Bimonadic adjunctions and the Explicit Basis property

M. Menni

Conicet - Universidad Nacional de La Plata - Argentina

Canonical restrictions

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Let $s : (A, a) \to (MA, \mu)$ a section of $a : (MA, \mu) \to (A, a)$.

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A \arrow[r, shift left=3, "\eta"] \arrow[r, shift right=3, "s"'] & MA \arrow[l, "a" description]
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Def. 3. The *canonical restriction of s* is the p.b. on the left or, equivalently, the equalizer on the right.

$$\begin{tikzcd}
A_s \arrow[r, "\overline{s}"] \arrow[d, "\overline{s}"'] & A \arrow[d, "\eta"] \\
A \arrow[r, "s"'] & MA
\end{tikzcd}
\qquad
\begin{tikzcd}
A_s \arrow[r, "\overline{s}"] & A \arrow[r, shift left=2, "\eta_A"] \arrow[r, shift right=2, "s"'] & MA
\end{tikzcd}$$

The Explicit Basis property

$$\begin{tikzcd}
A \arrow[r, shift left=5, "\eta"] \arrow[r, shift right=5, "s"'] & MA \arrow[l, "a" description]
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Def. 5. (M, η, μ) satisfies the *Explicit Basis* (EB) property if for every such section $s : (A, a) \to (MA, \mu)$ the map

$$MA_s \xrightarrow{M\overline{s}} MA \xrightarrow{a} A$$

is an iso. (I.e. the canonical restriction is a ‘basis’.)

The Explicit Basis property

$$\begin{tikzcd} A \arrow[r, shift left=4, "\eta"] \arrow[r, shift right=4, "s"'] & MA \arrow[l, "a" description] \end{tikzcd} \qquad\qquad \begin{tikzcd} A_s \arrow[r, "\overline{s}"] & A \arrow[r, shift left=2, "\eta_A"] \arrow[r, shift right=2, "s"'] & MA \end{tikzcd}$$

Def. 6. (M, η, μ) satisfies the *Explicit Basis* (EB) property if for every such section $s : (A, a) \to (MA, \mu)$ the map

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Example 3. Idempotent monads.

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Def. 7. (M, η, μ) satisfies the *Explicit Basis* (EB) property if for every such section $s : (A, a) \to (MA, \mu)$ the map

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Example 4. Idempotent monads.

Prop. 4. *If a monad is EB then idempotents split in the Kleisli category.*

Example: compact convex sets

compact convex set = compact convex subset of a locally convex Hausdorff real vector space.

$\mathbf{cConv}$ = category of compact convex sets and continuous affine functions between them.

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Prop. 9. $\mathbf{cConv} \to \mathbf{cHaus}$ *is EB*

Every canonical presentation $a : MA \to A$ has at most one section $s : A \to MA$ and...

... if it exists, its canonical restriction coincides with $\partial_e A \to A$.

Example: Myhill's combinatorial function

$\mathbf{Bij} \to \mathbf{Inj}$ determines an essential surjection

$$q : [\mathbf{Bij}, \mathbf{Set}] \to [\mathbf{Inj}, \mathbf{Set}]$$

and so, a monadic $q_! \dashv q^* : [\mathbf{Inj}, \mathbf{Set}] \to [\mathbf{Bij}, \mathbf{Set}]$.

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Prop. 12. *The induced monad on* $[\mathbf{Bij}, \mathbf{Set}]$ *is EB and the Kleisli category is the Schanuel topos.*

The EB prop. survives if we replace $\mathbf{Bij} \to \mathbf{Inj}$ with $\mathcal{M} \to \mathcal{C}$ where $(\mathcal{E}, \mathcal{M})$ is a factorization system on $\mathcal{C}$ with all $\mathcal{E}$-maps epi.

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(How ‘good’ the Kleisli category is depends on other things...)

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Cor. 5. *For* $\mathcal{C}$ *a monoid* $\mathcal{C}$-$\mathbf{Set} \to \mathbf{Set}$ *is EB iff* $\mathcal{C}$ *is reduced.*

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Cor. 13. *For* $\mathcal{C}$ *a monoid* $\mathcal{C}$-$\mathbf{Set} \to \mathbf{Set}$ *is EB iff* $\mathcal{C}$ *is reduced.*

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Note: in most examples of EB monads the ‘free algebra’ functor is comonadic.

“Is it possible that for some comonads the unit law implies the associative law ?”

