Graphically Factorizing the Tannaka Construction

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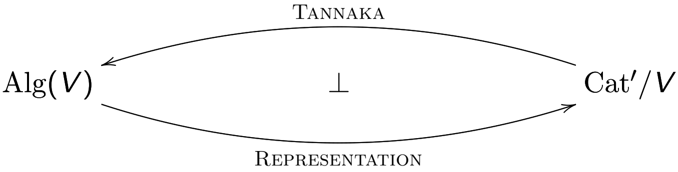
Introduction: Representation

There are a raft of functors which go by the name of "representation":

$$\begin{array}{c}\text{Algebraic}\\ \text{Gadgets}\\ \text{in } V\end{array} \xrightarrow{\quad\textsc{Representation}\quad} \begin{array}{c}\text{Categories}\\ \text{of Modules}\\ \text{over } V\end{array}$$

Introduction: Reconstruction

For well-behaved V, these functors have left adjoints—called "reconstruction" or simply "The Tannaka Construction".



Interesting Cases

Interesting cases of the Tannaka construction:

- Separable Frobenius monoidal functors into $V = R - \mathrm{mod}$ (Szlachanyi, 2002).
- Separable Frobenius monoidal functors from modular categories into $V = \mathrm{Vec}_k$ (Pfeiffer, 2009).
- Separable Frobenius monoidal functors into "general" V (M., 2011).

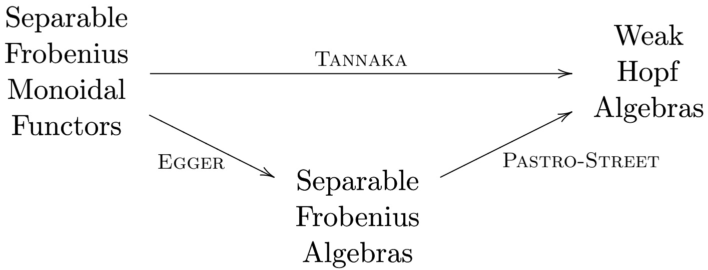
In all three cases, the Tannaka construction produces a *weak bialgebra* or a *weak Hopf algebra*.

Two Constructions

Egger (2008) gives a construction whereby Frobenius monoidal functors can be thought of as Frobenius monoids in a functor category. (see also Cockett and Seely 1999)

Pastro and Street (2009) give a construction of a weak Hopf algebra from a separable Frobenius algebra

In favourable cases, after overcoming some minor technical obstacles, we obtain:



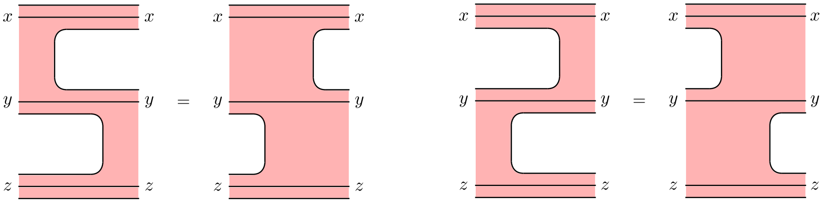
Many frames about graphical language for monoidal functors

Graphical Language for Monoidal Functors

Let F be a functor between monoidal categories; we can depict a monoidal structure on F:

Or a comonoidal structure on F:

Frobenius Monoidal Functors



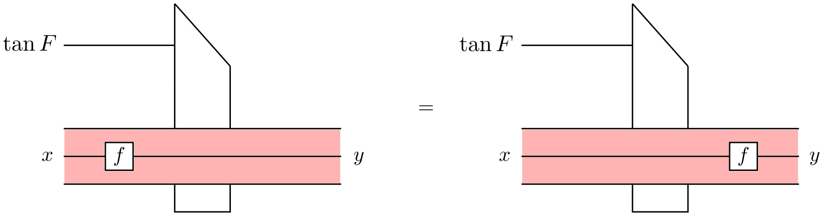
Many frames about the Tannaka construction

