A model structure for quasi-2-categories

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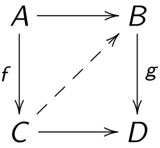
This is a report of joint work with Ross Street and Dominic Verity. All errors though are my own.

Orthogonality

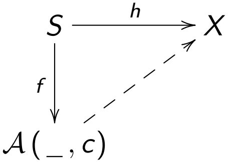
Let $\mathcal{L}$ and $\mathcal{R}$ be sets of morphisms in a category $\mathcal{A}$.
We write $\mathcal{L} \perp \mathcal{R}$ or $\mathcal{L} = \mathrm{llp}\,(\mathcal{R})$ or $\mathcal{R} = \mathrm{rlp}\,(\mathcal{L})$
or say that "$\mathcal{R}$ has the RLP with respect to $\mathcal{L}$"
when for every morphism $f \in \mathcal{L}$ and $g \in \mathcal{R}$
and every commutative square



there exists a diagonal (dashed) morphism
which makes the two triangles commute.

Horns (general)

Let $\widehat{\mathcal{A}} = [\mathcal{A}^{\mathrm{op}}, \mathit{Set}]$ be a presheaf category and let c be an object of $\mathcal{A}$. A (freestanding) horn is a morphism f from a designated set of morphisms with codomain $\mathcal{A}(\_, c)$.



A horn in X is a morphism $h\colon S \to X$. A filler for this horn is a diagonal morphism making the triangle commute.
We say that "X has the RLP with respect to f"
when there exists a filler for any h.

By Yoneda's Lemma, a horn in X corresponds to a collection of elements in X and a filler corresponds to a c-element in X.

Horns and higher categories

One approach to defining higher categories is to look for shape categories $\mathcal{A}$ and for definitions of horns in $\widehat{\mathcal{A}}$ such that

- Horns provide a notion of composable morphisms
- Fillers provide a notion of composite morphism

Higher categories are defined as those presheaves which have the RLP with respect to the horns (These presheaves satisfy a horn-filler condition).

Three examples

- Quasi-categories (Boardman & Vogt, Joyal)
- Weak complicial sets (Roberts, Street, Verity)
- Θ-sets (Joyal)

Horns in simplicial sets

$\mathcal{A} = \Delta$ is the skeletal category of finite linearly ordered sets with

- objects: $[n] = \{0, 1, \ldots, n\}$ for $n \geq 0$
- arrows: order preserving maps

Simplicial sets are contravariant functors $X \colon \Delta^{\mathrm{op}} \to Set$.
The elements x of X_n are called n-simplicies and have dimension n.

An n-simplex x has $n+1$ compatible faces of dimension $n-1$. This collection of faces is called a sphere and is the boundary of x. A horn is a sphere minus one of its faces. Suppose the missing face is the i-th face. When $i = 0, n$ the face is outer, e.g. first diagram. Otherwise it is inner, e.g. second diagram.

Kan complexes and quasi-categories

A simplicial set is a Kan complex if every horn has a filler and is a weak Kan complex if every inner horn has a filler.

Quasi-categories are weak Kan complexes and so have (not necessarily unique) fillers for all inner horns.
They also have

- objects as 0-simplices
- weak composition of 1-simplices given by 2-simplices
- a notion of homotopy given by certain 2-simplices
- homotopies between different composites
- 3-simplices witnessing associativity

See *Associativity Data in an* $(\infty, 1)$*-category* by Emily Riehl.

Model category

A model category is a category $\mathcal{A}$ with all small limits and colimits and with three sets $\mathcal{W}$, $\mathcal{C}$ and $\mathcal{F}$ of morphisms which contain the weak equivalences, cofibrations and fibrations (respectively).
The set $\mathcal{F} \cap \mathcal{W}$ contains the trivial fibrations and
the set $\mathcal{C} \cap \mathcal{W}$ contains the trivial cofibrations such that

M1 If two of f, g and gf are weak equivalences, so is the third

M2 Each of the three sets ($\mathcal{W}$, $\mathcal{C}$ and $\mathcal{F}$) is closed under retracts

M3 $\mathcal{C} \perp \mathcal{F} \cap \mathcal{W}$ and $\mathcal{C} \cap \mathcal{W} \perp \mathcal{F}$

M4 Every morphism can be factored into
- a trivial cofibration followed by a fibration
- a cofibration followed by a trivial fibration

Fibrant objects are those objects defined as having the RLP with respect to the trivial cofibrations.