Pre-coalgebras

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Def. 9. A *pre-coalgebra* is a pair (X, s) where $s : X \to CX$ is a map in $\mathcal{X}$ such that the diagram on the left below

$$\begin{tikzcd}
X \arrow[r, "s"] \arrow[dr, "id"'] & CX \arrow[d, "\varepsilon"] \\
& X
\end{tikzcd} \qquad \begin{tikzcd}
X \arrow[r, "s"] \arrow[d, "f"'] & CX \arrow[d, "Cf"] \\
X' \arrow[r, "s'"'] & CX'
\end{tikzcd}$$

commutes. A *morphism* $f : (X, s) \to (X', s')$ of pre-coalgebras is a map $f : X \to X'$ in $\mathcal{X}$ s.t. the diagram on the right above commutes.

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commutes. A *morphism* $f : (X, s) \to (X', s')$ of pre-coalgebras is a map $f : X \to X'$ in $\mathcal{X}$ s.t. the diagram on the right above commutes.

Let $\mathcal{X}_{\mathbf{C}}$ be the category of pre-coalgebras and morphisms between them.

The Redundant Coassociativity property

Clearly, the category $\mathcal{X}_{\mathrm{C}}$ of C-coalgebras is a full subcategory $\mathcal{X}_{\mathrm{C}} \to \underline{\mathcal{X}_{\mathrm{C}}}$.

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Example 21 (Wood)**.** Let $\mathcal{C}$ be a category with products and consider the comonadic $\mathcal{C}/I \to \mathcal{C}$. The resulting comonad $I \times (\_) : \mathcal{C} \to \mathcal{C}$ satisfies Redundant Coassociativity:

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$$\begin{tikzcd}
A \arrow[r, "s"] \arrow[dr, "id"'] & I \times A \arrow[d, "\varepsilon=\pi_1"] \\
 & A
\end{tikzcd}$$

so it is determined by the coalgebra $\pi_0 s : A \to I$.

The extended comparison

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Denote the standard comparison by $K : \mathcal{Y} \to \mathcal{X}_{\mathbf{C}}$ and ...
... extend it to a comparison $\underline{K} : \mathcal{Y} \to \underline{\mathcal{X}_{\mathbf{C}}}$

$$\begin{tikzcd}
& \mathcal{X} & \\
\mathcal{Y} \arrow[ur, "L"] \arrow[r, "K"] \arrow[rr, bend right, "\underline{K}"'] & \mathcal{X}_{\mathbf{C}} \arrow[u] \arrow[r] & \underline{\mathcal{X}_{\mathbf{C}}} \arrow[ul]
\end{tikzcd}$$

EB and RC

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& \mathcal{X} & \\
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Prop. 21. *If* $R : \mathcal{X} \to \mathcal{Y}$ *is monadic the following are equivalent:*

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 & \mathcal{X} & \\
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Prop. 22. *If $R : \mathcal{X} \to \mathcal{Y}$ is monadic the following are equivalent:*

1. $\underline{K} : \mathcal{Y} \to \underline{\mathcal{X}_{\mathbf{C}}}$ *is an equivalence.*

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$$\begin{tikzcd}
 & \mathcal{X} & \\
\mathcal{Y} \arrow[ur, "L"] \arrow[r, "K"] \arrow[rr, bend right, "\underline{K}"'] & \mathcal{X}_{\mathbf{C}} \arrow[u] \arrow[r] & \underline{\mathcal{X}_{\mathbf{C}}} \arrow[ul]
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Prop. 23. *If $R : \mathcal{X} \to \mathcal{Y}$ is monadic the following are equivalent:*

1. $\underline{K} : \mathcal{Y} \to \underline{\mathcal{X}_{\mathbf{C}}}$ *is an equivalence.*
2. $L : \mathcal{Y} \to \mathcal{X}$ *is comonadic and Redundant Coassociativity holds.*

EB and RC

$$\begin{tikzcd}
 & \mathcal{X} & \\
\mathcal{Y} \arrow[ur, "L"] \arrow[r, "K"] \arrow[rr, bend right, "\underline{K}"'] & \mathcal{X}_{\mathbf{C}} \arrow[u] \arrow[r] & \underline{\mathcal{X}_{\mathbf{C}}} \arrow[ul]
\end{tikzcd}$$

Prop. 24. *If* $R : \mathcal{X} \to \mathcal{Y}$ *is monadic the following are equivalent:*

1. $\underline{K} : \mathcal{Y} \to \underline{\mathcal{X}_{\mathbf{C}}}$ *is an equivalence.*
2. $L : \mathcal{Y} \to \mathcal{X}$ *is comonadic and Redundant Coassociativity holds.*
3. *The induced monad on* $\mathcal{Y}$ *reflects isos and is EB.*

Sketch of the proof

Every pre-coalgebra $(X, s : X \to LRX)$ induces a

$$\begin{tikzcd} RX \arrow[r, shift left=2, "\eta_R"] \arrow[r, shift right=2, "Rs"'] & RLRX \arrow[l, "R\varepsilon" description] \end{tikzcd}$$

in $\mathcal{Y}$, with $R\varepsilon$ as a common retraction of η_R and Rs.

