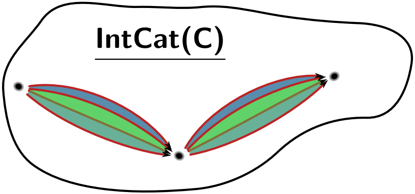
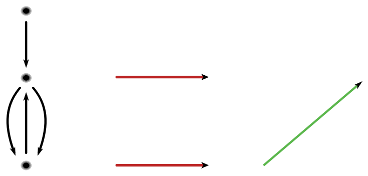


The collection of such internal entities forms a “rugby” category :



If $\underline{\mathbf{C}}$ is a "rugby" category, the list of internal objects becomes :



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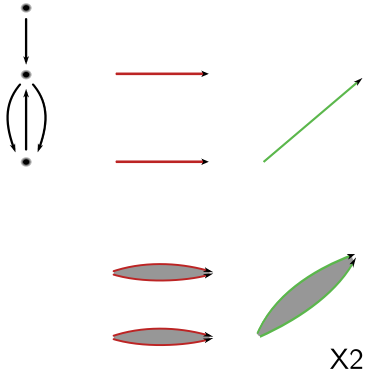

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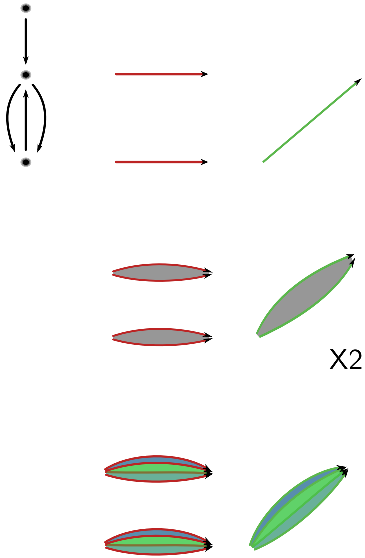

This process goes on, adding layers every time. Homsets look like hypercubes, giving 2-categories for n=1 (segment), "rugby" categories for n=2 (square) etc...

N^{pl} Categories

Definition

A (small) **double category** *is an internal category in* **Cat**.

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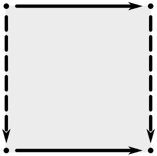
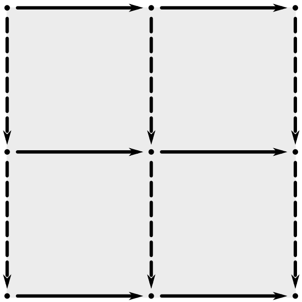
A general element in a double category is a square :



And it composes associatively in two directions, with two different units.

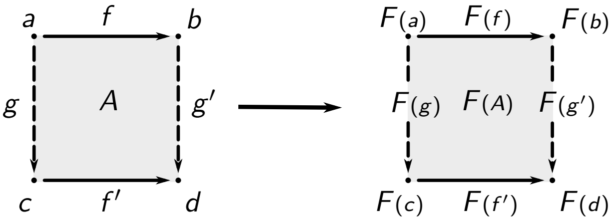
Moreover the two compositions interchange , i.e. the following diagram has a unique composition :



Definition

A **double functor** *is an internal functor in* **<u>Cat</u>**

It maps squares to squares respecting boundaries, units and composition.



Definition

A **horizontal double natural transformation** *is an internal natural transformation in* **Cat**