Tannaka Objects

Let F be a functor with rigid image. Then define

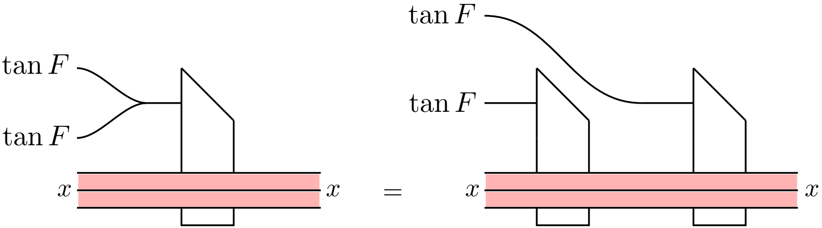
$$\tan F = \int_a Fa \oslash {}^*(Fa)$$

This is the (covariant) *Tannaka object* associated to F.
The Tannaka object for F acts universally on F, and the dinaturality of the end becomes the naturality of this action:

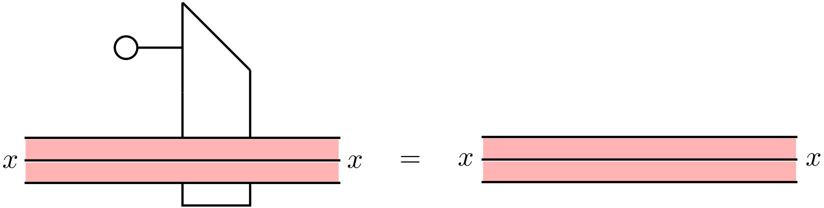


Algebra Structure

Define a multiplication on tan F by:

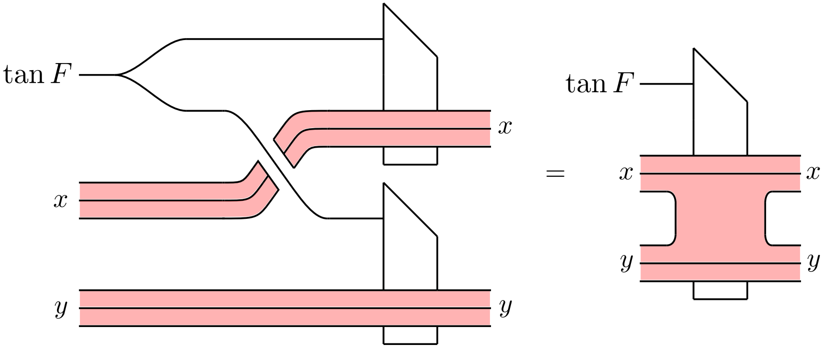


And a unit by:



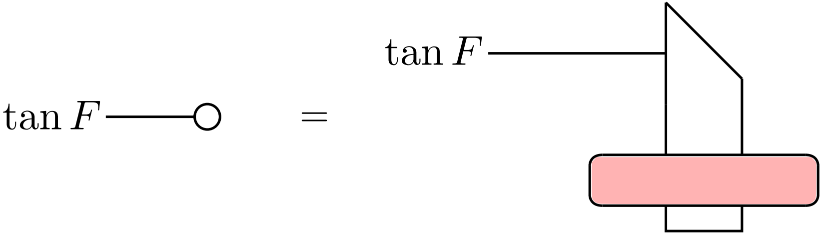
Coalgebra Structure

Suppose that tensoring with $\tan F$ preserves ends. Then we can define morphisms into $(\tan F)^{\otimes n}$ by giving an action of $\tan F$ on $F^{\otimes n}$. In particular, if F is monoidal and comonoidal, we can define a comultiplication:

