

# **Bimonadic adjunctions and the Explicit Basis property**

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# Canonical restrictions

Let  $(M, \eta, \mu)$  a monad on a category  $\mathcal{D}$ .

Let  $s : (A, a) \rightarrow (MA, \mu)$  a section of  $a : (MA, \mu) \rightarrow (A, a)$ .

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 \end{array}$$

**Def. 3.** The *canonical restriction of  $s$*  is the p.b. on the left or, equivalently, the equalizer on the right.

$$\begin{array}{ccc}
 A_s & \xrightarrow{\bar{s}} & A \\
 \bar{s} \downarrow & & \downarrow \eta \\
 A & \xrightarrow{s} & MA
 \end{array}
 \qquad
 \begin{array}{ccc}
 A_s & \xrightarrow{\bar{s}} & A \begin{array}{l} \xrightarrow{\eta_A} \\ \xrightarrow{s} \end{array} & MA
 \end{array}$$

# The Explicit Basis property

$$\begin{array}{ccc} & \xrightarrow{\eta} & \\ A & \xleftarrow{a} & MA \\ & \xrightarrow{s} & \end{array}$$

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$$A \begin{array}{c} \xrightarrow{\eta} \\ \xleftarrow{a} \\ \xrightarrow{s} \end{array} MA \qquad A_s \xrightarrow{\bar{s}} A \begin{array}{c} \xrightarrow{\eta_A} \\ \xrightarrow{s} \end{array} MA$$

**Def. 5.**  $(M, \eta, \mu)$  satisfies the *Explicit Basis* (EB) property if for every such section  $s : (A, a) \rightarrow (MA, \mu)$  the map

$$MA_s \xrightarrow{M\bar{s}} MA \xrightarrow{a} A$$

is an iso. (I.e. the canonical restriction is a ‘basis’.)

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**Example 3.** Idempotent monads.

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**Def. 7.**  $(M, \eta, \mu)$  satisfies the *Explicit Basis* (EB) property if for every such section  $s : (A, a) \rightarrow (MA, \mu)$  the map

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**Example 4.** Idempotent monads.

**Prop. 4.** *If a monad is EB then idempotents split in the Kleisli category.*



# Example: compact convex sets

compact convex set = compact convex subset of a locally convex Hausdorff real vector space.

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Every canonical presentation  $a : MA \rightarrow A$  has at most one section  $s : A \rightarrow MA$  and...

... if it exists, its canonical restriction coincides with  $\partial_e A \rightarrow A$ .

# Example: Myhill's combinatorial functions

$\mathbf{Bij} \rightarrow \mathbf{Inj}$  determines an essential surjection

$$q : [\mathbf{Bij}, \mathbf{Set}] \rightarrow [\mathbf{Inj}, \mathbf{Set}]$$

and so, a monadic  $q \dashv q^* : [\mathbf{Inj}, \mathbf{Set}] \rightarrow [\mathbf{Bij}, \mathbf{Set}]$ .

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**Prop. 12.** *The induced monad on  $[\mathbf{Bij}, \mathbf{Set}]$  is EB and the Kleisli category is the Schanuel topos.*

The EB prop. survives if we replace  $\mathbf{Bij} \rightarrow \mathbf{Inj}$  with  $\mathcal{M} \rightarrow \mathcal{C}$  where  $(\mathcal{E}, \mathcal{M})$  is a factorization system on  $\mathcal{C}$  with all  $\mathcal{E}$ -maps epi.



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(How 'good' the Kleisli category is depends on other things...)

