

# Graphically Factorizing the Tannaka Construction

Micah Blake McCurdy  
Macquarie University

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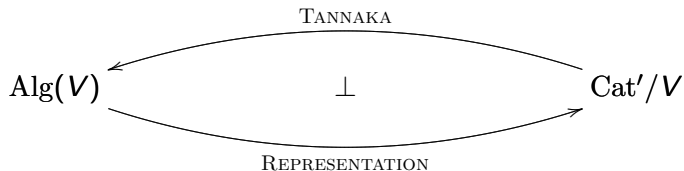
# Introduction: Representation

There are a raft of functors which go by the name of “representation”:



# Introduction: Reconstruction

For well-behaved  $V$ , these functors have left adjoints—called “reconstruction” or simply “The Tannaka Construction”.



# Interesting Cases

Interesting cases of the Tannaka construction:

- ▶ Separable Frobenius monoidal functors into  $V = R - \text{mod}$  (Szlachanyi, 2002).
- ▶ Separable Frobenius monoidal functors from modular categories into  $V = \text{Vec}_k$  (Pfeiffer, 2009).
- ▶ Separable Frobenius monoidal functors into “general”  $V$  (M., 2011).

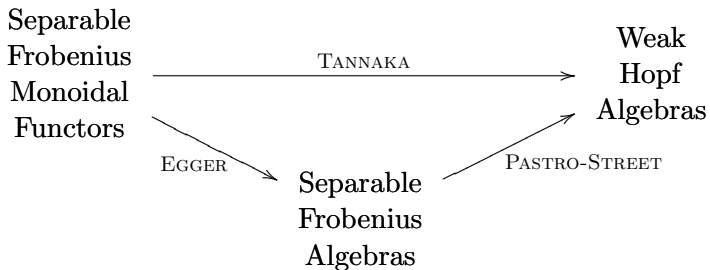
In all three cases, the Tannaka construction produces a *weak bialgebra* or a *weak Hopf algebra*.

## Two Constructions

Egger (2008) gives a construction whereby Frobenius monoidal functors can be thought of as Frobenius monoids in a functor category. (see also Cockett and Seely 1999)

Pastro and Street (2009) give a construction of a weak Hopf algebra from a separable Frobenius algebra

In favourable cases, after overcoming some minor technical obstacles, we obtain:



Many frames about graphical language for monoidal functors

# Graphical Language for Monoidal Functors

Let  $F$  be a functor between monoidal categories; we can depict a monoidal structure on  $F$ :

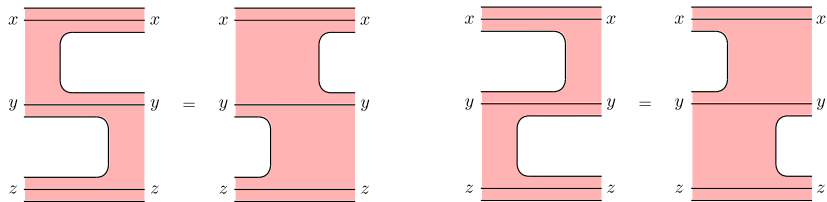


Or a comonoidal structure on  $F$ :





# Frobenius Monoidal Functors



Many frames about the Tannaka construction

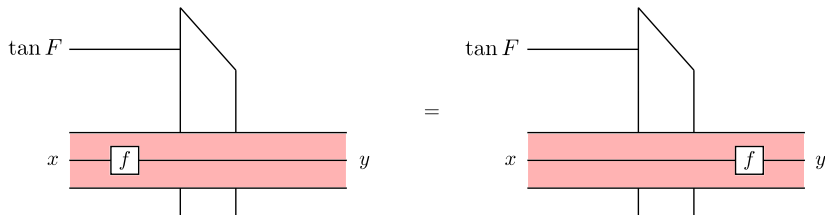
# Tannaka Objects

Let  $F$  be a functor with rigid image. Then define

$$\tan F = \int_a Fa \triangleleft^*(Fa)$$

This is the (covariant) *Tannaka object* associated to  $F$ .