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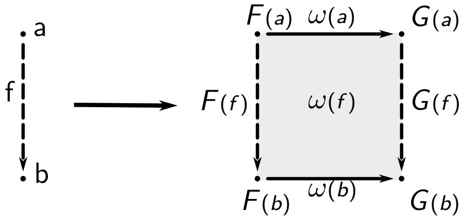
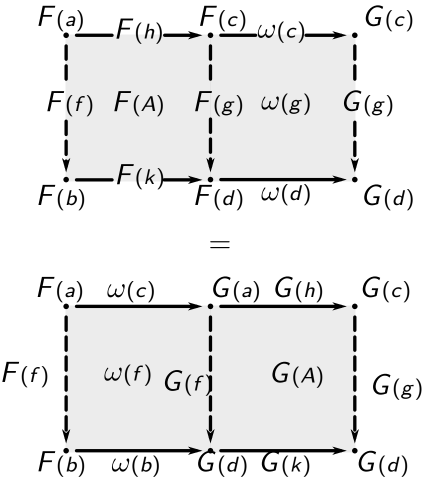
It associates squares to vertical morphisms :



In such a way that it intertwines functors horizontally :



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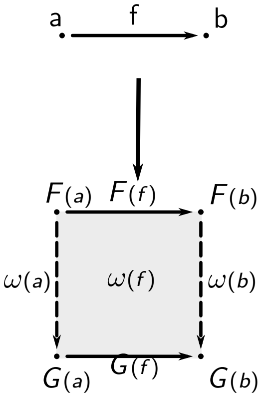

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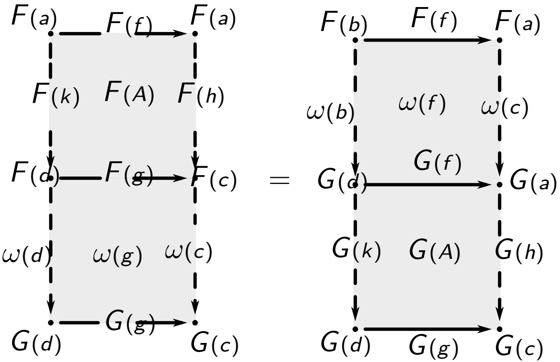

Definition

A **double comparison** *is an internal comparison in* **Cat**.

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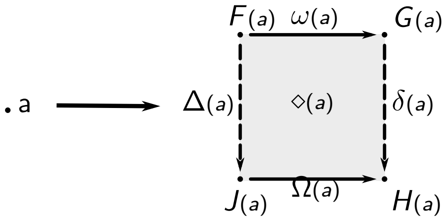
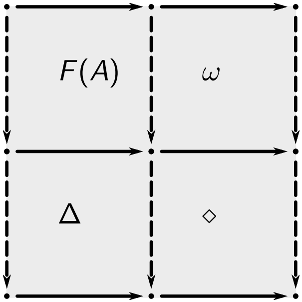
It associates squares to objects :



In such a way that all the following are equal :



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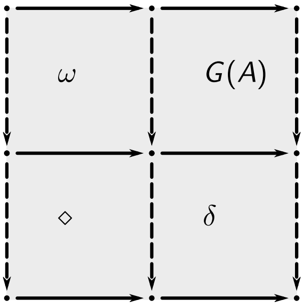

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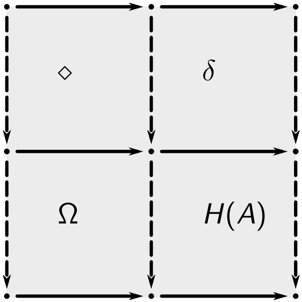

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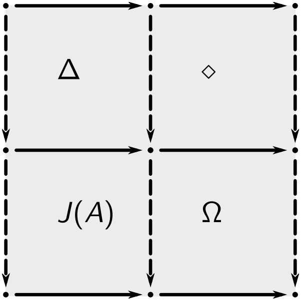

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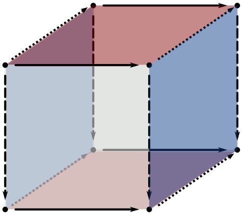
A general element in a double category is a cube :



And it composes associatively in three directions, with three different units.

A triple functor associates cubes to cubes.
A triple natural transformation associates cubes to squares.
A triple comparisons associates cubes to arrows.
The new entity associates cubes to objects.

Theorem

The categories of (strict) n-tuple categories are closed.

Which means, as we saw, that there is an internal Hom functor.