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**Prop. 16.** *The monadic  $i^* : \widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{C}}_0$  is EB iff  $\mathcal{C}$  is reduced.*

**Cor. 5.** *For  $\mathcal{C}$  a monoid  $\mathcal{C}\text{-Set} \rightarrow \text{Set}$  is EB iff  $\mathcal{C}$  is reduced.*

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**Cor. 8** (Janelidze).  *$G\text{-Set} \rightarrow \text{Set}$  is EB for any group  $G$ .*

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**Prop. 20.** *The monadic  $i^* : \widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{C}}_0$  is EB iff  $\mathcal{C}$  is reduced.*

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**Cor. 14** (Janelidze).  *$G\text{-Set} \rightarrow \text{Set}$  is EB for any group  $G$ .*

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Note: in most examples of EB monads the ‘free algebra’ functor is comonadic.



**“Is it possible that for some  
comonads the unit law implies the  
associative law ?”**

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**Def. 9.** A *pre-coalgebra* is a pair  $(X, s)$  where  $s : X \rightarrow CX$  is a map in  $\mathcal{X}$  such that the diagram on the left below

$$\begin{array}{ccc} X & \xrightarrow{s} & CX \\ & \searrow id & \downarrow \varepsilon \\ & & X \end{array}$$

$$\begin{array}{ccc} X & \xrightarrow{s} & CX \\ f \downarrow & & \downarrow Cf \\ X' & \xrightarrow{s'} & CX' \end{array}$$

commutes. A *morphism*  $f : (X, s) \rightarrow (X', s')$  of pre-coalgebras is a map  $f : X \rightarrow X'$  in  $\mathcal{X}$  s.t. the diagram on the right above commutes.

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Let  $\underline{\mathcal{X}}_{\mathbf{C}}$  be the category of pre-coalgebras and morphisms between them.

# The Redundant Coassociativity property

Clearly, the category  $\mathcal{X}_C$  of  $C$ -coalgebras is a full subcategory  $\mathcal{X}_C \rightarrow \underline{\mathcal{X}_C}$ .

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**Example 21** (Wood). Let  $\mathcal{C}$  be a category with products and consider the comonadic  $\mathcal{C}/I \rightarrow \mathcal{C}$ . The resulting comonad  $I \times (-) : \mathcal{C} \rightarrow \mathcal{C}$  satisfies Redundant Coassociativity:

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$$\begin{array}{ccc} A & \xrightarrow{s} & I \times A \\ & \searrow id & \downarrow \varepsilon = \pi_1 \\ & & A \end{array}$$

so it is determined by the coalgebra  $\pi_0 s : A \rightarrow I$ .



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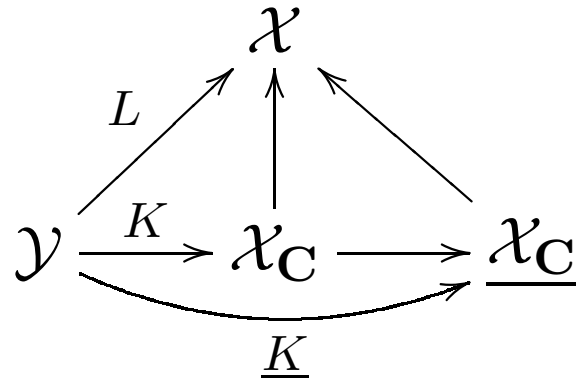
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... extend it to a comparison  $\underline{K} : \mathcal{Y} \rightarrow \underline{\mathcal{X}_C}$

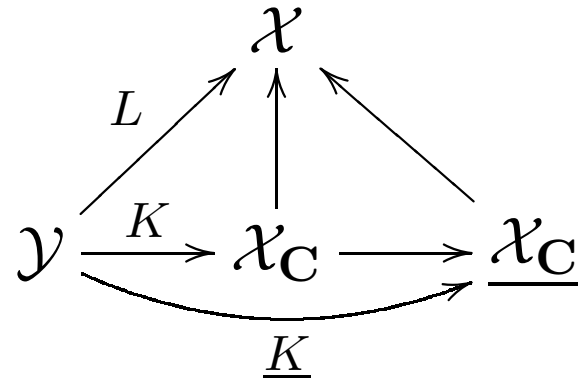
$$\begin{array}{ccccc} & & \mathcal{X} & & \\ & \nearrow L & \uparrow & \nwarrow & \\ \mathcal{Y} & \xrightarrow{K} & \mathcal{X}_C & \xrightarrow{\quad} & \underline{\mathcal{X}_C} \\ & \searrow \underline{K} & & & \end{array}$$

