Categorical Foundations for K-theory

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How is the K-theory of an object defined?

$$\text{Object} \longmapsto \text{Structured category} \xrightarrow[\text{machine}]{K\text{-theory}} \text{Infinite loop space/ Spectrum}$$

Possible structures:

- Quillen-exact,
- Waldhausen,
- Symmetric monoidal,
- One of these structures together with a topological enrichment.

Object	**Structured category**
Ring R	▪ Pseudo-coherent R-modules ▪ Finitely generated projective R-modules ▪ Finitely generated free R-modules
Space X	▪ ($\mathbb{R}$ or $\mathbb{C}$-)vector bundles over X
Ringed space $(X, \mathscr{O}_X)$	▪ Coherent $\mathscr{O}_X$-modules ▪ Locally free $\mathscr{O}_X$-modules of finite rank
Ring spectrum/ S-algebra R	▪ Semi-finite cell R-modules ▪ Finite cell R-modules

Back to questions

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2. What structured categories should be associated to such an object in order to define its K-theories?
3. How does this correspondance take the morphisms of these objects into account?

▸ Examples of K-theories

Op-fibred category setting

Let $\mathscr{B}$ be a category.

Notation

$OPFIB(\mathscr{B})$ is the 2-category of opfibrations over $\mathscr{B}$, opCartesian functors over $\mathscr{B}$ and natural transformations over $\mathscr{B}$.

Definition

A monoidal opfibred category is a monoidal object in the 2-Cartesian 2-monoidal category $OPFIB(\mathscr{B})$.

Notation

$MONOPFIB(\mathscr{B})$ is the 2-category of monoidal objects, strong monoidal morphisms and monoidal 2-cells in $OPFIB(\mathscr{B})$.

Op-indexed category setting

Notation

- *MONCAT is the 2-category of monoidal categories, strong monoidal functors and monoidal natural transformations.*
- *$MONCAT^{\mathscr{B}}$ is the corresponding 2-category of pseudo-functors, pseudo-natural transformations and modifications.*

Theorem

There is a 2-equivalence

$$MONOPFIB(\mathscr{B}) \simeq MONCAT^{\mathscr{B}}.$$

Monoids

Let $(\mathscr{E} \xrightarrow{P} \mathscr{B}, \otimes, u)$ be a monoidal opfibred category.

Definition

- A monoid in $\mathscr{E}$ is a monoid in any fibre of $\mathscr{E}$.
- A morphism of monoids $f\colon R \to S$ is a morphism in $\mathscr{E}$ such that

$$
\begin{array}{ccc}
R \otimes R & \xrightarrow{\phi \otimes \phi} & S \otimes S \\
{\scriptstyle \mu}\big\downarrow & & \big\downarrow{\scriptstyle \nu} \\
R & \xrightarrow[\phi]{} & S
\end{array}
\qquad\qquad
\begin{array}{ccc}
I_{P(R)} & \xrightarrow{u(P(\phi))} & I_{P(S)} \\
{\scriptstyle \eta}\big\downarrow & & \big\downarrow{\scriptstyle \lambda} \\
R & \xrightarrow[\phi]{} & S
\end{array}
$$

Modules

Definition

- A (right) **module** in $\mathscr{E}$ is a pair (R, M) where R is a monoid in $\mathscr{E}$ and M a (right) R-module in $\mathscr{E}_{P(R)}$.
- A **morphism of modules** is a pair $(\phi, \alpha)\colon (R, M) \to (S, N)$ where:
  - $\phi\colon R \to S$ and $\alpha\colon M \to N$ are morphisms in $\mathscr{E}$ such that $P(\phi) = P(\alpha)$,
  - $\phi\colon R \to S$ is a morphism of monoids in $\mathscr{E}$,
  - the following diagram commutes:

$$\begin{array}{ccc} M \otimes R & \xrightarrow{\alpha \otimes \phi} & N \otimes S \\ \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle \sigma} \\ M & \xrightarrow[\alpha]{} & N. \end{array}$$

An opfibration over an opfibration

Let $(\mathscr{E} \xrightarrow{P} \mathscr{B}, \otimes)$ be a monoidal opfibration.