The investigation of double categories is at its infancy but interesting examples emerge already. R.Brown used it to define non-commutative homotopy and though fairly unknown, some constructions in quantum groups known as Drinfeld' doubles are related to them as well. In the fully invertible situation, "double groups" have as famous special cases, crossed and bicrossed products of groups. Triple categories is as close as I got to virgin land.

Degeneracies

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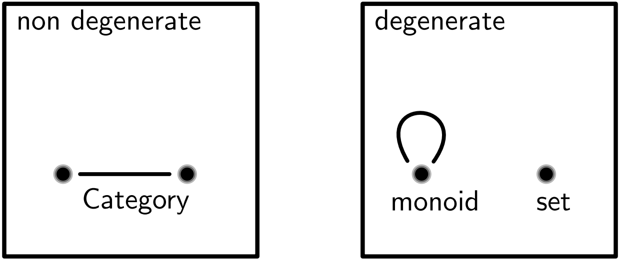

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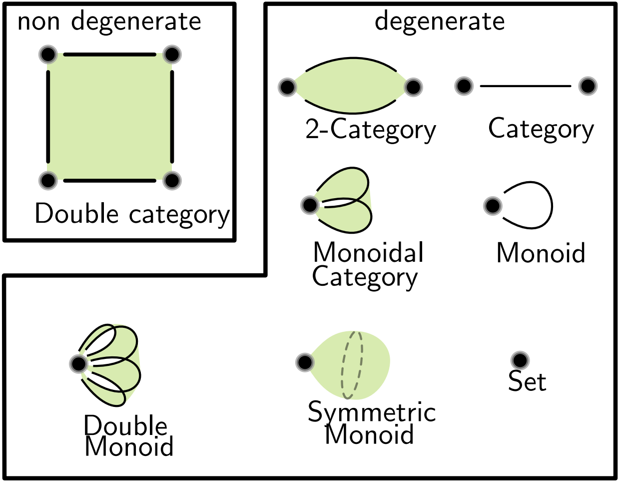

Weakening

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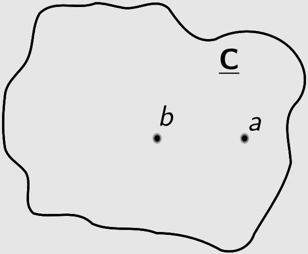

Objects

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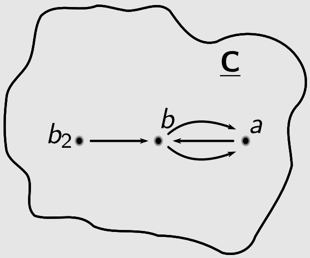

Objects
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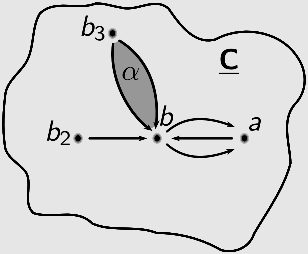

α - Assiciativity

Objects
Morphisms
2-Cells

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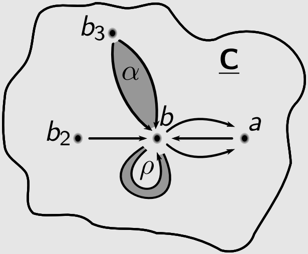

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Objects
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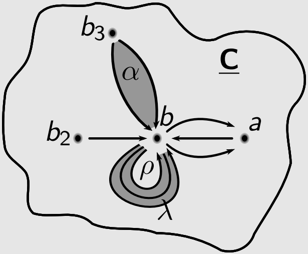

α - *Assiciativity*
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λ - *Left unit*

Objects
Morphisms
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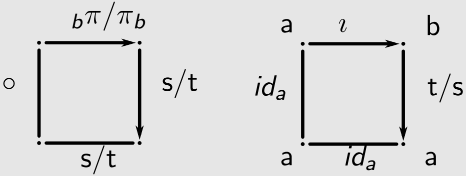
Commutative diagrams for arrows :