Cofibrantly generated I

Let $\mathcal{K}$ be a set of morphisms in a cocomplete category. Define

- cell $(\mathcal{K})$ as the closure of $\mathcal{K}$ under p.o. and t.f. composition
- cof $(\mathcal{K})$ as the closure of cell $(\mathcal{K})$ under retracts

A model category $\mathcal{A}$ is said to be cofibrantly generated if it is locally presentable and if there is a small set $\mathcal{I}$ and a small set $\mathcal{J}$ such that

- cof $(\mathcal{I})$ is the set of cofibrations
- cof $(\mathcal{J})$ is the set of trivial cofibrations
- $\mathcal{I}$ and $\mathcal{J}$ permit the small object argument

Cofibrantly generated II

In a cofibrantly generated model category

- $\mathrm{cof}(\mathcal{I}) = \mathrm{llp}(\mathrm{rlp}(\mathcal{I}))$
- $\mathrm{cof}(\mathcal{J}) = \mathrm{llp}(\mathrm{rlp}(\mathcal{J}))$

Hence the fibrations (and fibrant objects) have the RLP with respect to the <u>generating</u> trivial cofibrations.

We find weak higher categories as fibrant objects in a cofibrantly generated model category with generating trivial cofibrations the horns of interest.

Model structure for quasi-categories

The category of simplicial sets has
a cofibrantly generated model structure where

- Generating cofibrations $\mathcal{I}_\Delta$ are boundary inclusions into representables.
- Generating trivial cofibrations $\mathcal{J}_\Delta$ are the inner horns
- Cofibrations $\mathrm{cof}\,(\mathcal{I}_\Delta)$ are the monomorphisms
- Fibrant objects are quasi-categories

Extensions of quasi-categories

Two extensions of the concept of quasi-category

- Weak complicial sets
- Θ-sets

There is a cofibrantly generated model category whose fibrant objects are weak complicial sets. (See *Weak Complicial sets, A Simplicial Weak ω-Category Theory* by Dominic Verity).

Our work has been to construct a model structure on Θ_2-sets whose fibrant objects define quasi-2-categories.

Model structure for quasi-2-categories

We constructed a cofibrantly generated model structure on the presheaf category of Θ_2-sets where

- Generating trivial cofibrations $\mathcal{J}_\Theta$ are the horns
- Cofibrations are monomorphisms
- Fibrant objects are quasi-2-categories

The remainder of the talk will describe

- the generating trivial cofibrations (aka the horns)
- the concepts of homotopy and weak equivalence

on this model structure.

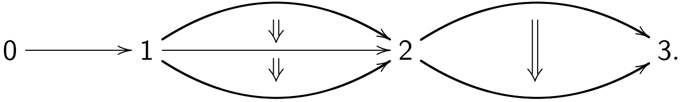
Theta

André provided the original definition of Θ as an extension of Δ. Equivalent definitions were given by Batanin-Street, Berger, Makkai-Zawadowski and Oury.

The category Θ_2 is a full subcategory of 2-*Cat*.
Let $\mathbf{m} = m_1, \ldots, m_n$ and let $[n; \mathbf{m}]$ denote the 2-category with

- objects $0, 1, ..., n$
- hom-sets $[n; \mathbf{m}](i-1, i) = [m_i]$

For example, the object $[3; 0, 2, 1]$ is



Θ_2 embedding — $\widehat{\Delta}(\_)$

Let $(\mathcal{A}, \otimes)$ be a monoidal category.
We define a pseudofunctor $\mathcal{A}(\_) : \Delta^{\mathrm{op}} \to \mathit{Cat}$

- the category $\mathcal{A}([n])$ is $\mathcal{A} \times \ldots \times \mathcal{A}$ (n times)
- the functor $\mathcal{A}(\gamma \colon [n] \to [m]) : \mathcal{A}([m]) \to \mathcal{A}([n])$ sends an object $(W_i)_{i=1}^{m}$ to

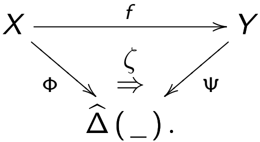
$$\left(\bigotimes_{j=\gamma(i-1)+1}^{\gamma(i)} W_j \right)_{i=1}^{n}$$

We only use $\mathcal{A} = \widehat{\Delta}$ with cartesian product; that is, we use $\widehat{\Delta}(\_)$.

Θ_2 embedding — $\widehat{\Delta} \wr \widehat{\Delta}$

The category $\widehat{\Delta} \wr \widehat{\Delta}$ has

- Objects as pairs $\left(X \colon \Delta^{\mathrm{op}} \to \mathit{Cat}, \Phi \colon X \to \widehat{\Delta}(\_)\right)$ with X_k a discrete category for all k.
- Morphisms as pairs $(f, \zeta) : (X; \Phi) \to (Y; \Psi)$ with $f \colon X \to Y$ and $\zeta \colon \Phi \to \Psi f$.