# EB and RC



**Prop. 21.** *If  $R : \mathcal{X} \rightarrow \mathcal{Y}$  is monadic the following are equivalent:*

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**Prop. 22.** *If  $R : \mathcal{X} \rightarrow \mathcal{Y}$  is monadic the following are equivalent:*

1.  $\underline{K} : \mathcal{Y} \rightarrow \underline{\mathcal{X}_C}$  *is an equivalence.*

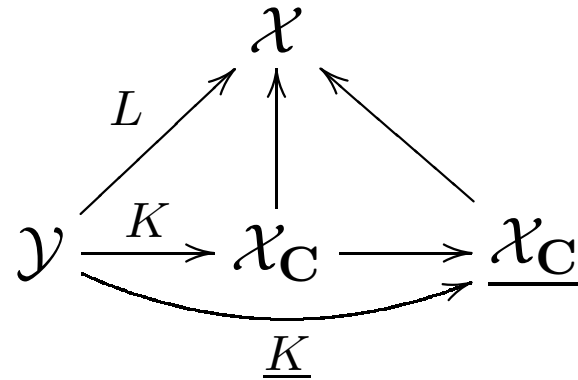
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$$\begin{array}{ccccc} & & \mathcal{X} & & \\ & \nearrow L & \uparrow & \nwarrow & \\ \mathcal{Y} & \xrightarrow{K} & \mathcal{X}_{\mathbf{C}} & \xrightarrow{\quad} & \underline{\mathcal{X}_{\mathbf{C}}} \\ & \searrow \underline{K} & & & \end{array}$$

**Prop. 23.** *If  $R : \mathcal{X} \rightarrow \mathcal{Y}$  is monadic the following are equivalent:*

1.  *$\underline{K} : \mathcal{Y} \rightarrow \underline{\mathcal{X}_{\mathbf{C}}}$  is an equivalence.*
2.  *$L : \mathcal{Y} \rightarrow \mathcal{X}$  is comonadic and Redundant Coassociativity holds.*

# EB and RC



**Prop. 24.** *If  $R : \mathcal{X} \rightarrow \mathcal{Y}$  is monadic the following are equivalent:*

1.  $\underline{K} : \mathcal{Y} \rightarrow \underline{\mathcal{X}_{\mathbf{C}}}$  is an equivalence.
2.  $L : \mathcal{Y} \rightarrow \mathcal{X}$  is comonadic and Redundant Coassociativity holds.
3. The induced monad on  $\mathcal{Y}$  reflects isos and is EB.



# Sketch of the proof

Every pre-coalgebra  $(X, s : X \rightarrow LRX)$  induces a

$$RX \begin{array}{c} \xrightarrow{\eta_R} \\ \xleftarrow{R\varepsilon} \\ \xrightarrow{Rs} \end{array} RLRX$$

in  $\mathcal{Y}$ , with  $R\varepsilon$  as a common retraction of  $\eta_R$  and  $Rs$ .

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in  $\mathcal{Y}$ , with  $R\varepsilon$  as a common retraction of  $\eta_R$  and  $Rs$ .