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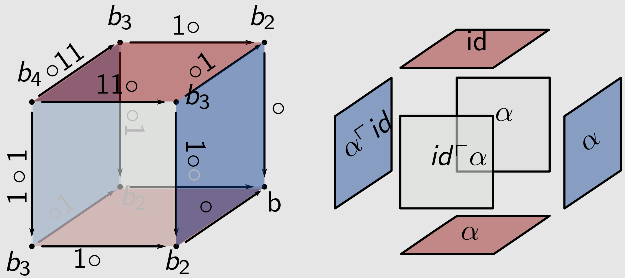

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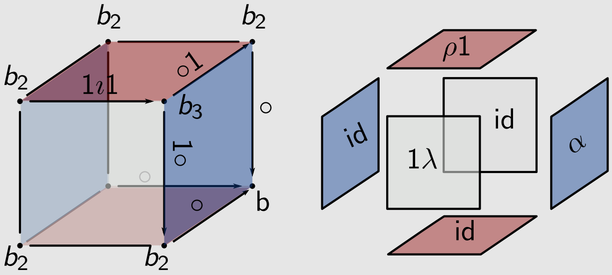

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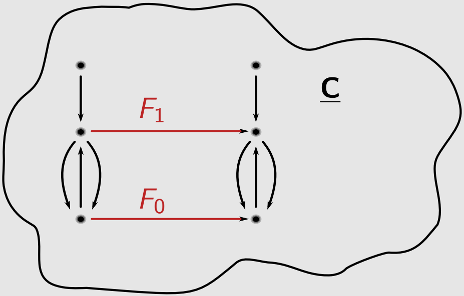

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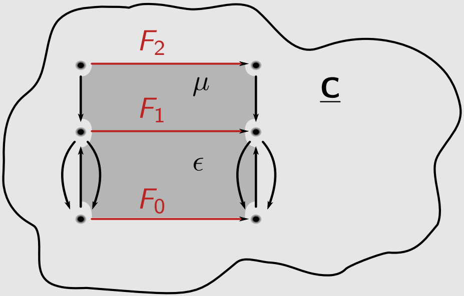
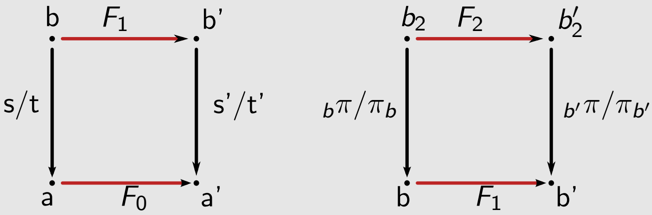


Morphisms
2-Cells

A set of commutative diagrams for arrows:



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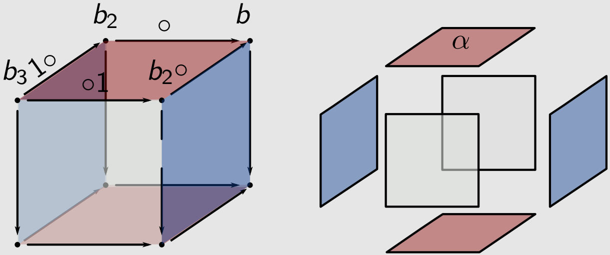

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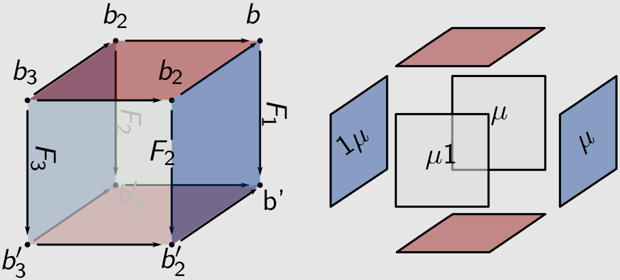