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in $\mathcal{Y}$, with $R\varepsilon$ as a common retraction of η_R and Rs.

Define the *canonical restriction* of (X, s) as the p.b./equalizer

$$\begin{tikzcd} X_s \arrow[r, "\overline{s}"] \arrow[d, "\overline{s}"'] & RX \arrow[d, "\eta_R"] \\ RX \arrow[r, "Rs"'] & RLRX \end{tikzcd} \qquad \begin{tikzcd} X_s \arrow[r, "\overline{s}"] & RX \arrow[r, shift left=2, "\eta"] \arrow[r, shift right=2, "Rs"'] & RLRX \end{tikzcd}$$

Sketch of proof (cont.)

The assignment $(X, s) \mapsto X_s$ extends to a right adjoint $\underline{N} : \underline{\mathcal{X}_{\mathbf{C}}} \to \mathcal{Y}$ to the extended comparison $\mathcal{Y} \to \underline{\mathcal{X}_{\mathbf{C}}}$.

$$\begin{tikzcd}
 & \mathcal{X} & \\
\mathcal{Y} \arrow[ur, bend left, "L"] \arrow[r, "K"] \arrow[rr, bend left, "\underline{K}"] & \mathcal{X}_{\mathbf{C}} \arrow[r] & \underline{\mathcal{X}_{\mathbf{C}}} \arrow[ll, bend left, "\underline{N}"]
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Sketch of proof (cont.)

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A slight extension of Beck's Thm. shows t.f.a.e:

1. $\underline{K} : \mathcal{Y} \to \underline{\mathcal{X}_{\mathbf{C}}}$ is an equivalence.
2. $L : \mathcal{Y} \to \mathcal{X}$ is comonadic and Redundant Coassociativity holds.

Sketch of the proof (cont.)

Finally, if $R : \mathcal{X} \to \mathcal{Y}$ is monadic, $\mathcal{X} = \mathcal{Y}^{\mathbf{M}}$ and a pre-coalgebra $((A, a), s)$ is just

$$\begin{tikzcd}
(A, a) \arrow[r, "s"] \arrow[dr, "id"'] & (MA, \mu) \arrow[d, "a"] & = & LR(A, a) \arrow[d, "\text{counit of } L \dashv R"] \\
& (A, a) & = & (A, a)
\end{tikzcd}$$

a section $s : (A, a) \to (MA, \mu)$ of the presentation of (A, a); and...

Sketch of the proof (cont.)

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... the adjoint $\underline{N} : \underline{\mathcal{X}_{\mathrm{C}}} \to \mathcal{Y}$ maps such a pre-coalgebra to its canonical restriction.

EB iff RC

Prop. 25. *If* $R : \mathcal{X} \to \mathcal{Y}$ *is monadic t.f.a.e.:*

1. $\underline{K} : \mathcal{Y} \to \underline{\mathcal{X}_{\mathbf{C}}}$ *is an equivalence.*
2. $L : \mathcal{Y} \to \mathcal{X}$ *is comonadic and Redundant Coassociativity holds.*
3. *The induced monad on* $\mathcal{Y}$ *reflects isos and is EB.*

EB iff RC

Prop. 26. *If* $R : \mathcal{X} \to \mathcal{Y}$ *is monadic t.f.a.e.:*

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2. $L : \mathcal{Y} \to \mathcal{X}$ *is comonadic and Redundant Coassociativity holds.*
3. *The induced monad on* $\mathcal{Y}$ *reflects isos and is EB.*

Cor. 16. *Let*

$$\begin{tikzcd}
\mathcal{X} \arrow[loop right, "\mathbf{C}"] \arrow[d, shift left, "\text{monadic}"] \\
\mathcal{Y} \arrow[loop right, "\mathbf{M}"] \arrow[u, shift left, "\text{comonadic}"]
\end{tikzcd}$$

EB iff RC

Prop. 27. *If* $R : \mathcal{X} \to \mathcal{Y}$ *is monadic t.f.a.e.:*

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2. $L : \mathcal{Y} \to \mathcal{X}$ *is comonadic and Redundant Coassociativity holds.*
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\end{tikzcd}$$