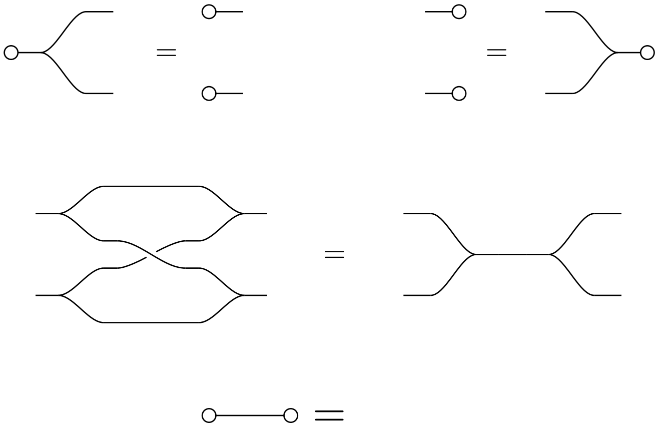


Coalgebra Structure

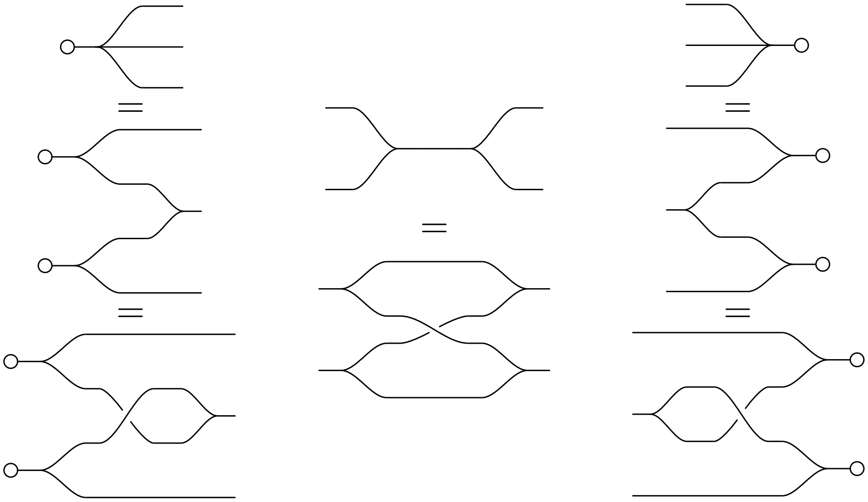
And a counit:



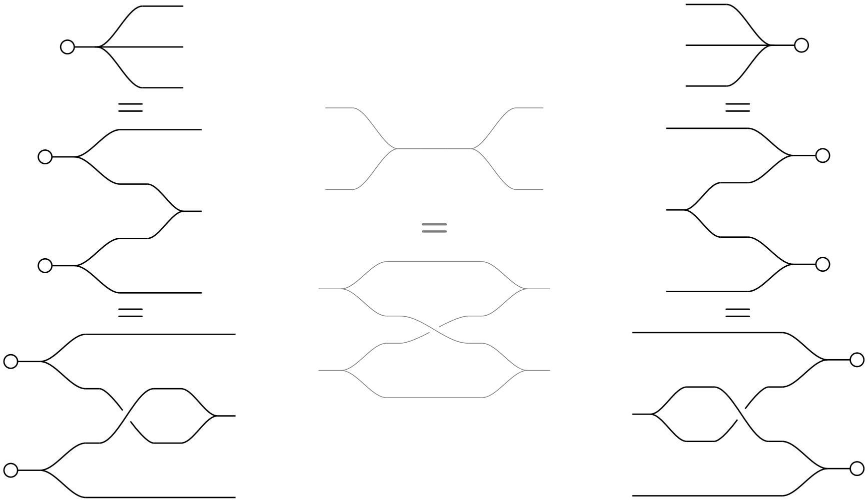
Bialgebras



Weak Bialgebras



“Relaxed” Weak Bialgebras



Many frames about Egger construction

Monoidal Functors as Monoids

Let J and K be monoidal categories, and let K have colimits of size J.
Then define a monoidal product on K^J by:

$$(f \owedge g)c = \operatorname*{colim}_{a \otimes b \longrightarrow c} fa \otimes gb$$

Then monoids in $(K^J, \owedge)$ correspond to monoidal functors from J to K.