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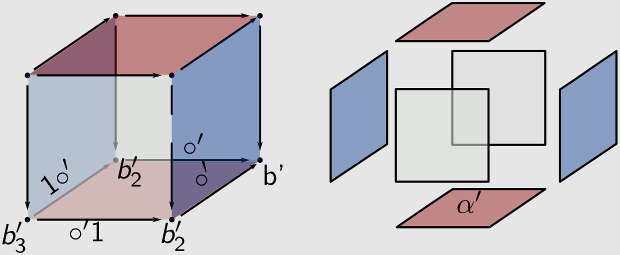

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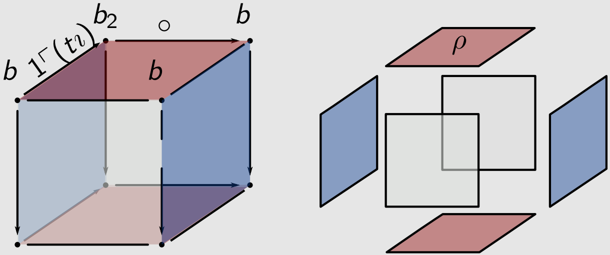

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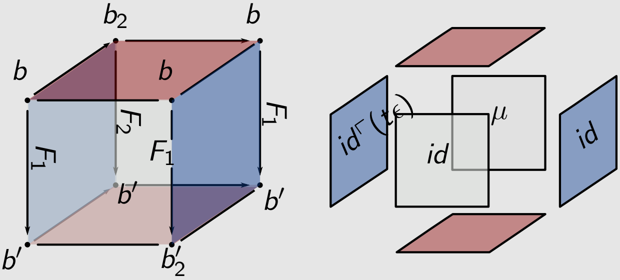

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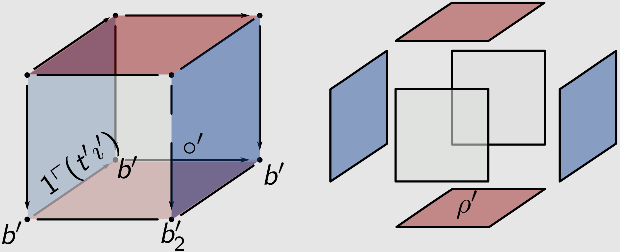
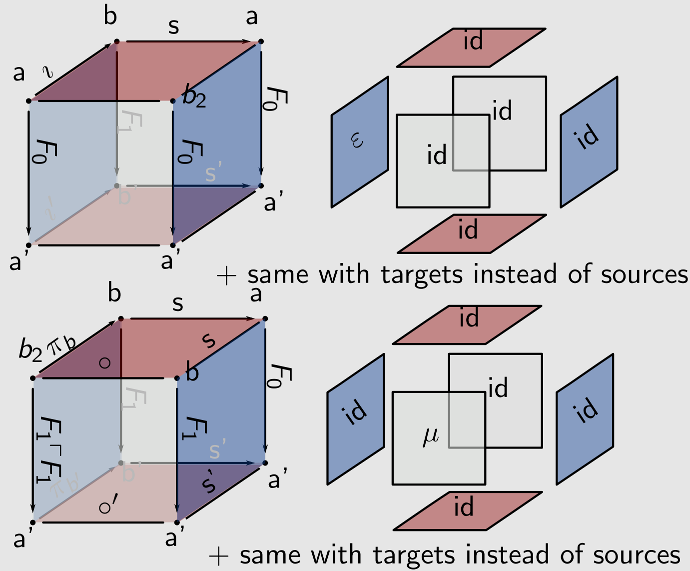
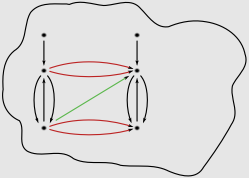


And a corresponding cube for λ.