Then: $\mathbf{M}$ *is EB* $\Leftrightarrow$ $\mathbf{C}$ *satisfies Redundant Coassociativity.*

EB iff RC

Prop. 28. *If $R : \mathcal{X} \to \mathcal{Y}$ is monadic t.f.a.e.:*

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2. $L : \mathcal{Y} \to \mathcal{X}$ *is comonadic and Redundant Coassociativity holds.*
3. *The induced monad on $\mathcal{Y}$ reflects isos and is EB.*

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Then: $\mathbf{M}$ *is EB* $\Leftrightarrow$ $\mathbf{C}$ *satisfies Redundant Coassociativity.*

A *CEB* monad is one that is iso-reflecting and EB.
Every CEB monad has a conservative underlying functor.

Two new examples

Example:Monadic Descent

$\mathcal{E}$ with pullbacks, $p : I \to J$ in $\mathcal{E}$, $\mathbf{M}$ the monad on $\mathcal{E}/I$ induced by $\Sigma_p \dashv p^* : \mathcal{E}/J \to \mathcal{E}/I$.

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Cor. 20. *If p is effective descent then $\mathbb{M}$ is CEB.*

Proof. The comonad on $\mathcal{E}/J$ determined by $\Sigma_p \dashv p^* : \mathcal{E}/J \to \mathcal{E}/I$ satisfies Redundant Coassociativity (Wood). □

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A little more effort shows:

Prop. 31. *$\mathbb{M}$ is CEB (anyway).*

I.e. the monadic category of ‘Descent data’ determines a CEB monad.

Example: modular categories

$\mathcal{D}$ with finite limits and coproducts.

Def. 15. $\mathcal{D}$ satisfies the *modular law* if for every $f : X \to Z$, the map

$$\begin{pmatrix} \langle in_0, f \rangle \\ in_1 \times Z \end{pmatrix} : X + (Y \times Z) \longrightarrow (X + Y) \times Z$$

is an iso for every Y in $\mathcal{D}$.

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Def. 17. $\mathcal{D}$ satisfies the *modular law* if for every $f : X \to Z$, the map

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is an iso for every Y in $\mathcal{D}$.

Def. 18 (Carboni'89)**.** $\mathcal{D}$ is called *modular* if $\mathcal{D}/U$ satisfies the modular law for every U, and

$$\begin{tikzcd}
X \arrow[r, "in_1"] \arrow[d, "f"'] & U + X \arrow[d, "U+f"] \\
U \arrow[r, "in_1"'] & U + U
\end{tikzcd}$$

is a pullback for every $f : X \to U$ in $\mathcal{D}$.

Example: modular categories

Prop. 32. *If $\mathcal{D}$ is modular then the monad induced by $D/\mathcal{D} \to \mathcal{D}$ is CEB for every D in $\mathcal{D}$.*

This is something that modular categories share with extensive categories.

Example: modular categories

Prop. 33. *If $\mathcal{D}$ is modular then the monad induced by $D/\mathcal{D} \to \mathcal{D}$ is CEB for every D in $\mathcal{D}$.*

This is something that modular categories share with extensive categories.

Lemma 3. *If $\mathcal{D}$ is modular, $1/\mathcal{D}$ is additive with kernels.*

Proof. Carboni's proof uses that for the monadic $1/\mathcal{D} \to \mathcal{D}$, every algebra is free. □

Example: modular categories

Prop. 34. *If* $\mathcal{D}$ *is modular then the monad induced by* $D/\mathcal{D} \to \mathcal{D}$ *is CEB for every* D *in* $\mathcal{D}$*.*

This is something that modular categories share with extensive categories.

Lemma 5. *If* $\mathcal{D}$ *is modular,* $1/\mathcal{D}$ *is additive with kernels.*

Proof. Carboni's proof uses that for the monadic $1/\mathcal{D} \to \mathcal{D}$, every algebra is free. ☐

Using additivity of $1/\mathcal{D}$ we get

Lemma 6. *Let* $F : \mathcal{D} \to 1/\mathcal{D}$ *be the left adjoint to* $1/\mathcal{D} \to \mathcal{D}$*. The induced comonad coincides with* $(F1) \times (\_) : 1/\mathcal{D} \to 1/\mathcal{D}$*.*

Modularity without descent

Fix $\mathcal{D}$ with finite limits and finite coproducts.
Let $\mathbb{M}$ be the monad on $\mathcal{D}$ induced by $1/\mathcal{D} \to \mathcal{D}$.