Define the *canonical restriction* of  $(X, s)$  as the p.b./equalizer

$$\begin{array}{ccc} X_s & \xrightarrow{\bar{s}} & RX \\ \bar{s} \downarrow & & \downarrow \eta_R \\ RX & \xrightarrow{Rs} & RLRX \end{array} \qquad \begin{array}{ccc} X_s & \xrightarrow{\bar{s}} & RX \\ & & \xrightarrow{\eta} \\ & & \xrightarrow{Rs} RLRX \end{array}$$

# Sketch of proof (cont.)

The assignment  $(X, s) \mapsto X_s$  extends to a right adjoint  $\underline{N} : \underline{\mathcal{X}}_{\mathbf{C}} \rightarrow \mathcal{Y}$  to the extended comparison  $\mathcal{Y} \rightarrow \underline{\mathcal{X}}_{\mathbf{C}}$ .

$$\begin{array}{ccccc} & & & \mathcal{X} & \\ & & L \curvearrowright & & \\ & & & \underline{K} & \\ \mathcal{Y} & \xrightarrow{K} & \mathcal{X}_{\mathbf{C}} & \longrightarrow & \underline{\mathcal{X}}_{\mathbf{C}} \\ & \xleftarrow{N} & & \xleftarrow{N} & \\ & & & & \end{array}$$

# Sketch of proof (cont.)

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$$\begin{array}{ccc} & & \mathcal{X} \\ & \nearrow L & \\ \mathcal{Y} & & \underline{\mathcal{X}}_{\mathbf{C}} \\ & \xrightarrow{\underline{K}} & \underline{\mathcal{X}}_{\mathbf{C}} \\ & \xleftarrow{\underline{N}} & \end{array}$$

A slight extension of Beck's Thm. shows t.f.a.e:

1.  $\underline{K} : \mathcal{Y} \rightarrow \underline{\mathcal{X}}_{\mathbf{C}}$  is an equivalence.
2.  $L : \mathcal{Y} \rightarrow \mathcal{X}$  is comonadic and Redundant Coassociativity holds.

# Sketch of the proof (cont.)

Finally, if  $R : \mathcal{X} \rightarrow \mathcal{Y}$  is monadic,  $\mathcal{X} = \mathcal{Y}^{\mathbf{M}}$  and a pre-coalgebra  $((A, a), s)$  is just

$$\begin{array}{ccc} (A, a) & \xrightarrow{s} & (MA, \mu) \\ & \searrow id & \downarrow a \\ & & (A, a) \end{array} \quad = \quad \begin{array}{ccc} LR(A, a) & & \\ & \downarrow \text{counit of } L \dashv R & \\ (A, a) & & \end{array}$$

a section  $s : (A, a) \rightarrow (MA, \mu)$  of the presentation of  $(A, a)$ ;  
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a section  $s : (A, a) \rightarrow (MA, \mu)$  of the presentation of  $(A, a)$ ; and...

... the adjoint  $\underline{N} : \underline{\mathcal{X}}_{\mathbf{C}} \rightarrow \mathcal{Y}$  maps such a pre-coalgebra to its canonical restriction.

# EB iff RC

**Prop. 25.** *If  $R : \mathcal{X} \rightarrow \mathcal{Y}$  is monadic t.f.a.e.:*

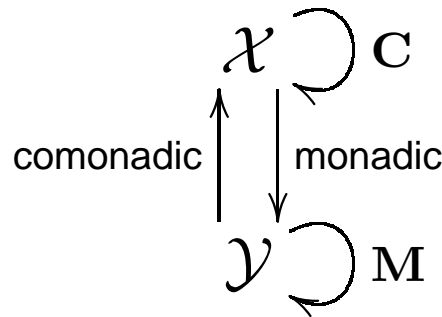
1.  $\underline{K} : \mathcal{Y} \rightarrow \underline{\mathcal{X}}_{\mathbf{C}}$  is an equivalence.
2.  $L : \mathcal{Y} \rightarrow \mathcal{X}$  is comonadic and Redundant Coassociativity holds.
3. The induced monad on  $\mathcal{Y}$  reflects isos and is EB.