The Tannaka object for  $F$  acts universally on  $F$ , and the dinaturality of the end becomes the naturality of this action:



# Algebra Structure

Define a multiplication on  $\tan F$  by:

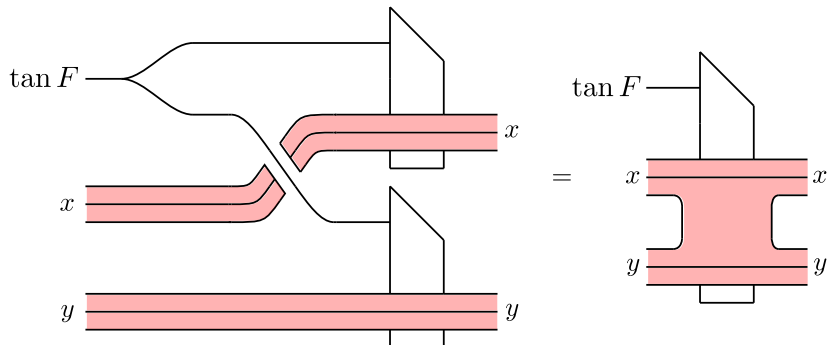
The diagram shows an equality between two configurations of a red horizontal bar with a double line, labeled  $x$  at both ends. On the left, a trapezoidal shape is attached to the top of the bar. Two lines labeled  $\tan F$  branch from the top-left corner of the trapezoid to the left. On the right, the same red bar is shown, but with two trapezoidal shapes attached to its top surface. A line labeled  $\tan F$  connects the top-left corner of the first trapezoid to the top-left corner of the second trapezoid. A curved line labeled  $\tan F$  connects the top-right corner of the first trapezoid to the top-left corner of the second trapezoid.

And a unit by:

The diagram shows an equality between two configurations of a red horizontal bar with a double line, labeled  $x$  at both ends. On the left, a trapezoidal shape is attached to the top of the bar. A line connects the top-left corner of the trapezoid to a small circle. On the right, the red bar is shown without any trapezoidal shapes attached to it.

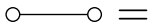
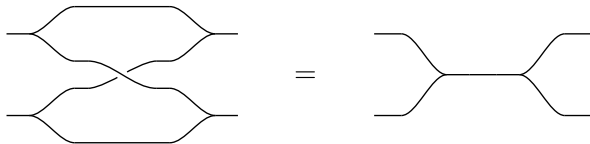
# Coalgebra Structure

Suppose that tensoring with  $\tan F$  preserves ends. Then we can define morphisms into  $(\tan F)^{\otimes n}$  by giving an action of  $\tan F$  on  $F^{\otimes n}$ . In particular, if  $F$  is monoidal and comonoidal, we can define a comultiplication:

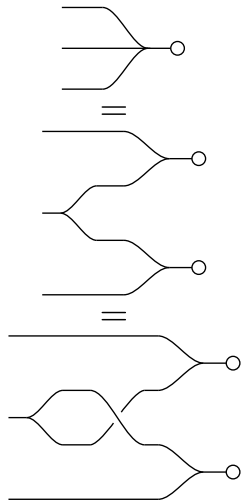
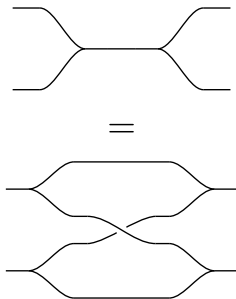
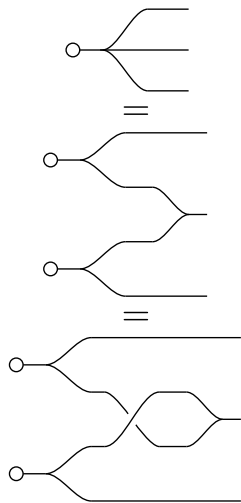




# Bialgebras

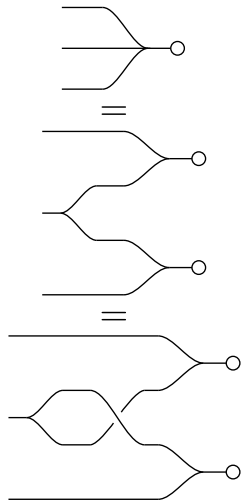
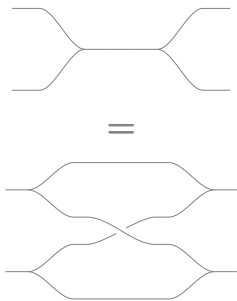
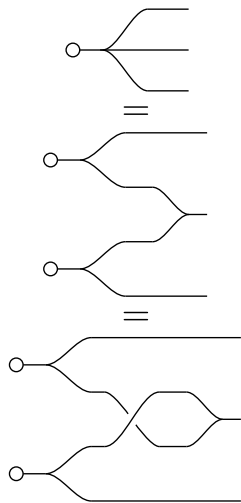


# Weak Bialgebras





# "Relaxed" Weak Bialgebras



Many frames about Egger construction

# Monoidal Functors as Monoids

Let  $J$  and  $K$  be monoidal categories, and let  $K$  have colimits of size  $J$ .