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Let $\mathbb{M}$ be the monad on $\mathcal{D}$ induced by $1/\mathcal{D} \to \mathcal{D}$.

Prop. 36. *The following are equivalent:*

1. *$\mathcal{D}$ is modular.*

Modularity without descent

Fix $\mathcal{D}$ with finite limits and finite coproducts.
Let $\mathbb{M}$ be the monad on $\mathcal{D}$ induced by $1/\mathcal{D} \to \mathcal{D}$.

Prop. 37. *The following are equivalent:*

1. *$\mathcal{D}$ is modular.*
2. *$1/\mathcal{D}$ is additive and $\mathcal{D} \to 1/\mathcal{D}$ is comonadic. (Carboni-Janelidze)*

Modularity without descent

Fix $\mathcal{D}$ with finite limits and finite coproducts.
Let $\mathbb{M}$ be the monad on $\mathcal{D}$ induced by $1/\mathcal{D} \to \mathcal{D}$.

Prop. 38. *The following are equivalent:*

1. *$\mathcal{D}$ is modular.*
2. *$1/\mathcal{D}$ is additive and $\mathcal{D} \to 1/\mathcal{D}$ is comonadic. (Carboni-Janelidze)*
3. *$1/\mathcal{D}$ is additive and $\mathbb{M}$ is CEB.*

Modularity without descent

Fix $\mathcal{D}$ with finite limits and finite coproducts.
Let M be the monad on $\mathcal{D}$ induced by $1/\mathcal{D} \to \mathcal{D}$.

Prop. 39. *The following are equivalent:*

1. $\mathcal{D}$ *is modular.*
2. $1/\mathcal{D}$ *is additive and* $\mathcal{D} \to 1/\mathcal{D}$ *is comonadic. (Carboni-Janelidze)*
3. $1/\mathcal{D}$ *is additive and* M *is CEB.*

In this case, the canonical $\mathcal{D} \to (1/\mathcal{D})/(F1)$ *is an equivalence.*

Modularity without descent

Fix $\mathcal{D}$ with finite limits and finite coproducts.
Let M be the monad on $\mathcal{D}$ induced by $1/\mathcal{D} \to \mathcal{D}$.

Prop. 40. *The following are equivalent:*

1. *$\mathcal{D}$ is modular.*
2. *$1/\mathcal{D}$ is additive and $\mathcal{D} \to 1/\mathcal{D}$ is comonadic. (Carboni-Janelidze)*
3. *$1/\mathcal{D}$ is additive and M is CEB.*

In this case, the canonical $\mathcal{D} \to (1/\mathcal{D})/(F1)$ is an equivalence.

Cor. 27. *Assume that $1/\mathcal{D}$ is additive. Then:*

Modularity without descent

Fix $\mathcal{D}$ with finite limits and finite coproducts.
Let M be the monad on $\mathcal{D}$ induced by $1/\mathcal{D} \to \mathcal{D}$.

Prop. 41. *The following are equivalent:*

1. *$\mathcal{D}$ is modular.*
2. *$1/\mathcal{D}$ is additive and $\mathcal{D} \to 1/\mathcal{D}$ is comonadic. (Carboni-Janelidze)*
3. *$1/\mathcal{D}$ is additive and M is CEB.*

In this case, the canonical $\mathcal{D} \to (1/\mathcal{D})/(F1)$ is an equivalence.

Cor. 28. *Assume that $1/\mathcal{D}$ is additive. Then:*

1. *$\mathcal{D}$ is additive if and only if M is trivial.*

Modularity without descent

Fix $\mathcal{D}$ with finite limits and finite coproducts.
Let $\mathbb{M}$ be the monad on $\mathcal{D}$ induced by $1/\mathcal{D} \to \mathcal{D}$.

Prop. 42. *The following are equivalent:*

1. *$\mathcal{D}$ is modular.*
2. *$1/\mathcal{D}$ is additive and $\mathcal{D} \to 1/\mathcal{D}$ is comonadic. (Carboni-Janelidze)*
3. *$1/\mathcal{D}$ is additive and $\mathbb{M}$ is CEB.*

In this case, the canonical $\mathcal{D} \to (1/\mathcal{D})/(F1)$ is an equivalence.

Cor. 29. *Assume that $1/\mathcal{D}$ is additive. Then:*

1. *$\mathcal{D}$ is additive if and only if $\mathbb{M}$ is trivial.*
2. *$\mathcal{D}$ is modular if and only if $\mathbb{M}$ is CEB.*

That’s it, thanks.