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**Prop. 26.** *If  $R : \mathcal{X} \rightarrow \mathcal{Y}$  is monadic t.f.a.e.:*

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**Cor. 16.** *Let*



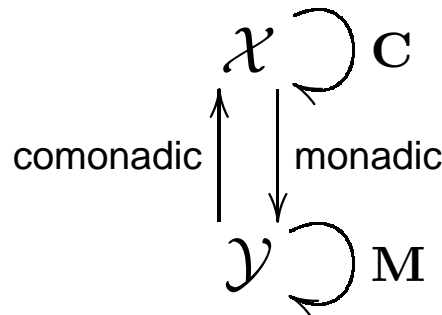


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**Prop. 27.** *If  $R : \mathcal{X} \rightarrow \mathcal{Y}$  is monadic t.f.a.e.:*

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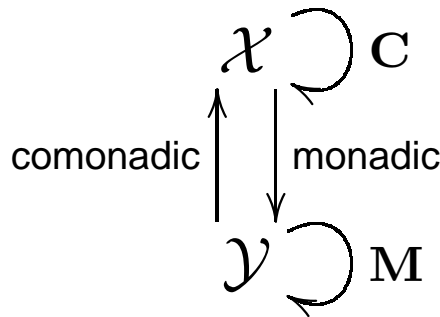
*Then:  $\mathbf{M}$  is EB  $\Leftrightarrow \mathbf{C}$  satisfies Redundant Coassociativity.*

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**Prop. 28.** *If  $R : \mathcal{X} \rightarrow \mathcal{Y}$  is monadic t.f.a.e.:*

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3. The induced monad on  $\mathcal{Y}$  reflects isos and is EB.

**Cor. 18.** *Let*



*Then:  $\mathbf{M}$  is EB  $\Leftrightarrow \mathbf{C}$  satisfies Redundant Coassociativity.*

A **CEB** monad is one that is iso-reflecting and EB.  
 Every CEB monad has a conservative underlying functor.

# Two new examples

# Example: Monadic Descent

$\mathcal{E}$  with pullbacks,  $p : I \rightarrow J$  in  $\mathcal{E}$ ,  $\mathbb{M}$  the monad on  $\mathcal{E}/I$  induced by  $\Sigma_p \dashv p^* : \mathcal{E}/J \rightarrow \mathcal{E}/I$ .

# Example: Monadic Descent

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**Cor. 20.** *If  $p$  is effective descent then  $\mathbb{M}$  is CEB.*

*Proof.* The comonad on  $\mathcal{E}/J$  determined by  $\Sigma_p \dashv p^* : \mathcal{E}/J \rightarrow \mathcal{E}/I$  satisfies Redundant Coassociativity (Wood). □

# Example: Monadic Descent

$\mathcal{E}$  with pullbacks,  $p : I \rightarrow J$  in  $\mathcal{E}$ ,  $\mathbb{M}$  the monad on  $\mathcal{E}/I$  induced by  $\Sigma_p \dashv p^* : \mathcal{E}/J \rightarrow \mathcal{E}/I$ .

**Cor. 21.** *If  $p$  is effective descent then  $\mathbb{M}$  is CEB.*

*Proof.* The comonad on  $\mathcal{E}/J$  determined by  $\Sigma_p \dashv p^* : \mathcal{E}/J \rightarrow \mathcal{E}/I$  satisfies Redundant Coassociativity (Wood). □

A little more effort shows:

**Prop. 31.**  *$\mathbb{M}$  is CEB (anyway).*

I.e. the monadic category of ‘Descent data’ determines a CEB monad.

# Example: modular categories

$\mathcal{D}$  with finite limits and coproducts.

**Def. 15.**  $\mathcal{D}$  satisfies the *modular law* if for every  $f : X \rightarrow Z$ , the map

$$\left( \begin{array}{c} \langle in_0, f \rangle \\ in_1 \times Z \end{array} \right) : X + (Y \times Z) \longrightarrow (X + Y) \times Z$$

is an iso for every  $Y$  in  $\mathcal{D}$ .

# Example: modular categories

$\mathcal{D}$  with finite limits and coproducts.