Then define a monoidal product on  $K^J$  by:

$$(f \otimes g)c = \operatorname{colim}_{a \otimes b \rightarrow c} fa \otimes gb$$

Then monoids in  $(K^J, \otimes)$  correspond to monoidal functors from  $J$  to  $K$ .

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Then comonoids in  $(K^J, \otimes)$  correspond to comonoidal functors from  $J$  to  $K$ .

# Frobenius Monoidal Functors as Frobenius Monoids

With these two tensor products,  $K^J$  is a *linearly distributive* category, that is, there are coherent morphisms

$$\delta: f \otimes (g \otimes h) \longrightarrow (f \otimes g) \otimes h$$

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Frobenius monoidal functors from  $J$  to  $K$  correspond to Frobenius monoids in  $K^J$ , considered as a linearly distributive category. (Egger, see also Day, and also Cockett and Seely).

Many frames about Pastro Street construction.

# Frobenius to Weak Hopf

Fix an ambient braided category, and let:

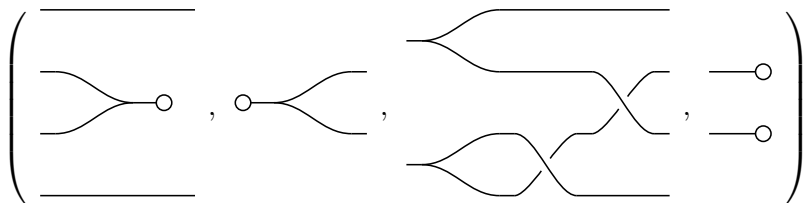
$$\left( \begin{array}{c} \text{---} \\ \text{---} \end{array} \curvearrowright, \text{---} \circ, \left. \begin{array}{c} \text{---} \\ \text{---} \end{array} \rightleftarrows, \circ \text{---} \right)$$

be a separable Frobenius algebra structure on an object  $A$ ; that is, satisfying:

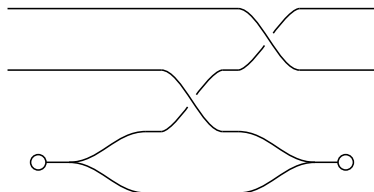
The diagram shows two equations. The first equation is a chain of three diagrams connected by equals signs. The first diagram has two horizontal lines on the left that merge into one on the right, and two horizontal lines on the right that merge into one on the left. The second diagram has two horizontal lines on the left that cross each other, and two horizontal lines on the right that cross each other. The third diagram has two horizontal lines on the left that merge into one on the right, and two horizontal lines on the right that merge into one on the left. The second equation shows a diagram with two horizontal lines on the left that merge into one on the right, and two horizontal lines on the right that merge into one on the left, followed by an equals sign and a single horizontal line.

## Frobenius to Weak Hopf

Then the following construction (Pastro and Street 2009, see also Bohm and Szlachanyi 1999) gives a weak bialgebra structure on  $A \otimes A$ :



And an antipode making  $A \otimes A$  into a weak Hopf algebra can be defined by:



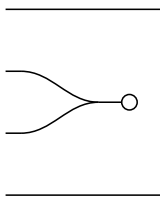


If  $A$  is not separable, then this gives a “relaxed” weak bialgebra.

Suppose that the domain of  $F$  is also rigid; then we can calculate:

$$\begin{aligned}(F \vee F)I &= \lim_{I \rightarrow a \otimes b} Fa \otimes Fb \\ &\simeq \lim_{I \xrightarrow{\eta} a \otimes *a} Fa \otimes F(*a) \\ &\simeq \int_a Fa \otimes F(*a) \\ &\simeq \int_a Fa \otimes *(Fa) \\ &= \tan F\end{aligned}$$

Then, it so happens that the definitions of the above construction coincide, for instance, consider:



$$(F \otimes F) \otimes (F \otimes F) \xrightarrow{\delta^2} F \otimes (F \otimes F) \otimes F \xrightarrow{\mu; \epsilon} F \otimes \perp \otimes F \xrightarrow{\sim} F \otimes F$$

Now evaluate this at  $I$  and precompose with an inclusion:

$$(F \vee F)I \otimes (F \vee F)I \longrightarrow [(F \vee F) \otimes (F \vee F)]I \xrightarrow{\left( \begin{array}{c} \overline{\quad} \\ \text{>} \\ \overline{\quad} \end{array} \right)_I} (F \vee F)I$$

This is isomorphic to a map of the form

$$\tan F \otimes \tan F \longrightarrow \tan F$$