Proposition

Suppose P has opfibred reflexive coequalizers and that the functors $- \otimes E \colon \mathscr{E}_B \to \mathscr{E}_B$, for all $B \in \mathscr{B}$ and $E \in \mathscr{E}_B$, preserve reflexive coequalizers. Then, there is an opfibration over an opfibration.

$$\begin{array}{c} Mod\mathscr{E} \\ \big\downarrow \\ Mon\mathscr{E} \\ \big\downarrow{\scriptstyle P} \\ \mathscr{B} \end{array}$$

Example

Sheaves of modules over ringed spaces

Monoidal opfibration of sheaves of abelian groups over spaces:

$$Sh \to Top^{op}.$$

Modules and commutative monoids in there:

$$Mod(Sh) \to Comm(Sh) \to Top^{op}.$$

Dual gives sheaves of modules over ringed spaces:

$$\mathscr{O}\text{-}Mod \to Ringed \to Top.$$

Sites

Definitions

Let $\mathscr{C}$ be a category.

- A covering of an object $C \in \mathscr{C}$ is a set R of arrows of codomain C.
- A covering function J on $\mathscr{C}$ is a function that assigns a class of coverings $J(C)$ to each object $C \in \mathscr{C}$.
- A site is a pair $(\mathscr{C}, J)$ where J is a covering function on the category $\mathscr{C}$.

Locally trivial objects

Let $(\mathscr{B}, J)$ be a site containing all identity-singleton coverings.

Definitions

- A subcategory of trivial objects is a full and replete subcategory $Triv \subset \mathscr{B}$.

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- An object $B \in \mathscr{B}$ is locally trivial if it can be covered by a J-covering R whose domains are trivial. Full subcategory of locally trivial objects $\mathit{Loc} \subset \mathscr{B}$.

$$\mathit{Triv} \subset \mathit{Loc} \subset \mathscr{B}.$$

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$$Triv \subset Loc \subset \mathscr{B}.$$

- Site (Loc, J_l) whose coverings are J-coverings with trivial domains.

Example

Topological manifolds

Site (Top, J)	Open subset pretopology.
Trivial objects	Euclidean spaces.
Locally trivial objects	Topological manifolds.
J_l	Coverings of topological manifols by open euclidean spaces.

Fibred sites

Definitions

- A fibred site is a fibration $\mathscr{E} \xrightarrow{P} \mathscr{B}$ together with a site $(\mathscr{B}, J)$ on its base.

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- A fibred site is a fibration $\mathscr{E} \xrightarrow{P} \mathscr{B}$ together with a site $(\mathscr{B}, J)$ on its base.
- The induced covering function is the covering function $J_{\mathscr{E}}$ on $\mathscr{E}$ whose coverings are Cartesian lifts of J-coverings.

Locally trivial objects

Definition

Let $P\colon \mathscr{E} \to (\mathscr{B}, J)$ be a fibred site. A subfibration of trivial objects of P is a "globally" replete and full subfibration

$$\begin{array}{ccc} Triv_t & \hookrightarrow & \mathscr{E} \\ {\scriptstyle Triv}\downarrow & & \downarrow{\scriptstyle P} \\ Triv_b & \hookrightarrow & \mathscr{B}. \end{array}$$

One can then consider the subfunctor of locally trivial objects

$$\begin{array}{ccccc} Loc(Triv_t, J_{\mathscr{E}}) & =: & Loc_t & \hookrightarrow & \mathscr{E} \\ & & {\scriptstyle Loc}\downarrow & & \downarrow{\scriptstyle P} \\ Loc(Triv_b, J) & =: & Loc_b & \hookrightarrow & \mathscr{B}. \end{array}$$

Fibrational properties of locally trivial objects

Proposition

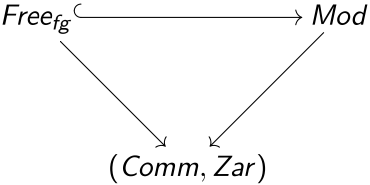
Let $P\colon \mathscr{E} \to (\mathscr{B}, J)$ be a fibred site with J containing all identity-singleton coverings. Let $Triv \subset P$ a subfibration of trivial objects. Suppose that J_l is a coverage.