**Def. 17.**  $\mathcal{D}$  satisfies the *modular law* if for every  $f : X \rightarrow Z$ , the map

$$\begin{pmatrix} \langle in_0, f \rangle \\ in_1 \times Z \end{pmatrix} : X + (Y \times Z) \longrightarrow (X + Y) \times Z$$

is an iso for every  $Y$  in  $\mathcal{D}$ .

**Def. 18** (Carboni'89).  $\mathcal{D}$  is called *modular* if  $\mathcal{D}/U$  satisfies the modular law for every  $U$ , and

$$\begin{array}{ccc} X & \xrightarrow{in_1} & U + X \\ f \downarrow & & \downarrow U+f \\ U & \xrightarrow{in_1} & U + U \end{array}$$

is a pullback for every  $f : X \rightarrow U$  in  $\mathcal{D}$ .



# Example: modular categories

**Prop. 32.** *If  $\mathcal{D}$  is modular then the monad induced by  $D/\mathcal{D} \rightarrow \mathcal{D}$  is CEB for every  $D$  in  $\mathcal{D}$ .*

This is something that modular categories share with extensive categories.

# Example: modular categories

**Prop. 33.** *If  $\mathcal{D}$  is modular then the monad induced by  $D/\mathcal{D} \rightarrow \mathcal{D}$  is CEB for every  $D$  in  $\mathcal{D}$ .*

This is something that modular categories share with extensive categories.

**Lemma 3.** *If  $\mathcal{D}$  is modular,  $1/\mathcal{D}$  is additive with kernels.*

*Proof.* Carboni's proof uses that for the monadic  $1/\mathcal{D} \rightarrow \mathcal{D}$ , every algebra is free. □

# Example: modular categories

**Prop. 34.** *If  $\mathcal{D}$  is modular then the monad induced by  $D/\mathcal{D} \rightarrow \mathcal{D}$  is CEB for every  $D$  in  $\mathcal{D}$ .*

This is something that modular categories share with extensive categories.

**Lemma 5.** *If  $\mathcal{D}$  is modular,  $1/\mathcal{D}$  is additive with kernels.*

*Proof.* Carboni's proof uses that for the monadic  $1/\mathcal{D} \rightarrow \mathcal{D}$ , every algebra is free. □

Using additivity of  $1/\mathcal{D}$  we get

**Lemma 6.** *Let  $F : \mathcal{D} \rightarrow 1/\mathcal{D}$  be the left adjoint to  $1/\mathcal{D} \rightarrow \mathcal{D}$ . The induced comonad coincides with  $(F1) \times (-) : 1/\mathcal{D} \rightarrow 1/\mathcal{D}$ .*

# Modularity without descent

Fix  $\mathcal{D}$  with finite limits and finite coproducts.  
Let  $\mathbb{M}$  be the monad on  $\mathcal{D}$  induced by  $1/\mathcal{D} \rightarrow \mathcal{D}$ .

# Modularity without descent

Fix  $\mathcal{D}$  with finite limits and finite coproducts.

Let  $\mathbb{M}$  be the monad on  $\mathcal{D}$  induced by  $1/\mathcal{D} \rightarrow \mathcal{D}$ .

**Prop. 36.** *The following are equivalent:*

1.  $\mathcal{D}$  is modular.

# Modularity without descent

Fix  $\mathcal{D}$  with finite limits and finite coproducts.

Let  $\mathbb{M}$  be the monad on  $\mathcal{D}$  induced by  $1/\mathcal{D} \rightarrow \mathcal{D}$ .

**Prop. 37.** *The following are equivalent:*

1.  $\mathcal{D}$  is modular.
2.  $1/\mathcal{D}$  is additive and  $\mathcal{D} \rightarrow 1/\mathcal{D}$  is comonadic. (Carboni-Janelidze)

# Modularity without descent

Fix  $\mathcal{D}$  with finite limits and finite coproducts.

Let  $\mathbb{M}$  be the monad on  $\mathcal{D}$  induced by  $1/\mathcal{D} \rightarrow \mathcal{D}$ .