b
s
a
a
b_2
F_0
F_1
F_0
F_0
s'
b'
a'
a'
a'
id
id
ε
id
id
id
+ same with targets instead of sources
b
s
a
b_2 π b
s
o
b
F_0
F_1
$F_1 \ulcorner F_1$
F_1
s'
b'
π b'
s'
o'
a'
a'
a'
id
id
id
μ
id
id
+ same with targets instead of sources

Definition

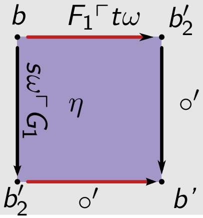
A **weak internal natural transformation** *between two weak internal functors consists of :*



A morphism

Definition

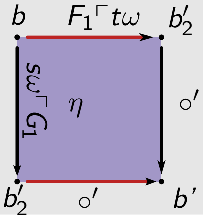
A **weak internal natural transformation** *between two weak internal functors consists of :*



A morphism
A 2-Cell

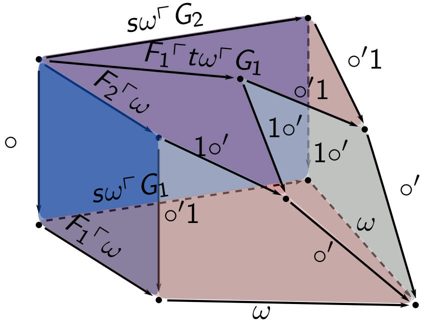
Definition

A **weak internal natural transformation** *between two weak internal functors consists of :*

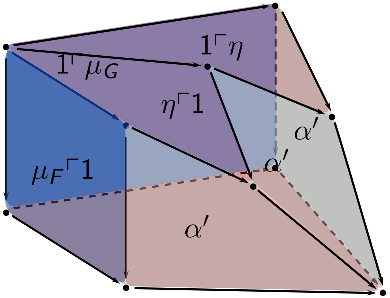


A morphism
A 2-Cell
Relations

The relations are :

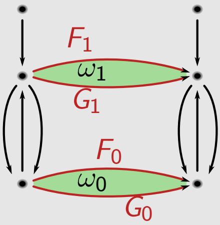


The relations are :



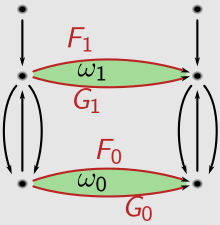
Definition

A **weak internal natural cell** *between two weak internal functors consists of :*



Definition

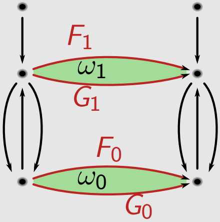
A **weak internal natural cell** *between two weak internal functors consists of :*



2-Cells

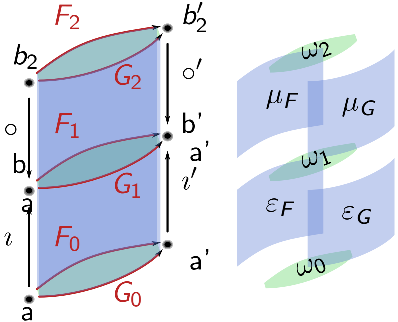
Definition

A **weak internal natural cell** *between two weak internal functors consists of :*



2-Cells
Relations

Contrary to the natural transformation case, the relations are very easy to visualize :



Conjecture (Majard '10)

The categories of weak n-tuple categories are weakly enriched over themselves.

Conclusion

A systematic description of weak n-tuple categories is needed. Although the case of weak double categories is fully understood, very little work has been done on the other versions of fully weak n-tuple categories.

References

N.Martins.Ferriera, *Pseudo-Categories*, arXiv:math/0604549v1 (2006)

R.Pare and M.Grandis, *Limits in Double Categories*, Cahiers de Topologie et Geometrie Differentielle Categorique, XL(3), 1999, 162-220

R.Brown, G.Mosa, *Double Categories, 2-Categories, Thin Structures and Connections*, Theory and Applications of Categories, Vol. 5, No. 7, 1999, pp. 163175.

Thank you