Θ_2 embedding

Define an embedding of Θ_2 into $\widehat{\Delta} \wr \widehat{\Delta}$ by

- sending $[n; \mathbf{m}]$
- to the pseudonatural transformation $\Delta[n] \to \widehat{\Delta}(\_)$ which sends $\mathrm{Id}_{[n]}$ to $(\Delta[m_1], \ldots, \Delta[m_n])$

Box functor and representables

The Kan construction on the embedding $\Theta_2 \xrightarrow{\text{emb}} \widehat{\Delta} \wr \widehat{\Delta}$ induces a functor $\Box \colon \widehat{\Delta} \wr \widehat{\Delta} \to \widehat{\Theta_2}$ defined

- by sending $\left(X \colon \Delta^{\mathrm{op}} \to \mathit{Cat}, \Phi \colon X \to \widehat{\Delta}\,(\_)\right)$
- to $\widehat{\Delta} \wr \widehat{\Delta} \left(\text{emb}\,(\_)\,, X \xrightarrow{\Phi} \widehat{\Delta}\,(\_)\right)$

Define a functor $\Box_n \colon \widehat{\Delta}/_{\Delta[n]} \times \widehat{\Delta} \times \ldots \times \widehat{\Delta} \to \widehat{\Theta_2}$

- by sending $\left(A \xrightarrow{f} \Delta[n], X_1, \ldots, X_n\right)$
- to $\Box \left(A \xrightarrow{\Phi \circ f} \widehat{\Delta}\,(\_)\right)$

where Φ is defined by sending $\mathrm{Id}_{[n]}$ to $(X_1, \ldots, X_n)$.
The representable $\Theta[n; \mathbf{m}]$ is isomorphic to

$$\Box_n \left(\mathrm{Id}_{\Delta[n]}, \Delta[m_1], \ldots, \Delta[m_n]\right).$$

Abstract corner tensor (Street)

Let $\mathcal{V}$ be a complete cocomplete closed symmetric monoidal category. Suppose we have a functor $\wedge\colon \mathcal{T} \otimes \ldots \otimes \mathcal{T} \to \mathcal{T}$.
Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ and $\mathcal{C}$ be categories which admit enough colimits to allow tensoring with homs of $\mathcal{T}$ and taking coends over $\mathcal{T}$.
Suppose we have a functor

$$\square\colon \mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_n \to \mathcal{C}.$$

Using Day convolution, define

$$\overline{\square}\colon\ [\mathcal{T}, \mathcal{A}_1] \otimes \ldots \otimes [\mathcal{T}, \mathcal{A}_n] \to [\mathcal{T}, \mathcal{C}]$$

by putting $\overline{\square}\,(M_1, \ldots, M_n)\,t$ isomorphic to

$$\int^{u_1, \ldots, u_n} \mathcal{T}\left(\wedge\left(u_1, \ldots, u_n\right), t\right) \otimes \square\left(M_1 u_1, \ldots, M_n u_n\right).$$

Corner tensor (specialized)

Let $\mathcal{T}$ be the category $\mathbf{2}$ and let $\wedge$ be the infimum.
Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ and $\mathcal{C}$ be cocomplete categories and let

$$\square \colon \mathcal{A}_1 \times \ldots \times \mathcal{A}_n \to \mathcal{C}.$$

We have a functor $\overline{\square} \colon \mathcal{A}_1^{\mathbf{2}} \times \ldots \times \mathcal{A}_n^{\mathbf{2}} \to \mathcal{C}^{\mathbf{2}}$ defined for a family of morphisms $f_1, \ldots, f_n$ by

$$\overline{\square}\,(f_1, \ldots, f_n)\, t \cong \int^{u_0, \ldots, u_n \in \mathbf{2}} \mathbf{2}\,(u_0 \wedge \ldots \wedge u_n, t) \cdot \square\,(f_1 u_0, \ldots, f_n u_n)\,.$$

Using the product functor for $\square$ we have what is known as the pushout smash (Hess) and pushout product (Hovey).

Categorical nerve

The categorical nerve $N_\Delta \mathcal{C}$ of $\mathcal{C}$ is defined by

$$(N_\Delta \mathcal{C})_n = Cat([n], \mathcal{C}).$$

Let $\mathbb{I}$ denote the chaotic category with two distinct objects. Let $\mathbb{I}_\Delta$ denote $N_\Delta \mathbb{I}$ which is the interval object for the model structure on simplicial sets whose fibrant objects are quasi-categories.