Then, $Loc \subset P$ is a subfibration.

$$Triv \subset Loc \subset P.$$

Example

Finitely generated projective modules



Example

Finitely generated projective modules

$$
\begin{array}{ccccc}
Free_{fg} & \hookrightarrow & Proj_{fg} & \hookrightarrow & Mod \\
& \searrow & \downarrow & \swarrow & \\
& & (Comm, Zar) & &
\end{array}
$$

Example

Finitely generated projective modules

Other examples:

- vector bundles,
- G-torsors,
- locally constant sheaves (of rings, abelian groups, . . .),
- schemes,
- locally free sheaves of modules,
- . . .

Locally trivial modules

Monoidal opfibred category $\mathscr{E} \xrightarrow{P} \mathscr{B}$ with previous conditions about reflexive coequalizers in the fibres.

$$\begin{array}{ccc} & & (Mod\mathscr{E})^{op} \\ & & \downarrow \\ (\mathscr{C}, J) & \longrightarrow & (Mon\mathscr{E})^{op} \end{array}$$

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$$\begin{array}{ccc} Mod_{\mathscr{C}} & \longrightarrow & (Mod\mathscr{E})^{op} \\ \downarrow & & \downarrow \\ (\mathscr{C}, J) & \longrightarrow & (Mon\mathscr{E})^{op} \end{array}$$

Locally trivial modules

Monoidal opfibred category $\mathscr{E} \xrightarrow{P} \mathscr{B}$ with previous conditions about reflexive coequalizers in the fibres.

$$\begin{array}{ccccc}
Triv_t & \hookrightarrow & Mod_{\mathscr{C}} & \longrightarrow & (Mod\mathscr{E})^{op} \\
\downarrow & & \downarrow & & \downarrow \\
Triv_b & \hookrightarrow & (\mathscr{C}, J) & \longrightarrow & (Mon\mathscr{E})^{op}
\end{array}$$

Locally trivial modules

Monoidal opfibred category $\mathscr{E} \xrightarrow{P} \mathscr{B}$ with previous conditions about reflexive coequalizers in the fibres.

$$\begin{array}{ccccccc} Triv_t & \hookrightarrow & Loc_t & \hookrightarrow & Mod_{\mathscr{C}} & \longrightarrow & (Mod\mathscr{E})^{op} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Triv_b & \hookrightarrow & Loc_b & \hookrightarrow & (\mathscr{C}, J) & \longrightarrow & (Mon\mathscr{E})^{op} \end{array}$$

Example

Coherent sheaves of modules over schemes

$$
\begin{array}{ccccccc}
Mod_{fp}^{op} & \xrightarrow{(\mathrm{Spec},\mathscr{O},\tilde{\ })} & Coh & \hookrightarrow & \mathscr{O}\text{-}Mod_{l} & \longrightarrow & \mathscr{O}\text{-}Mod \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Comm^{op} & \xrightarrow[(\mathrm{Spec},\mathscr{O})]{} & Sch & \hookrightarrow & (LRinged, Zar) & \hookrightarrow & Ringed
\end{array}
$$

Modules in a monoidal abelian opfibration

Theorem (Ardizzoni, 2004)

Let $\mathscr{V}$ be a monoidal category whose underlying category is abelian. Let R be a monoid in $\mathscr{V}$. Suppose that the functor $- \otimes R$ preserve finite colimits.
Then, the category Mod_R of R-modules in $\mathscr{V}$ is abelian.

Definition

We call a monoidal category that is abelian and such that each $- \otimes A$ preserves finite colimits a (right) monoidal abelian category.

Corollary

Let $\mathscr{E} \to \mathscr{B}$ be monoidal bifibration whose fibres are monoidal abelian categories. Then, there is a bifibration $Mod(\mathscr{E}) \to Mon(\mathscr{E})$ whose fibres are abelian and direct image functors additive.

Exact categories of locally trivial objects

Proposition

Suppose:

- $P\colon \mathscr{E} \to \mathscr{B}$ *fibred in abelian categories and additive functors.*
- P *comes with a subfibration of trivial objects* $Triv \subset P$ *whose fibres* $(Triv_t)_B \subset \mathscr{E}_B$ *are Quillen-exact subcategories for all* $B \in Triv_b$.
- *Site* $(\mathscr{B}, J)$ *such that* J_l *is a coverage satisfying axiom (L).*
- *The inverse image functors of* P *are exact over* J_l*-coverings.*

Then for each $B \in Loc_b$, $(Loc_t)_B \subset \mathscr{E}_B$ *is a Quillen-exact subcategory.*