**Prop. 38.** *The following are equivalent:*

1.  $\mathcal{D}$  is modular.
2.  $1/\mathcal{D}$  is additive and  $\mathcal{D} \rightarrow 1/\mathcal{D}$  is comonadic. (Carboni-Janelidze)
3.  $1/\mathcal{D}$  is additive and  $\mathbb{M}$  is CEB.

# Modularity without descent

Fix  $\mathcal{D}$  with finite limits and finite coproducts.

Let  $\mathbb{M}$  be the monad on  $\mathcal{D}$  induced by  $1/\mathcal{D} \rightarrow \mathcal{D}$ .

**Prop. 39.** *The following are equivalent:*

1.  $\mathcal{D}$  is modular.
2.  $1/\mathcal{D}$  is additive and  $\mathcal{D} \rightarrow 1/\mathcal{D}$  is comonadic. (Carboni-Janelidze)
3.  $1/\mathcal{D}$  is additive and  $\mathbb{M}$  is CEB.

*In this case, the canonical  $\mathcal{D} \rightarrow (1/\mathcal{D})/(F1)$  is an equivalence.*



# Modularity without descent

Fix  $\mathcal{D}$  with finite limits and finite coproducts.

Let  $\mathbb{M}$  be the monad on  $\mathcal{D}$  induced by  $1/\mathcal{D} \rightarrow \mathcal{D}$ .

**Prop. 40.** *The following are equivalent:*

1.  $\mathcal{D}$  is modular.
2.  $1/\mathcal{D}$  is additive and  $\mathcal{D} \rightarrow 1/\mathcal{D}$  is comonadic. (Carboni-Janelidze)
3.  $1/\mathcal{D}$  is additive and  $\mathbb{M}$  is CEB.

*In this case, the canonical  $\mathcal{D} \rightarrow (1/\mathcal{D})/(F1)$  is an equivalence.*

**Cor. 27.** *Assume that  $1/\mathcal{D}$  is additive. Then:*

# Modularity without descent

Fix  $\mathcal{D}$  with finite limits and finite coproducts.

Let  $\mathbb{M}$  be the monad on  $\mathcal{D}$  induced by  $1/\mathcal{D} \rightarrow \mathcal{D}$ .

**Prop. 41.** *The following are equivalent:*

1.  $\mathcal{D}$  is modular.
2.  $1/\mathcal{D}$  is additive and  $\mathcal{D} \rightarrow 1/\mathcal{D}$  is comonadic. (Carboni-Janelidze)
3.  $1/\mathcal{D}$  is additive and  $\mathbb{M}$  is CEB.

*In this case, the canonical  $\mathcal{D} \rightarrow (1/\mathcal{D})/(F1)$  is an equivalence.*

**Cor. 28.** *Assume that  $1/\mathcal{D}$  is additive. Then:*

1.  $\mathcal{D}$  is additive if and only if  $\mathbb{M}$  is trivial.

# Modularity without descent

Fix  $\mathcal{D}$  with finite limits and finite coproducts.  
Let  $\mathbb{M}$  be the monad on  $\mathcal{D}$  induced by  $1/\mathcal{D} \rightarrow \mathcal{D}$ .

**Prop. 42.** *The following are equivalent:*

1.  $\mathcal{D}$  is modular.
2.  $1/\mathcal{D}$  is additive and  $\mathcal{D} \rightarrow 1/\mathcal{D}$  is comonadic. (Carboni-Janelidze)
3.  $1/\mathcal{D}$  is additive and  $\mathbb{M}$  is CEB.

*In this case, the canonical  $\mathcal{D} \rightarrow (1/\mathcal{D})/(F1)$  is an equivalence.*

**Cor. 29.** *Assume that  $1/\mathcal{D}$  is additive. Then:*

1.  $\mathcal{D}$  is additive if and only if  $\mathbb{M}$  is trivial.
2.  $\mathcal{D}$  is modular if and only if  $\mathbb{M}$  is CEB.

**That's it, thanks.**