Comonoidal Functors as Comonoids

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Frobenius Monoidal Functors as Frobenius Monoids

With these two tensor products, K^J is a *linearly distributive* category, that is, there are coherent morphisms

$$\delta\colon f \owedge (g \ovee h) \longrightarrow (f \owedge g) \ovee h$$

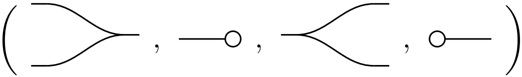
$$\delta\colon (f \ovee g) \owedge h \longrightarrow f \ovee (g \owedge h)$$

Frobenius monoidal functors from J to K correspond to Frobenius monoids in K^J, considered as a linearly distributive category. (Egger, see also Day, and also Cockett and Seely).

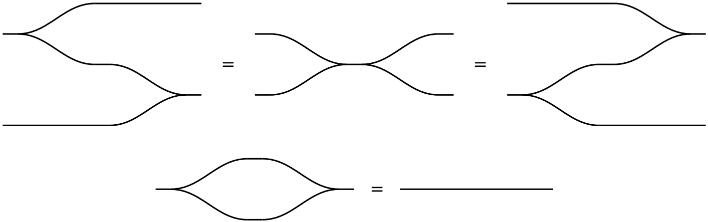
Many frames about Pastro Street construction.

Frobenius to Weak Hopf

Fix an ambient braided category, and let:

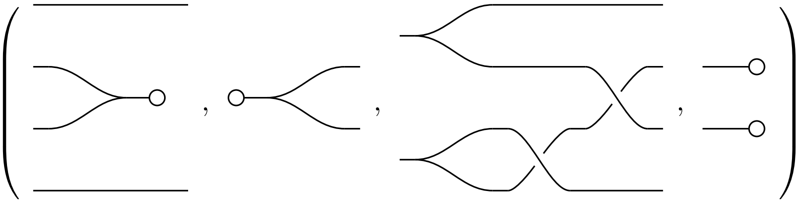


be a separable Frobenius algebra structure on an object A; that is, satisfying:

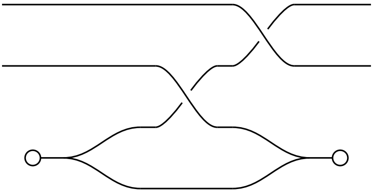


Frobenius to Weak Hopf

Then the following construction (Pastro and Street 2009, see also Bohm and Szlachanyi 1999) gives a weak bialgebra structure on $A \otimes A$:



And an antipode making $A \otimes A$ into a weak Hopf algebra can be defined by:

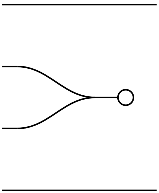


If A is not separable, then this gives a "relaxed" weak bialgebra.

Suppose that the domain of F is also rigid; then we can calculate:

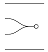
$$\begin{aligned} (F \varovee F)I &= \lim_{I \longrightarrow a \otimes b} Fa \otimes Fb \\ &\simeq \lim_{I \xrightarrow{\eta} a \otimes {}^* a} Fa \otimes F({}^* a) \\ &\simeq \int_a Fa \otimes F({}^* a) \\ &\simeq \int_a Fa \otimes {}^*(Fa) \\ &= \tan F \end{aligned}$$

Then, it so happens that the definitions of the above construction coincide, for instance, consider:



$$(F \ovee F) \owedge (F \owedge F) \xrightarrow{\delta^2} F \ovee (F \owedge F) \ovee F \xrightarrow{\mu;\epsilon} F \ovee \bot \ovee F \xrightarrow{\sim} F \ovee F$$

Now evaluate this at I and precompose with an inclusion:



$$(F \ovee F)I \otimes (F \ovee F)I \longrightarrow [(F \ovee F) \owedge (F \ovee F)]I \xrightarrow{\left(\quad\right)_I} (F \ovee F)I$$

This is isomorphic to a map of the form

$$\tan F \otimes \tan F \longrightarrow \tan F$$