Horns

Let

- $h^{(n)k}$, $h^{(m_i)k}$ be horns of $\widehat{\Delta}$
- $e\colon \Delta[0] \to \mathbb{I}_\Delta$ be the equivalence extension of $\widehat{\Delta}$
- the morphisms b^n, b^{m_i} be boundary inclusions of $\widehat{\Delta}$

A *horizontal horn* is

$$\overline{\square_n}\left(h^{(n)k}, b^{m_1}, \dots, b^{m_n}\right)$$

A *vertical horn* is

$$\overline{\square_n}\left(b^n, b^{m_1}, \dots, b^{m_{h-1}}, h^{(m_h)k}, b^{m_{h+1}}, \dots, b^{m_n}\right)$$

A *vertical equivalence extensions* is

$$\overline{\square_n}\left(b^n, b^{m_1}, \dots, b^{m_{k-1}}, e, b^{m_{k+1}}, \dots, b^{m_n}\right).$$

Quasi-2-categories

Quasi-2-categories are defined as the fibrant objects and so are those objects having the RLP with respect to these morphisms.

Oriented simplices in 2-cat

Ross Street introduced

the free n-category $\mathcal{O}_n$ on the oriented n-simplex $\Delta[n]$.

Define a functor $\overline{\mathcal{O}} \colon \Delta \to 2\text{-}Cat$ where the 2-category $\overline{\mathcal{O}}_n$ is obtained from the n-th oriental $\mathcal{O}_n$ by identifying all cells of dimension strictly greater than 2 and inverting those of dim'n 2.

Θ_2-nerve of 2-categories

We use the Kan process on the inclusion of Θ_2 into 2-*Cat* to define a functor $N_2\colon \text{2-}Cat \to \widehat{\Theta_2}$ by

$$(N_2\mathcal{A})_{[p;\mathbf{q}]} = \text{2-}Cat\left([p;\mathbf{q}],\mathcal{A}\right).$$

Homotopy coherent nerve

We use the Kan process on the composite

$$\Delta \xrightarrow{\overline{\mathcal{O}}} 2\text{-}Cat \xrightarrow{N_2} \widehat{\Theta_2}.$$

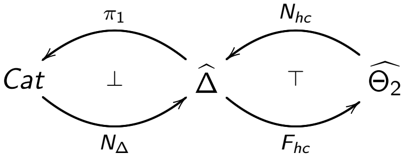
to obtain an adjunction

$$\widehat{\Theta_2} \underset{N_{hc}=\widetilde{K}}{\overset{F_{hc}=\_\otimes K}{\rightleftarrows}} \widehat{\Delta} \quad (\perp)$$

and define a *homotopy coherent nerve* $N_{hc}\colon \widehat{\Theta_2} \to \widehat{\Delta}$.

Adjunctions and enrichment

We have a pair of adjunctions



and enrich $\widehat{\Theta_2}$ over $\widehat{\Delta}$ to obtain the $\widehat{\Delta}$-category $(N_{hc})_* \widehat{\Theta_2}$
and enrich $\widehat{\Theta_2}$ over Cat to obtain the 2-category $(\pi_1 N_{hc})_* \widehat{\Theta_2}$.

We do so to interpret homotopy and homotopy equivalence in terms of quasi-isomorphisms and isomorphisms (respectively) in these two enriched categories.

Interval object and the horiz. equivalence extension

We define the *interval object* of $\widehat{\Theta_2}$ as $F_{hc}\mathbb{I}_\Delta$ denoted $\mathbb{I}_\Theta$.

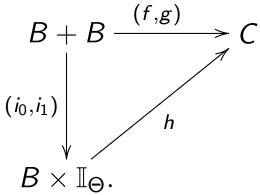
The *elementary horizontal equivalence* is
the morphism $E\colon \Theta[0;] \to \mathbb{I}_\Theta$ defined by sending $*$ to 0.

The *horizontal equivalence extensions* are morphisms
of the form $E \mathbin{\overline{\times}} b$ with b a boundary morphism.

The fibrations are those morphisms with the RLP with respect to the horns and equivalence extensions (horizontal and vertical).

Homotopy

A homotopy between parallel morphisms $f, g \colon B \to C$ is a morphism $h \colon \mathbb{I}_\Theta \to [B, C]$ such that



A pair of morphisms $f \colon X \to Y$ and $g \colon Y \to X$ are homotopy equivalences when gf is homotopic to Id_X and fg is homotopic to Id_Y.

More homotopy

Morphisms f and g are homotopic

if and only if

they are vertices of an adjoint quasi-isomorphism in $N_{hc}[B, C]$

if and only if

they are isomorphic in $\pi_1 N_{hc}[B, C]$.

Morphisms g and h are mutual homotopy inverses if and only if they are mutually inverse 1-cells in the 2-category $(\pi_1 N_{hc})_* \widehat{\Theta_2}$.

The homotopy relation is an equivalence relation.

Weak equivalences

A morphism $w\colon U \to V$ of $\widehat{\Theta_2}$ is a *weak equivalence* when

$$w^*\colon [V, A] \to [U, A]$$

is a homotopy equivalence for all fibrant objects A.